Lecturecise 9
Solutions to exercises

2013

Exercise

We call a relation $r \subseteq S \times S$ functional if $\forall x, y, z \in S.(x, y) \in r \land (x, z) \in r \rightarrow y = z$.
For each of the following statements either give a counterexample or prove it.
In the following, assume $Q \subset S$.

(i) for any r, $wp(r, S \setminus Q) = S \setminus wp(r, Q)$

(ii) if r is functional, $wp(r, S \setminus Q) = S \setminus wp(r, Q)$

(iii) for any r, $wp(r, Q) = sp(Q, r^{-1})$

(iv) if r is functional, $wp(r, Q) = sp(Q, r^{-1})$

(v) for any r, $wp(r, Q_1 \cup Q_2) = wp(r, Q_1) \cup wp(r, Q_2)$

(vi) if r is functional, $wp(r, Q_1 \cup Q_2) = wp(r, Q_1) \cup wp(r, Q_2)$

(vii) for any r, $wp(r_1 \cup r_2, Q) = wp(r_1, Q) \cup wp(r_2, Q)$

(viii) Alice has the following conjecture: For all sets S and relations $r \subseteq S \times S$ it holds:

$$\Big(S \neq \emptyset \land dom(r) = S \land \triangle_S \cap r = \emptyset\Big) \rightarrow \Big(r \circ r \cap ((S \times S) \setminus r) \neq \emptyset\Big)$$

She tried many sets and relations and did not find any counterexample. Is her conjecture true?
If so, prove it, otherwise provide a counterexample for which S is smallest.

Exercise Solution

(i) counterexample: S = {a, b, c, d} and r = {(a, b), (b, c), (c, a)}

(ii) as above, r is functional

(iii) counterexample: S = { a, b} and r ={(a, b)} r-1 = {(b, a)} Q={b}

(iv) counterexample as in (iii)

(v) counterexample: S = { a, b} and r = {(a, b), (a, a)} and Q1 = {a} and Q2 = {b}

(vi) true:

$$
\begin{aligned}
wp(r, Q_1 \cup Q_2) &\Leftrightarrow \{s | \forall s'.(s, s') \in r \rightarrow (s' \in Q_1 \vee s' \in Q_2)\} \\
&\Leftrightarrow \{s | \forall s'.(s, s') \notin r \vee s' \in Q_1 \vee s' \in Q_2\} \\
&\Leftrightarrow \{s | \exists s'.(s, s') \notin r \vee s' \in Q_1 \vee s' \in Q_2\} \text{ (since r functional)} \\
&\Leftrightarrow \{s | (\exists s'.(s, s') \notin r \vee s' \in Q_1) \vee (\exists s'.(s, s') \notin r \vee s' \in Q_2)\} \\
&\Leftrightarrow wp(r, Q_1) \cup wp(r, Q_2)
\end{aligned}
$$

(vii) counterexample: S= {a, b} and r1 = {(a, b)} and r2 = {(b, a)} and Q = {b}

Exercise Solution

(viii) $S = \mathbb{N}\{0\}$

$r = \{(s, s')|\exists c.s + c = s'\}$, i.e. (1, 2), (1, 3), (1, 4), ..., (2, 3), (2, 4), ...

$S \neq \emptyset \wedge dom(r) = S \wedge \triangle_S \cap r = \emptyset$ is satisfied

$r \circ r \cap ((S \times S) \setminus r) \neq \emptyset$ is equivalent to $r \circ r \subseteq r$, i.e. the relation is transitive, which is the case for our r.

S cannot be finite, since r is closed under composition, i.e. it is a transitive closure. If S was finite, we'd necessarily have a loop somewhere violating the $\triangle_S \cap r = \emptyset$ requirement.