Exercises 1

1 PL validity

For each of the following propositional logic formulae determine whether it is valid or not. If it is valid prove it, otherwise give a counterexample.

(i) $P \wedge Q \to P \to Q$

(ii) $(P \to Q) \vee P \wedge \neg Q$

(iii) $(P \to Q \to R) \to P \to R$

(iv) $(P \to Q \vee R) \to P \to R$

(v) $\neg(P \vee Q) \to R \to \neg R \to Q$

2 FOL validity

For each of the following predicate logic formulae determine whether it is valid or not. If it is valid prove it, otherwise give a counterexample.

(i) $(\forall x, y.p(x, y) \to p(y, x)) \to \forall z.p(z, z)$

(ii) $\forall x, y.p(x, y) \to p(y, x) \to \forall z.p(z, z)$

(iii) $(\exists x.p(x)) \to \forall y.p(y)$

(iv) $(\forall x.p(x)) \to \exists y.p(y)$

(v) $\exists x, y.(p(x, y) \to (p(y, x) \to \forall z.p(z, z)))$

3 FOL Normal forms

Put the following formulae into prenex normal form:

(i) $(\forall x.\exists y.p(x, y)) \to \forall x.p(x, x)$

(ii) $\exists z.(\forall x.\exists y.p(x, y)) \to \forall x.p(x, z)$

(iii) $\forall w.\neg(\exists x, y.\forall z.p(x, z) \to q(y, z)) \wedge \exists z.p(w, z)$

4 Redundant logical connectives

a) Given $\top, \wedge$ and $\neg$, prove that $\bot, \vee, \rightarrow$ and $\leftrightarrow$ are redundant logical connectives. That is, show that each of $\bot$, $F_1 \vee F_2$, $F_1 \rightarrow F_2$ and $F_1 \leftrightarrow F_2$ is equivalent to a formula that uses only $F_1, F_2, \top, \wedge$ and $\neg$.

b) Now extend propositional formulas with a NAND operator, denoted $\bar{\wedge}$ and defined by

$$x \mathbin{\bar{\wedge}} y = \neg(x \wedge y)$$

Show that for each propositional formula F there exists an equivalent formula that uses $\bar{\wedge}$ as the only operator.

5 Complexity of normal forms

a) We define the size of a formula as the number of nodes in its syntax tree. For example, $size(P \wedge \neg R) = 4$, where P and R are propositional variables.

Now consider propositional formulas containing only $\wedge, \vee, \rightarrow, \neg$ and the recursive definition of NNF for propositional logic.

Find an integer constant K such that, for every such formula G we have:

$$size(NNF(G)) \leq K \cdot size(G)$$

Once you guess the value K, prove that the above inequation holds for this K, using mathematical induction.

b) Prove that there is no polynomial-time algorithm for transforming a propositional formula into an equivalent formula in conjunctive normal form. You do not need to use any deep results of complexity theory.

Specifically, prove that there exists an infinite family of formulas $F_1, F_2, \ldots$ such that for each n, every algorithm that transforms F_n to CNF needs exponential time. (Note that it is not enough to prove that one particular algorithm will take exponential time, you need to prove that every algorithm would need exponential time.)

6 Relations

Prove the following or give a counterexample.

(i) $(r \cup s) \circ t = (r \circ t) \cup (s \circ t)$

(ii) $(r \cap s) \circ t = (r \circ t) \cap (s \circ t)$

(iii) $(r_1 \circ r_2)^{-1} = (r_2^{-1} \circ r_1^{-1})$

(iv) $S \bullet r = ran(\triangle_S \circ r)$

(v) If $r_1 \subseteq r_1'$ then $r_1 \circ r_2 \subseteq r_1' \circ r_2$ and $r_2 \circ r_1 \subseteq r_2 \circ r_1'$.

(vi) If $r_1 \subseteq r_1'$ then $r_1 \cup r_2 \subseteq r_1' \cup r_2$ and $r_2 \cup r_1 \subseteq r_2 \cup r_1'$.

7 Composition of partial functions

Given two partial functions r_1 and r_2, show that $r = r_1 \circ r_2$ is also a partial function.

8 Transitive relations

Given a relation $r \subseteq A \times A$, prove that r is transitive if and only if $r \circ r \subseteq r$.

9 Symmetric relations

Recall that a relation $r \subseteq A \times A$ is symmetric if $\forall x, y \in A.\ (x, y) \in r \rightarrow (y, x) \in r$.
Now let r be an arbitrary relation. Prove that $r^{-1} \circ (r \cup r^{-1})^* \circ r$ is symmetric.

10 Transitive closure

Recall that we define the powers of a relation $r \subseteq A \times A$ as follows:

$$r^0 = \triangle_A, \quad r^1 = r, \quad \text{and} \quad r^{n+1} = r^n \circ r$$

We showed that the *reflexive and transitive closure* $r^* = \bigcup_{n \geq 0} r^n$ is the smallest reflexive and transitive relation on A containing r. Show that for any relation r on a set A, $(r \cup r^{-1})^*$ is the least equivalence relation containing r. Precisely, show that

(i) $(r \cup r^{-1})^*$ is an equivalence relation, and

(ii) if s is an equivalence relation containing r, then $(r \cup r^{-1})^* \subseteq s$.

11 Monotonicity of relation composition

Let $E(r_1, r_2, ..., r_n)$ be a relation composed of relations r_i with an arbitrary combination of relation composition and union, e.g. one possible expression could be $(r_1 \circ r_2) \cup r_3$. Show that this operation is monotone, that is show that for any i $r_i \subseteq r_i' \quad \rightarrow \quad E(r_1, r_2, ..., r_i, ..., r_n) \subseteq E(r_1, r_2, ..., r_i', ..., r_n)$