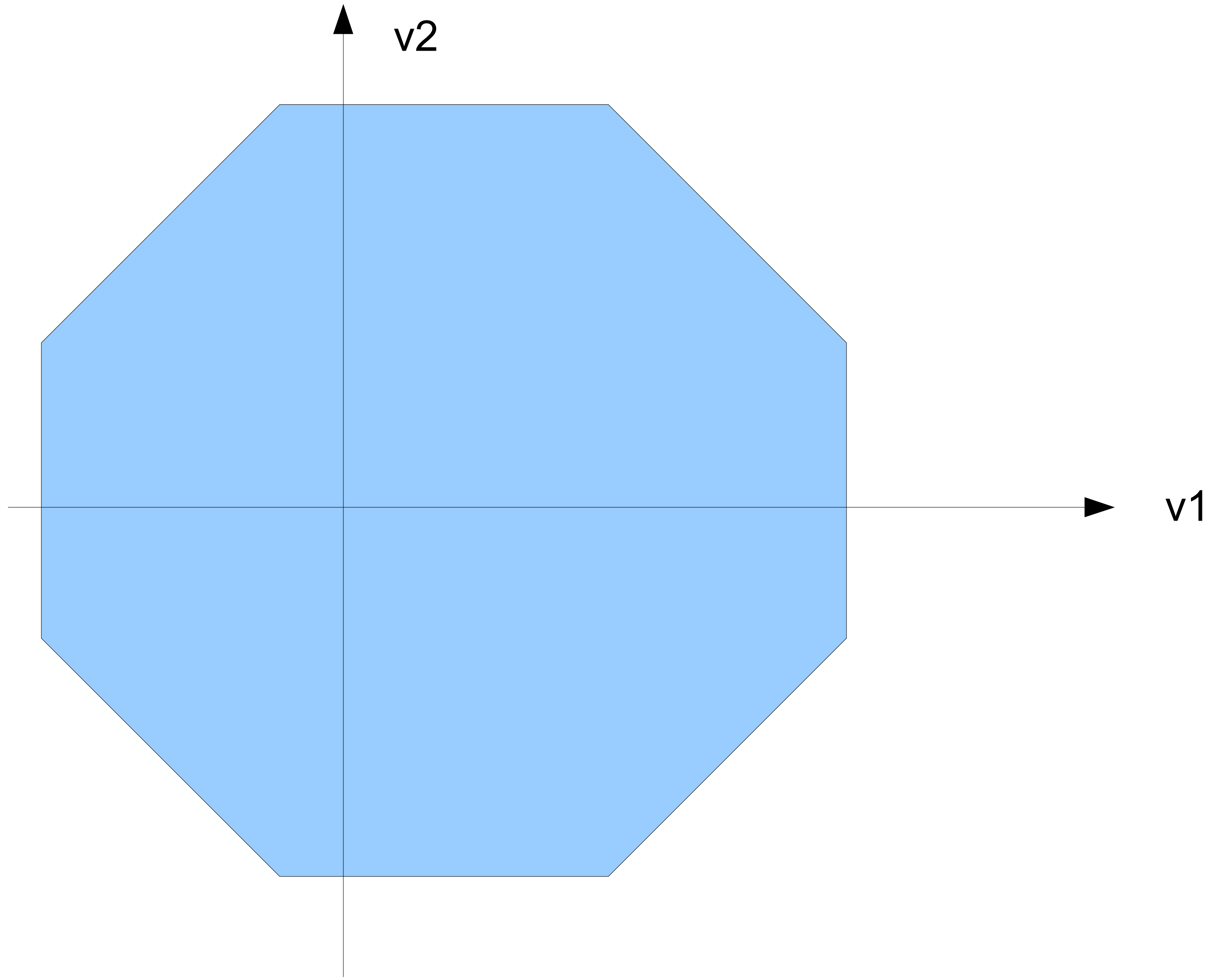
The Octagon Abstract Domain



The Difference Bound Matrix

$$\mathbf{m}_{ij} \triangleq \begin{cases} c & \text{if } (v_j - v_i \leq c) \in C, \\ +\infty & \text{elsewhere} \quad . \end{cases}$$

