Homework 2 - Relations

March 2, 2012

The solution to the following problems should be fairly formal, that is, they should be of similar formality as in the exercises. You can use facts shown or stated in class, but make it obvious when you apply them (for example by doing your proofs step by step, or stating explicitly which rules you use, if you use several at a time).

Problem 1

Given a relation $r \subseteq A \times A$, prove that r is transitive if and only if $r \circ r \subseteq r$.

Problem 2

Recall that a relation $r \subseteq A \times A$ is symmetric if $\forall x, y \in A.\ (x, y) \in r \rightarrow (y, x) \in r$.
Now let r be an arbitrary relation. Prove that $r^{-1} \circ (r \cup r^{-1})^* \circ r$ is symmetric.

Hint: consider what it means to have $(x, y) \in r^*$ for some concrete (x, y).

Problem 3

Recall the guarded command language from Lecture 2 and in particular the constructs for sequential composition and the if-statement:

$$\text{s1; s2} \quad \rightsquigarrow \quad r_{s1} \circ r_{s2} \tag{1}$$

$$(\text{ assume(F); s1 }) \ [] \ (\text{ assume}(\neg\text{F}); \text{s2 }) \quad \rightsquigarrow \quad (\triangle_F \circ r_{s1}) \cup (\triangle_{\neg F} \circ r_{s2}) \tag{2}$$

Also recall that a relation $r \subseteq A \times B$ is a total function if and only if
$\forall x, y, z.\ (x, y) \in r \ \wedge \ (x, z) \in r \rightarrow y = z$ and $\forall x.\ \exists y.\ (x, y) \in r$.
Show that if `s1` and `s2` are deterministic statements (i.e. total functions), then their composition using

(i) sequential composition

(ii) if-statement

remains deterministic. From this, prove by induction on the syntax tree that relational expressions built from functions using sequential composition and if-statements remain a function.

Problem 4

Recall that we define the powers of a relation $r \subseteq A \times A$ as follows:

$$r^0 = \triangle_A, \quad r^1 = r, \quad \text{and} \quad r^{n+1} = r^n \circ r$$

We showed that the *reflexive and transitive closure* $r^* = \bigcup_{n \geq 0} r^n$ is the smallest reflexive and transitive relation on A containing r. Show that for any relation r on a set A, $(r \cup r^{-1})^*$ is the least equivalence relation containing r. Precisely, show that

(i) $(r \cup r^{-1})^*$ is an equivalence relation, and

(ii) if s is an equivalence relation containing r, then $(r \cup r^{-1})^* \subseteq s$.