Lecturecise 5
Paths, Triples, Postconditions, Preconditions

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Loop-Free Programs as Relations: Summary

command c	$R(c)$	$\rho(c)$
$(x = t)$	$x' = t \wedge \bigwedge_{v \in V \setminus \{x\}} v' = v$	
$c_1 \,;\, c_2$	$\exists \bar{z}.\ \ R(c_1)[\bar{x}' := \bar{z}] \wedge R(c_2)[\bar{x} := \bar{z}]$	$\rho(c_1) \circ \rho(c_2)$
$\mathbf{if}(*)\ c_1\ \mathbf{else}\ c_2$	$R(c_1) \vee R(c_2)$	$\rho(c_1) \cup \rho(c_2)$
$\mathbf{assume}(\mathbf{F})$	$F \wedge \bigwedge_{v \in V} v' = v$	$\Delta_{S(F)}$

$\rho(v_i = t) = \{((v_1, \ldots, v_i, \ldots, v_n), (v_1, \ldots, v_i', \ldots, v_n) \mid v_i' = t\}$
$S(F) = \{\bar{v} \mid F\}, \quad \Delta_A = \{(\vec{v}, \vec{v}) \mid \vec{v} \in A\}$ (diagonal relation on A)
Δ (without subscript) is identity on entire set of states (no-op)
We always have: $\rho(c) = \{(\bar{v}, \bar{v}') \mid R(c)\}$
Shorthands:

$\mathbf{if}(*)\ c_1\ \mathbf{else}\ c_2$	$c_1\ []\ c_2$
$\mathbf{assume}(F)$	$[F]$

Examples:

$$\mathbf{if}\ (F)\ c_1\ \mathbf{else}\ c_2 \quad \equiv \quad [F]; c_1\ \ []\ \ [\neg F]; c_2$$
$$\mathbf{if}\ (F)\ c \quad \equiv \quad [F]; c\ \ []\ \ [\neg F]$$

Program Paths

Loop-Free Programs

c - a loop-free program whose assignments, havocs, and assumes are $c_1, \ldots, c_n$

The relation $\rho(c)$ is of the form $E(\rho(c_1), \ldots, \rho(c_n))$; it composes meanings of $c_1, \ldots, c_n$ using union ($\cup$) and composition ($\circ$)

```
(if (x > 0)
   x = x − 1
else
   x = 0
);
(if (y > 0)
   y = y − 1
else
   y = x + 1
)
```

```
([x > 0]; x = x − 1
  []
 ([¬(x>0)]; x = 0)
);
([y > 0]; y = y − 1
  []
 [¬(y>0)]; y = x+1
)
```

$$
\begin{aligned}
&\big(\Delta_{S(x>0)} \circ \rho(x = x - 1) \\
&\quad \cup \\
&\ \Delta_{S(\neg(x>0))} \circ \rho(x = 0) \\
&\big) \circ \\
&\big(\Delta_{S(y>0)} \circ \rho(y = y - 1) \\
&\quad \cup \\
&\ \Delta_{S(\neg(y>0))} \circ \rho(y = x + 1) \\
&\big)
\end{aligned}
$$

Note: $\circ$ binds stronger than $\cup$, so $r \circ s \cup t = (r \circ s) \cup t$

Normal Form for Loop-Free Programs

Composition distributes through union:

$$(r_1 \cup r_2) \circ (s_1 \cup s_2) = r_1 \circ s_1 \;\cup\; r_1 \circ s_2 \;\cup\; r_2 \circ s_1 \;\cup\; r_2 \circ s_2$$

Example corresponding to two if-else statements one after another:

$$\begin{array}{l} (\Delta_1 \circ r_1 \\ \quad \cup \\ \;\Delta_2 \circ r_2 \\ ) \circ \\ (\Delta_3 \circ r_3 \\ \quad \cup \\ \;\Delta_4 \circ r_4 \\ ) \end{array} \qquad \equiv \qquad \begin{array}{l} \Delta_1 \circ r_1 \circ \Delta_3 \circ r_3 \cup \\ \Delta_1 \circ r_1 \circ \Delta_4 \circ r_4 \cup \\ \Delta_2 \circ r_2 \circ \Delta_3 \circ r_3 \cup \\ \Delta_2 \circ r_2 \circ \Delta_4 \circ r_4 \end{array}$$

Sequential composition of basic statements is called basic path.
Loop-free code describes finitely many (exponentially many) paths.