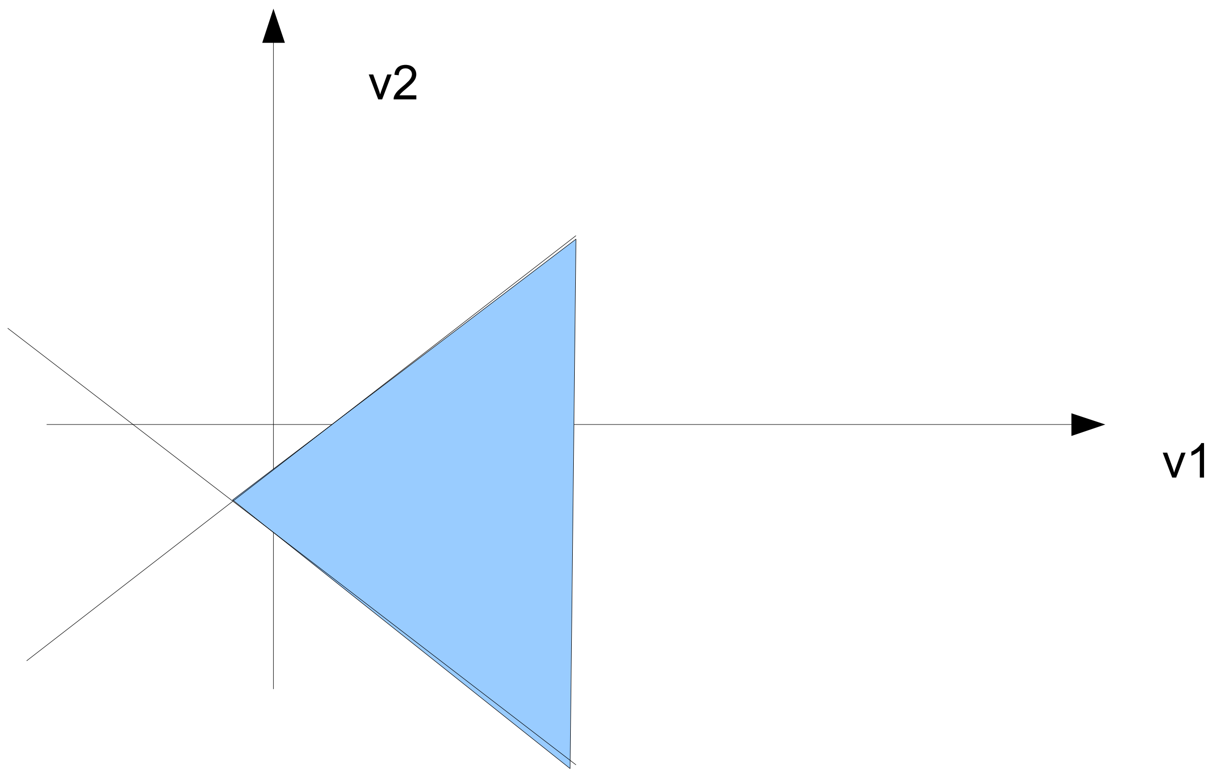
$\mathbf{m}$ is called a *Difference-Bound Matrix (DBM)*.

The V-domain

$$\mathcal{D}(\mathbf{m}) \stackrel{\triangle}{=} \{(s_0, \ldots, s_{N-1}) \in \mathbb{I}^N \mid \forall i, j,\ s_j - s_i \leq \mathbf{m}_{ij}\} \ .$$

The V-domain

$$\mathcal{D}(\mathbf{m}) \stackrel{\triangle}{=} \{(s_0, \ldots, s_{N-1}) \in \mathbb{I}^N \mid \forall i, j,\ s_j - s_i \leq \mathbf{m}_{ij}\} \ .$$



Two DBM's with the same set concretisation

$$
\begin{array}{cc|ccc}
 & & & j & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & +\infty & 4 & 3 \\
i & 2 & -1 & +\infty & +\infty \\
 & 3 & -1 & 1 & +\infty
\end{array}
\qquad
\begin{array}{cc|ccc}
 & & & j & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 0 & 4 & 3 \\
i & 2 & -1 & 0 & +\infty \\
 & 3 & -1 & 1 & 0
\end{array}
$$

Introducing V- and V+

$$\mathcal{V}^+ \; = \; \{v_0, \ldots, v_{N-1}\}$$

$$(\pm v_i \pm v_j \leq c) \qquad v_i, v_j \in \mathcal{V}^+ \qquad c \in \mathbb{I}$$

$$\mathcal{V} \; = \{ \; v_0^+, \; v_0^-, \; \ldots, \; v_{N-1}^+, \; v_{N-1}^- \; \}$$

The V+ - Domain

$$\mathcal{D}^+(\mathbf{m}^+) \triangleq \left\{ \begin{array}{l} (s_0, \ldots, s_{N-1}) \in \mathbb{I}^N \mid \\ (s_0, -s_0, \ldots, s_{N-1}, -s_{N-1}) \in \mathcal{D}(\mathbf{m}^+) \end{array} \right\}.$$

$$\mathbf{m}^+ \trianglelefteq \ \mathbf{n}^+ \Longrightarrow \mathcal{D}^+(\mathbf{m}^+) \subseteq \mathcal{D}^+(\mathbf{n}^+)$$

