Quantifiers Elimination Solutions

1 $\exists y.\exists x.\ x < -2 \wedge 1 - 5y < x \wedge 1 + y < 13x$

Starting by eliminating x, we try to ensure coefficient 1 or -1 for x and replace it with a new variable z.(Note that the formula is already in DNF and the literals are normalized.)

$$\exists y.\exists x.\ x < -2 \wedge 1 - 5y < x \wedge 1 + y < 13x$$
$$\Updownarrow$$
$$\exists y.\exists x.\ 13x < -26 \wedge 13 - 65y < 13x \wedge 1 + y < 13x$$
$$\Updownarrow$$
$$\exists y.\exists z.\ z < -26 \wedge 13 - 65y < z \wedge 1 + y < z \wedge 13|z$$

Now, we try to eliminate z and then y:

$$\exists y.\exists z.\ z < -26 \wedge 13 - 65y < z \wedge 1 + y < z \wedge 13|z$$
$$\Updownarrow$$
$$\exists y.\ \vee_{i=1}^{13} (13 - 65y < -26 - i \wedge 1 + y < -26 - i \wedge 13|26 + i)$$
$$\Updownarrow$$
$$\exists y.\ 13 - 65y < -39 \wedge 1 + y < -39$$
$$\Updownarrow$$
$$\exists y.\ 52 < 65y \wedge y < -40$$
$$\Updownarrow$$
$$\exists y.\ 4 < 5y \wedge y < -40$$
$$\Updownarrow$$
$$false$$

2 $Is\ x < y + 2 \wedge y < x + 1 \wedge x = 3k \wedge (y = 6p + 1 \vee y = 6p - 1)\ satisfiable?$

Yes, a possible answer is:

$$x = 6;\ k = 2;\ y = 5;\ p = 1$$

More general, all possible answers can be written as follows:

$$\forall p,\ x = 6p;\ y = 6p - 1;\ k = 2p$$

3 $\exists x.\ (3x + 1 < 10 \vee 7x - 6 < 7) \wedge 2|x$

We try to simplify the formula:

$$\exists x.\ (3x + 1 < 10 \vee 7x - 6 < 7) \wedge 2|x$$
$$\Updownarrow$$
$$\exists x.\ (x < 3 \vee 7x < 13) \wedge 2|x$$
$$\Updownarrow \text{ because } x \text{ is an integer}$$
$$\exists x.\ (x < 3 \vee x < 2) \wedge 2|x$$
$$\Updownarrow$$
$$\exists x.\ x < 3 \wedge 2|x$$
$$\Updownarrow$$
$$\vee_{i=0}^{i=1}\ (i < 3 \wedge 2|i)$$
$$\Updownarrow$$
$$True$$

4 $\exists y.\exists x.\ 5x + 7y = a \wedge x \leq y \wedge 0 \leq x$

If $x'; y'$ satisfied the formula and $x \leq 7$, then $x' - 7; y' + 5$ also satisfies it. Hence:

$$\exists y.\exists x.\ 5x + 7y = a \wedge x \leq y \wedge 0 \leq x$$
$$\Updownarrow$$
$$\exists y.\ \vee_{i=0}^{6}\ (5i + 7y = a \wedge i \leq y)$$

To eliminate y, note that actually $y = \frac{a-5i}{7}$:

$$\exists y.\ \vee_{i=0}^{6}\ (5i + 7y = a \wedge i \leq y)$$
$$\Updownarrow$$
$$\vee_{i=0}^{6}(7|a - 5i \wedge 12i \leq a)$$