Properties of Program Contexts

Some Properties of Relations

$$(p_1 \subseteq p_2) \rightarrow (p_1 \circ p) \subseteq (p_2 \circ p)$$

$$(p_1 \subseteq p_2) \rightarrow (p \circ p_1) \subseteq (p \circ p_2)$$

$$(p_1 \subseteq p_2) \wedge (q_1 \subseteq q_2) \quad \rightarrow \quad (p_1 \cup q_1) \subseteq (p_2 \cup q_2)$$

$$(p_1 \cup p_2) \circ q = (p_1 \circ q) \cup (p_2 \circ q)$$

Monotonicity of Expressions using $\cup$ and $\circ$

For a program with k integer variables, $S = \mathbb{Z}^k$
Consider relations that are subsets of $S \times S$ (i.e. S^2)
The set of all such relations is

$$C = \{r \mid r \subseteq S^2\}$$

Let $E(r)$ be given by any expression built from relation r and some additional relations $b_1, \ldots, b_n$, using $\cup$ and $\circ$.
Example: $E(r) = (b_1 \circ r) \cup (r \circ b_2)$
$E(r)$ is function $C \to C$, maps relations to relations
Claim: E is monotonic function on C:

$$r_1 \subseteq r_2 \to E(r_1) \subseteq E(r_2)$$

Prove of disprove.

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Prove of disprove.
Proof: induction on the expression tree defining E, using monotonicity properties of $\cup$ and $\circ$

Union-Distributivity of Expressions using $\cup$ and $\circ$

Claim: E distributes over unions, that is, if $r_i, i \in I$ is a family of relations,

$$E(\bigcup_{i \in I} r_i) = \bigcup_{i \in I} E(r_i)$$

Prove or disprove.

Union-Distributivity of Expressions using $\cup$ and $\circ$

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Prove or disprove.
False. Take $E(r) = r \circ r$ and consider relations r_1, r_2. The claim becomes

$$(r_1 \cup r_2) \circ (r_1 \cup r_2) = r_1 \circ r_1 \cup r_2 \circ r_2$$

that is,

$$r_1 \circ r_1 \cup r_1 \circ r_2 \cup r_2 \circ r_1 \cup r_2 \circ r_2 = r_1 \circ r_1 \cup r_2 \circ r_2$$

Taking, for example, $r_1 = \{(1,2)\}$, $r_2 = \{(2,3)\}$ we obtain

$$\{(1,3)\} = \emptyset \qquad (\mathit{false})$$

Union "Distributivity" in One Direction

Lemma:

$$E(\bigcup_{i \in I} r_i) \supseteq \bigcup_{i \in I} E(r_i)$$

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Proof. Let $r = \bigcup_{i \in I} r_i$. Note that, for every i, $r_i \subseteq r$. We have shown that E is monotonic, so $E(r_i) \subseteq E(r)$. Since all $E(r_i)$ are included in $E(r)$, so is their union, so

$$\bigcup E(r_i) \subseteq E(r)$$

as desired.

Union-Distributivity - Refined

Does distributivity

$$E(\bigcup_{i \in I} r_i) = \bigcup_{i \in I} E(r_i)$$

hold, for each of these cases

1. If $E(r)$ is given by an expression containing r at most once?

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3. If $E(r)$ contains r any number of times, but $r_i, i \in I$ is a **directed family** of relations: for each i, j there exists k such that $r_i \cup r_j \subseteq r_k$, and I is possibly uncountably infinite.

Union-Distributivity - Refined

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3. If $E(r)$ contains r any number of times, but $r_i, i \in I$ is a **directed family** of relations: for each i, j there exists k such that $r_i \cup r_j \subseteq r_k$, and I is possibly uncountably infinite. Induction. Generalizes the previous case.

About Strength and Weakness

Putting Conditions on Sets Makes them Smaller

Let P_1 and P_2 be formulas ("conditions") whose free variables are among $\bar{x}$. Those variables may denote program state.
When we say "condition P_1 is stronger than condition P_2" it simply means

$$\forall \bar{x}.\ (P_1 \rightarrow P_2)$$

- if we know P_1, we immediately get (conclude) P_2
- if we know P_2 we need not be able to conclude P_1

Stronger condition = smaller set: if P_1 is stronger than P_2 then

$$\{\bar{x} \mid P_1\} \subseteq \{\bar{x} \mid P_2\}$$

- strongest possible condition: "false" $\rightsquigarrow$ smallest set: $\emptyset$
- weakest condition: "true" $\rightsquigarrow$ biggest set: set of all tuples

Hoare Triples

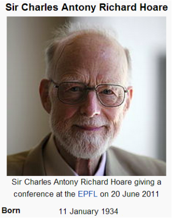
About Hoare Logic

We have seen how to translate programs into relations. We will use these relations in a proof system called Hoare logic. Hoare logic is a way of inserting annotations into code to make proofs about (imperative) program behavior simpler.