DBM Coherence

$$\textit{Theorem 1:}\ \mathbf{m}^+ \text{ is coherent } \iff \forall i, j,\ \mathbf{m}^+_{ij} = \mathbf{m}^+_{\bar{j}\,\bar{\imath}}\ .$$

Octagon Constraints

constraint over $\mathcal{V}^+$	constraint(s) over $\mathcal{V}$
$v_i - v_j \leq c \quad (i \neq j)$	$v_i^+ - v_j^+ \leq c, \quad v_j^- - v_i^- \leq c$
$v_i + v_j \leq c \quad (i \neq j)$	$v_i^+ - v_j^- \leq c, \quad v_j^+ - v_i^- \leq c$
$-v_i - v_j \leq c \quad (i \neq j)$	$v_j^- - v_i^+ \leq c, \quad v_i^- - v_j^+ \leq c$
$v_i \leq c$	$v_i^+ - v_i^- \leq 2c$
$v_i \geq c$	$v_i^- - v_i^+ \leq -2$

The Potential Graph

$$\mathcal{G}(\mathbf{m}) = \{\mathcal{V}, \mathcal{A}, w\}$$

$$\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}, \qquad w \in \mathcal{A} \mapsto \mathbb{I},$$

$$\mathcal{A} \triangleq \{(v_i, v_j) \mid \mathbf{m}_{ij} < +\infty\}, \qquad w((v_i, v_j)) \triangleq \mathbf{m}_{ij} \ .$$

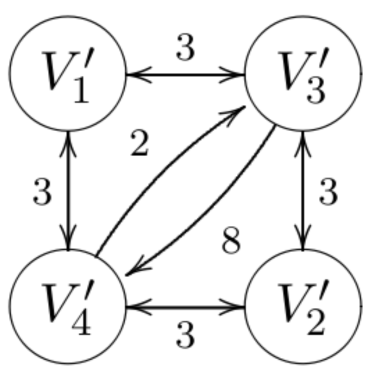
Representing the constraints

(a) $$\begin{cases} V_1 + V_2 \leq 3 \\ V_2 - V_1 \leq 3 \\ V_1 - V_2 \leq 3 \\ -V_1 - V_2 \leq 3 \\ 2V_2 \leq 2 \\ -2V_2 \leq 8 \end{cases}$$

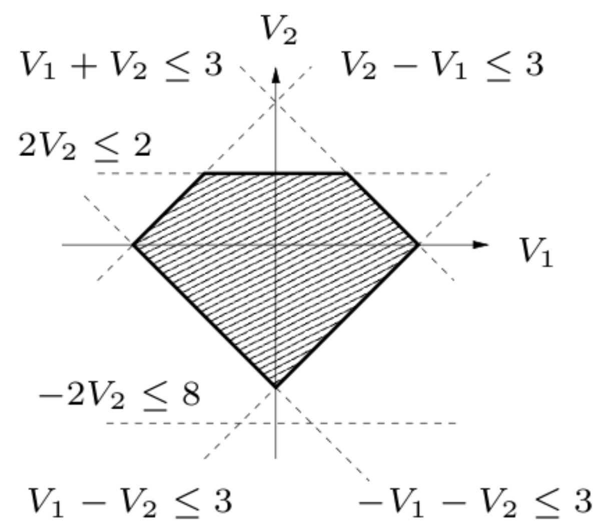
(b)

	j → 1	2	3	4
i = 1	$+\infty$	$+\infty$	3	3
2	$+\infty$	$+\infty$	3	3
3	3	3	$+\infty$	8
4	3	3	2	$+\infty$

(c)



(d)



