Exercises 2

First-order logic

Reminder:
Propositional formulas are expressions built from

- constants true and false
- propositional variables, such as p, q, p_1, q_1, ...
- logical operators ($\wedge, \vee, \neg, \rightarrow, \leftrightarrow$)

On top of propositional operations, **first-order logic** adds:

- constructs to represent the structure of propositions:
  - equality ($=$)
  - predicate symbols (P, Q, ...)
  - function symbols (f, g, ...)
  - first-order variables (x, y, ...) denoting entities in some domain D
- quantifiers forall ($\forall$), exists ($\exists$)

Normal Forms:

- Negation Normal Form (NNF) only allows the logical connectives $\neg, \wedge, \vee$, where $\neg$ is only applied to atomic formulas.
- Conjunctive Normal Form (CNF): $\bigwedge_i \bigvee_j A$ (A: atomic formula)
- Disjunctive Normal Form (DNF): $\bigvee_i \bigwedge_j A$ (A: atomic formula)

A FOL **interpretation** is a pair D_I, α_I, consisting of a domain D_I and an asignment α_I. These define a mapping from

- variable symbols to values from D_I
- function symbols to functions within the domain $f_I : D_I^n \rightarrow D_I$
- predicate symbols to predicates within the domain $p_I : D_I^n \rightarrow \{\text{true}, \text{false}\}$

Problem 1: Dreadbury mansion

Consider the following:

1. Someone who lives in Dreadbury Mansion killed Aunt Agatha.
2. Agatha, the butler, and Charles live in Dreadbury Mansion, and are the only people who live therein.
3. A killer always hates his victim, and is never richer than his victim.
4. Charles hates no one that Aunt Agatha hates.
5. Agatha hates everyone except the butler.
6. The butler hates everyone not richer than Aunt Agatha.
7. The butler hates everyone Aunt Agatha hates.
8. No one hates everyone.
9. Agatha is not the butler.

Who killed Aunt Agatha?
Formalize these statements in first-order logic, by defining the variables, predicate and function symbols first.

Problem 2: Interpretations [1]

Consider the formula

$$\forall x, y.\ (A(x) \wedge B(x, y)) \rightarrow A(y)$$

Determine, whether it is valid, satisfiable or unsatisfiable under the following interpretations.

(i) The domain of the natural numbers, where A is interpreted as `even`, and B is interpreted as `equals`.

(ii) The domain of the natural numbers, where A is interpreted as `even`, and B is interpreted as `is an integer divisor of`.

(iii) The domain of the natural numbers, where A is interpreted as `even`, and B is interpreted as `is an integer multiple of`.

(iv) The domain of the Booleans, true,false, where A is interpreted as `negation`, and B is interpreted as `xor`.

[1] Adapted from *Exercises for First-Order Logic*, Ian Barland, John Greiner, Fuching Chi, http://cnx.org/content/m12353/latest/.

Problem 3: Normal forms

A formula is in prenex normal form (PNF) if all of its quantifiers appear at the beginning of the formula[2]:

$$Q_1 x_1 \dots Q_n x_n.\ F$$

and F is quantifier free. Every FOL formula F can be transformed into an equivalent formula F in PNF. To compute an equivalent PNF F' of F,

1. Convert F into NNF formula F_1.
2. When multiple quantified variables have the same name, rename them to fresh variables, resulting in F_2.
3. Remove all quantifiers from F_2 to produce quantifier-free formula F_3.
4. Add the quantifiers before F_3,

   $$F_4 : Q_1 x_1 \dots . Q_n x_n . F_3$$

   where the Q_i are the quantifiers such that if Q_j is in the scope of Q_i in F_1, then $i < j$. F_4 is equivalent to F.

Apply this procedure to

(i) $\forall x.\ \neg(\exists y.\ p(x, y) \wedge p(x, z)) \vee \exists y.\ p(x, y)$

(ii) $\forall w.\ \neg(\exists x, y.\ \forall z.\ p(x, z) \to q(y, z)) \wedge \exists z.\ p(w, z)$

Problem 4: Partial functions and Relation composition

Given two partial functions r_1 and r_2, show that $r = r_1 \circ r_2$ is also a partial function.

Problem 5: Monotonicity

Let $E(r_1, r_2, ..., r_n)$ be a relation composed of relations r_i with an arbitrary combination of relation composition and union, e.g. one possible expression could be $(r_1 \circ r_2) \cup r_3$. Show that this operation is monotone, that is show that for any i

$$r_i \subseteq r_i' \quad \to \quad E(r_1, r_2, ..., r_i, ..., r_n) \subseteq E(r_1, r_2, ..., r_i', ..., r_n)$$

[2] Definition from *Calculus of Computation*, A.R.Bradley and Zohar Manna, 2007