Example proof:

```
//{0 <= y}
i = y;
//{0 <= y & i = y}
r = 0;
//{0 <= y & i = y & r = 0}
while //{r = (y−i)∗x & 0 <= i}
 (i > 0) (
  //{r = (y−i)∗x & 0 < i}
  r = r + x;
  //{r = (y−i+1)∗x & 0 < i}
  i = i − 1
  //{r = (y−i)∗x & 0 <= i}
)
//{r = x ∗ y}
```

Hoare Triple and Friends



$P, Q \subseteq S \qquad r \subseteq S \times S$

Hoare Triple:

$$\{P\}\ r\ \{Q\} \iff \forall s, s' \in S.\, (s \in P \land (s, s') \in r \rightarrow s' \in Q)$$

$\{P\}$ does not denote a singleton set containing P but is just a notation for an "assertion" around a command. Likewise for $\{Q\}$.

Strongest postcondition:

$$sp(P, r) = \{s' \mid \exists s.\, s \in P \land (s, s') \in r\}$$

Weakest precondition:

$$wp(r, Q) = \{s \mid \forall s'.(s, s') \in r \rightarrow s' \in Q\}$$

Exercise: Which Hoare triples are valid?

Assume all variables to be over integers.

1. $\{j = a\}$ j :=j+1 $\{a = j + 1\}$
2. $\{i = j\}$ i:=j+i $\{i > j\}$
3. $\{j = a + b\}$ i:=b; j:=a $\{j = 2 * a\}$
4. $\{i > j\}$ j:=i+1; i:=j+1 $\{i > j\}$
5. $\{i \mathrel{!}= j\}$ **if** i>j then m:=i−j **else** m:=j−i $\{m > 0\}$
6. $\{i = 3*j\}$ **if** i>j then m:=i−j **else** m:=j−i $\{m-2*j=0\}$

Postconditions and Their Strength

What is the relationship between these postconditions?

$$\{x = 5\} \quad x := x + 2 \quad \{\mathbf{x > 0}\}$$

$$\{x = 5\} \quad x := x + 2 \quad \{\mathbf{x = 7}\}$$

Postconditions and Their Strength

What is the relationship between these postconditions?

$$\{x = 5\} \quad x := x + 2 \quad \{\mathbf{x} > \mathbf{0}\}$$

$$\{x = 5\} \quad x := x + 2 \quad \{\mathbf{x} = \mathbf{7}\}$$

- weakest conditions (predicates) correspond to largest sets
- strongest conditions (predicates) correspond to smallest sets

that satisfy a given property.

(Graphically, a stronger condition $x > 0 \land y > 0$ denotes one quadrant in plane, whereas a weaker condition $x > 0$ denotes the entire half-plane.)

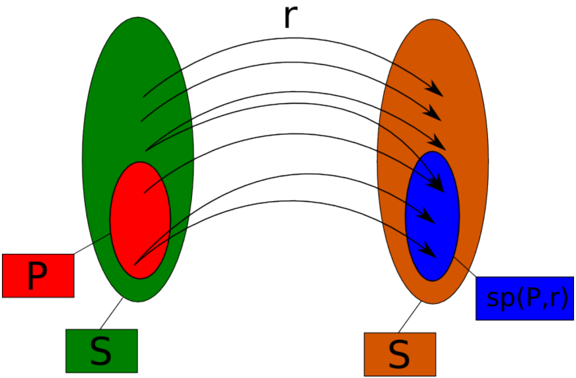
Strongest Postconditions

Strongest Postcondition

Definition: For $P \subseteq S$, $r \subseteq S \times S$,

$$sp(P, r) = \{s' \mid \exists s. s \in P \land (s, s') \in r\}$$

This is simply the relation image of a set.