Emptiness Test

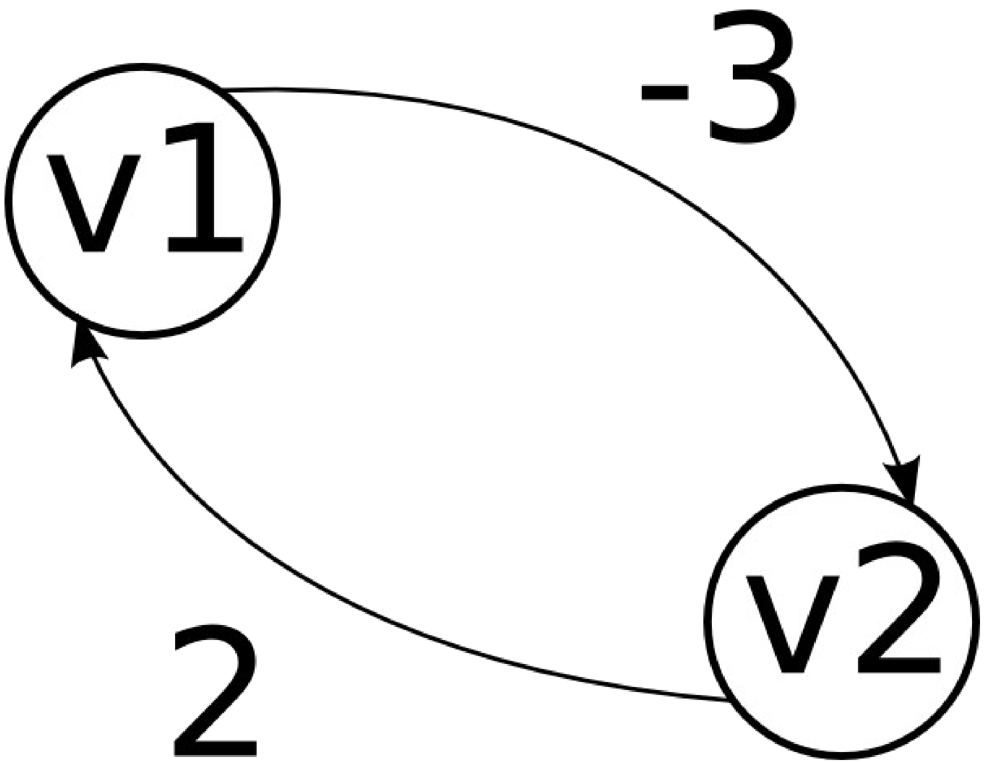
Theorem 2:

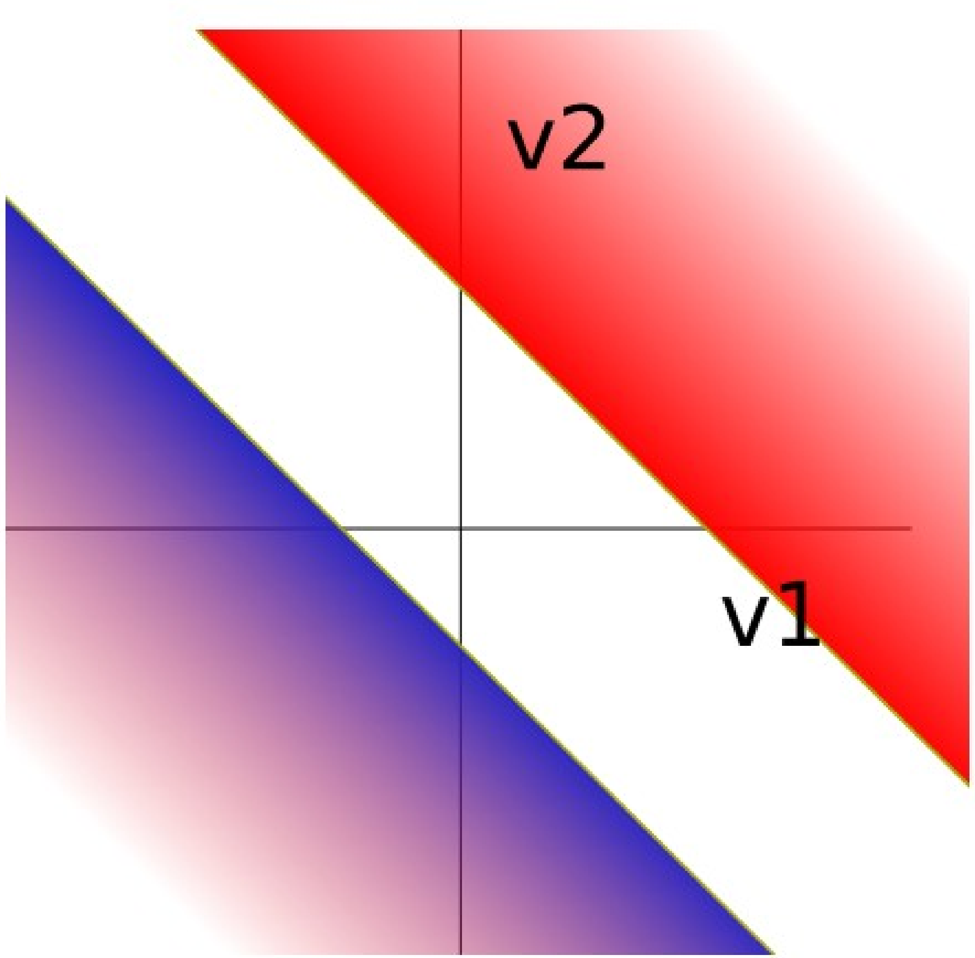
. $\mathcal{D}(\mathbf{m}) = \emptyset \iff \mathcal{G}(\mathbf{m})$ has a cycle with a strictly negative weight.

. If $\mathbb{I} \neq \mathbb{Z}$, then $\mathcal{D}(\mathbf{m}^+) = \emptyset \iff \mathcal{D}^+(\mathbf{m}^+) = \emptyset$.
If $\mathbb{I} = \mathbb{Z}$, then $\mathcal{D}(\mathbf{m}^+) = \emptyset \implies \mathcal{D}^+(\mathbf{m}^+) = \emptyset$, but the converse is false

Empty Set example

- V2 – V1 <= -3
- V1 – V2 <= 2





Order

$$\mathbf{m} \trianglelefteq \mathbf{n} \stackrel{\triangle}{\iff} \forall i, j,\ \mathbf{m}_{ij} \leq \mathbf{n}_{ij} \quad .$$

Implicit constraints

- V1 – V3 <= 4
- V1 – V2 <= 1
- V2 – V3 <= 2
- => V1 – V3 <= 3

Closure

$$\begin{cases} \mathbf{m}^*_{ii} \stackrel{\triangle}{=} 0, \\ \mathbf{m}^*_{ij} \stackrel{\triangle}{=} \min\limits_{\substack{1 \leq M \\ \langle i=i_1, i_2, \ldots, i_M=j \rangle}} \sum\limits_{k=1}^{M-1} \mathbf{m}_{i_k i_{k+1}} & \text{if } i \neq j \end{cases} .$$

Theorem 3

Theorem 3:

1. $\mathbf{m} = \mathbf{m}^* \iff \forall i, j, k,\ \mathbf{m}_{ij} \leq \mathbf{m}_{ik} + \mathbf{m}_{kj}$ and $\forall i,\ \mathbf{m}_{ii} = 0$ *(Local Definition)*.
2. $\forall i, j,$ if $\mathbf{m}^*_{ij} \neq +\infty$, then $\exists (s_0, \ldots, s_{N-1}) \in \mathcal{D}(\mathbf{m})$ such that $s_j - s_i = \mathbf{m}^*_{ij}$ *(Saturation)*.
3. $\mathbf{m}^* = \inf_{\trianglelefteq} \{\mathbf{n} \mid \mathcal{D}(\mathbf{n}) = \mathcal{D}(\mathbf{m})\}$ *(Normal Form)*.

Strong Closure

Definition 1: $\mathbf{m}^+$ is *strongly closed* if and only if

- $\mathbf{m}^+$ is *coherent*: $\forall i, j,\ \mathbf{m}^+_{ij} = \mathbf{m}^+_{\bar{\jmath}\,\bar{\imath}}$;
- $\mathbf{m}^+$ is *closed*: $\forall i,\ \mathbf{m}^+_{ii} = 0$ and $\forall i, j, k,\ \mathbf{m}^+_{ij} \leq \mathbf{m}^+_{ik} + \mathbf{m}^+_{kj}$;
- $\forall i, j,\ \mathbf{m}^+_{ij} \leq (\mathbf{m}^+_{i\,\bar{\imath}} + \mathbf{m}^+_{\bar{\jmath} j})/2$.

Strong Closure Theorem

Theorem 4:

1. $\mathbf{m}^+ = (\mathbf{m}^+)^\bullet \iff \mathbf{m}^+$ is strongly closed.
2. $\forall i, j,$ if $(\mathbf{m}^+)^\bullet_{ij} \neq +\infty$, then $\exists (s_0, \ldots, s_{2N-1}) \in \mathcal{D}(\mathbf{m}^+)$ such that $\forall k,\ s_{2k} = -s_{2k+1}$ and $s_j - s_i = (\mathbf{m}^+)^\bullet_{ij}$ *(Saturation)*.
3. $(\mathbf{m}^+)^\bullet = \inf_{\trianglelefteq}\{\mathbf{n}^+ \mid \mathcal{D}^+(\mathbf{n}^+) = \mathcal{D}^+(\mathbf{m}^+)\}$ *(Normal Form)*.

Equality and Inclusion Testing

Theorem 5:

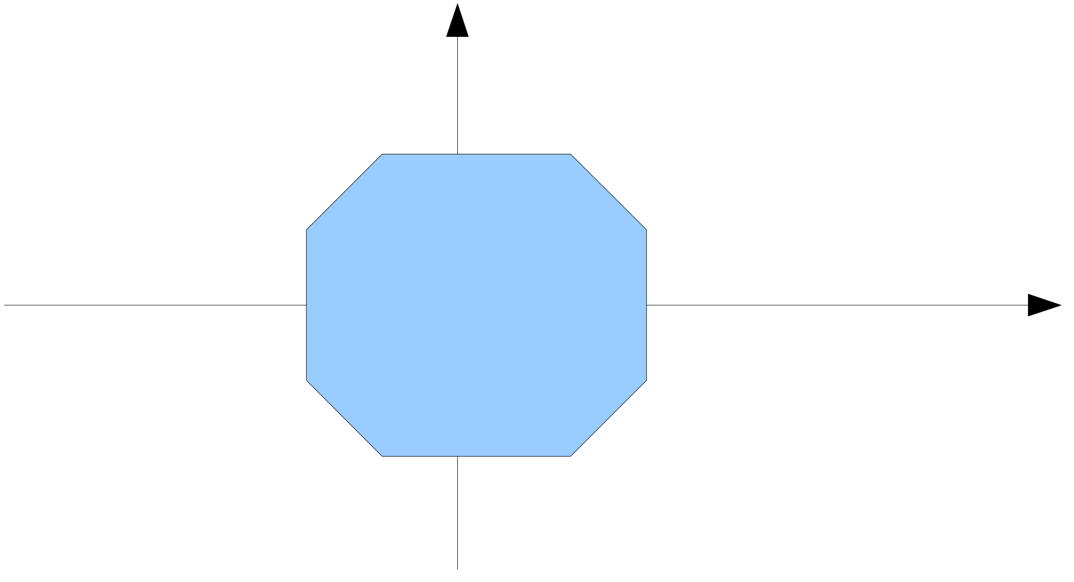
1. $\mathcal{D}^+(\mathbf{m}^+) \subseteq \mathcal{D}^+(\mathbf{n}^+) \iff (\mathbf{m}^+)^\bullet \trianglelefteq \mathbf{n}^+$;
2. $\mathcal{D}^+(\mathbf{m}^+) = \mathcal{D}^+(\mathbf{n}^+) \iff (\mathbf{m}^+)^\bullet = (\mathbf{n}^+)^\bullet$.

Projection

Theorem 6:

$$\{\ t \mid \exists(s_0, \ldots, s_{N-1}) \in \mathcal{D}^+(\mathbf{m}^+) \text{ such that } s_i = t\ \}$$
$$= [\ -(\mathbf{m}^+)^{\bullet}_{2i\ 2i+1}/2,\ (\mathbf{m}^+)^{\bullet}_{2i+1\ 2i}/2\ ]$$

(interval bounds are included only if finite).

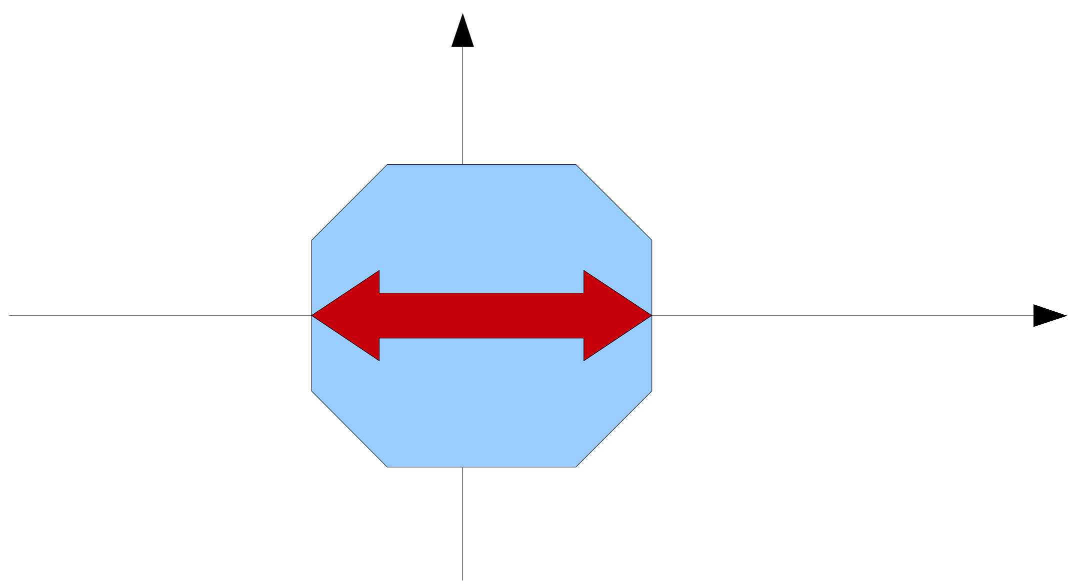


Projection

Theorem 6:

$$\{\ t \mid \exists (s_0, \ldots, s_{N-1}) \in \mathcal{D}^+(\mathbf{m}^+) \text{ such that } s_i = t\ \}$$
$$= [\ -(\mathbf{m}^+)^\bullet_{2i\ 2i+1}/2,\ (\mathbf{m}^+)^\bullet_{2i+1\ 2i}/2\ ]$$