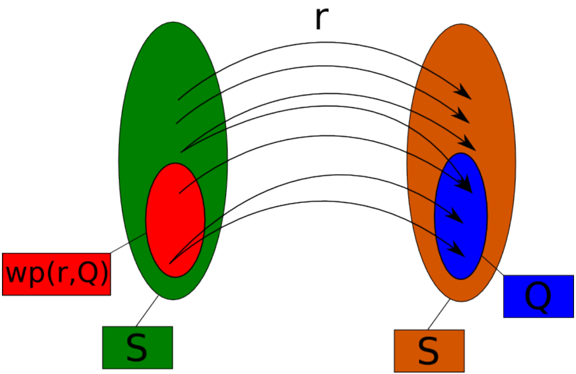
Weakest Preconditions

Weakest Precondition

Definition: for $Q \subseteq S$, $r \subseteq S \times S$,

$$wp(r, Q) = \{s \mid \forall s'.(s, s') \in r \to s' \in Q\}$$

Note that this is in general not the same as $sp(Q, r^{-1})$ when then relation is non-deterministic or partial.



Three Forms of Hoare Triple

Lemma: the following three conditions are equivalent:

- $\{P\}r\{Q\}$
- $P \subseteq wp(r, Q)$
- $sp(P, r) \subseteq Q$

Three Forms of Hoare Triple

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Proof. The three conditions expand into the following three formulas

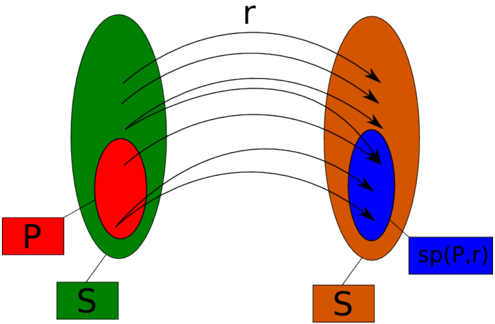
- $\forall s, s'. \ [(s \in P \land (s, s') \in r) \rightarrow s' \in Q]$
- $\forall s. \ [s \in P \rightarrow (\forall s'.(s, s') \in Q)]$
- $\forall s'. \ [(\exists s. \ s \in P \land (s, s') \in P) \rightarrow s' \in Q]$

which are easy to show equivalent using basic first-order logic properties.

Lemma: Characterization of sp

$sp(P, r)$ is the the smallest set Q such that $\{P\}r\{Q\}$, that is:

- $\{P\}r\{sp(P, r)\}$
- $\forall Q \subseteq S.\ \{P\}r\{Q\} \to sp(P, r) \subseteq Q$



$$\{P\}\ r\ \{Q\} \Leftrightarrow \forall s, s' \in S. (s \in P \land (s, s') \in r \to s' \in Q)$$
$$sp(P, r) = \{s' \mid \exists s. s \in P \land (s, s') \in r\}$$

Proof of Lemma: Characterization of sp

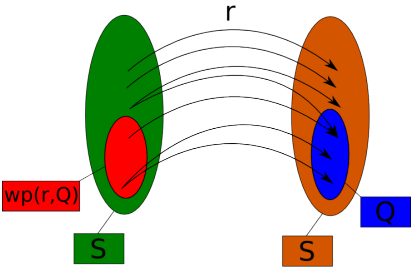
Apply Three Forms of Hoare triple. The two conditions then reduce to:

- $sp(P, r) \subseteq sp(P, r)$
- $\forall P \subseteq S.\ sp(P, r) \subseteq Q \rightarrow sp(P, r) \subseteq Q$

Lemma: Characterization of wp

$wp(r, Q)$ is the largest set P such that $\{P\}r\{Q\}$, that is:

- $\{wp(r, Q)\}r\{Q\}$
- $\forall P \subseteq S.\ \{P\}r\{Q\} \rightarrow P \subseteq wp(r, Q)$



$$\{P\}\ r\ \{Q\} \Leftrightarrow \forall s, s' \in S.\ \left(s \in P \land (s, s') \in r \rightarrow s' \in Q\right)$$
$$wp(r, Q) = \{s \mid \forall s'.(s, s') \in r \rightarrow s' \in Q\}$$

Proof of Lemma: Characterization of wp

Apply Three Forms of Hoare triple. The two conditions then reduce to:

- $wp(r, Q) \subseteq wp(r, Q)$
- $\forall P \subseteq S.\ P \subseteq wp(r, Q) \to P \subseteq wp(r, Q)$

Exercise: Postcondition of inverse versus wp

Lemma:

$$S \setminus wp(r, Q) = sp(S \setminus Q, r^{-1})$$

In other words, when instead of good states we look at the completement set of “error states”, then *wp* corresponds to doing *sp* backwards.

Note that $r^{-1} = \{(y, x) \mid (x, y) \in r\}$ and is always defined.

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Proof of the lemma: Expand both sides and apply basic first-order logic properties.