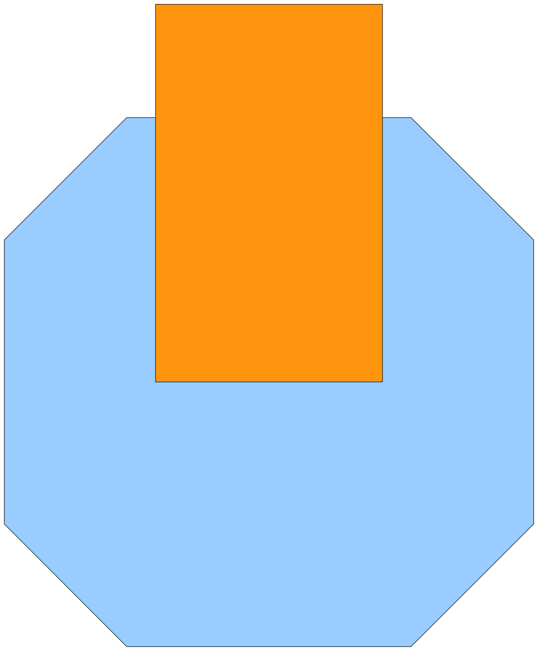
(interval bounds are included only if finite).



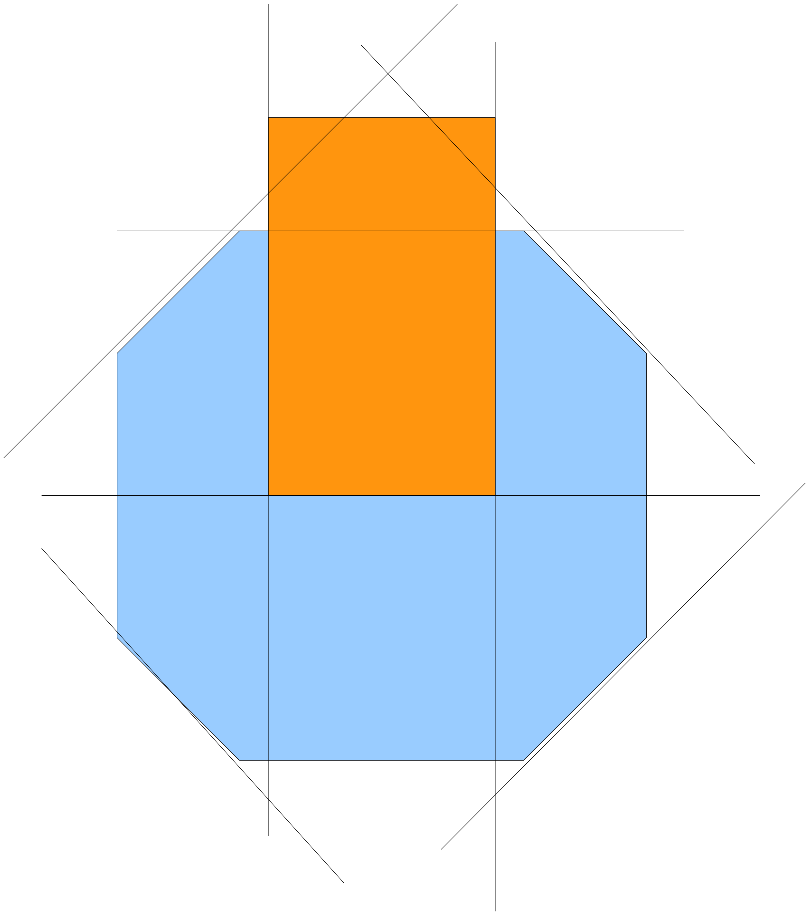
Least upper bound and greatest upper bound

$$
\begin{aligned}
[\mathbf{m}^+ \wedge \mathbf{n}^+]_{ij} &\stackrel{\triangle}{=} \min(\mathbf{m}^+_{ij}, \mathbf{n}^+_{ij}); \\
[\mathbf{m}^+ \vee \mathbf{n}^+]_{ij} &\stackrel{\triangle}{=} \max(\mathbf{m}^+_{ij}, \mathbf{n}^+_{ij}) \quad .
\end{aligned}
$$

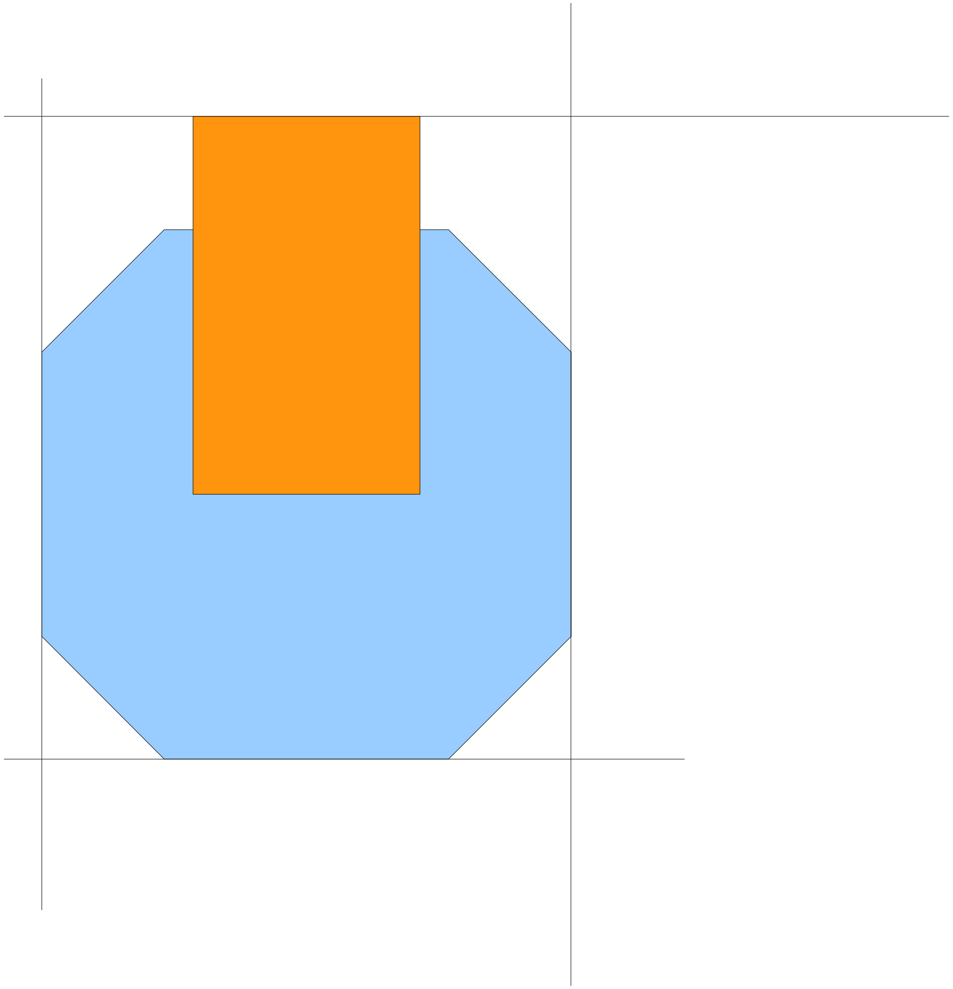
Min



Min



Max



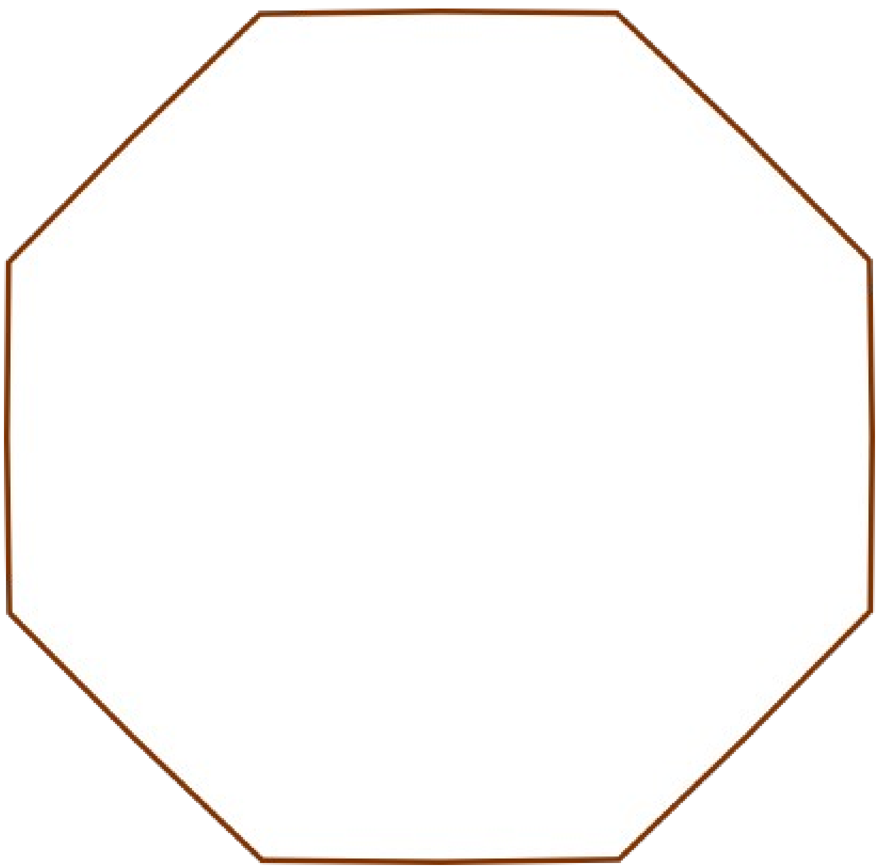
Union and Intersection

Theorem 7:

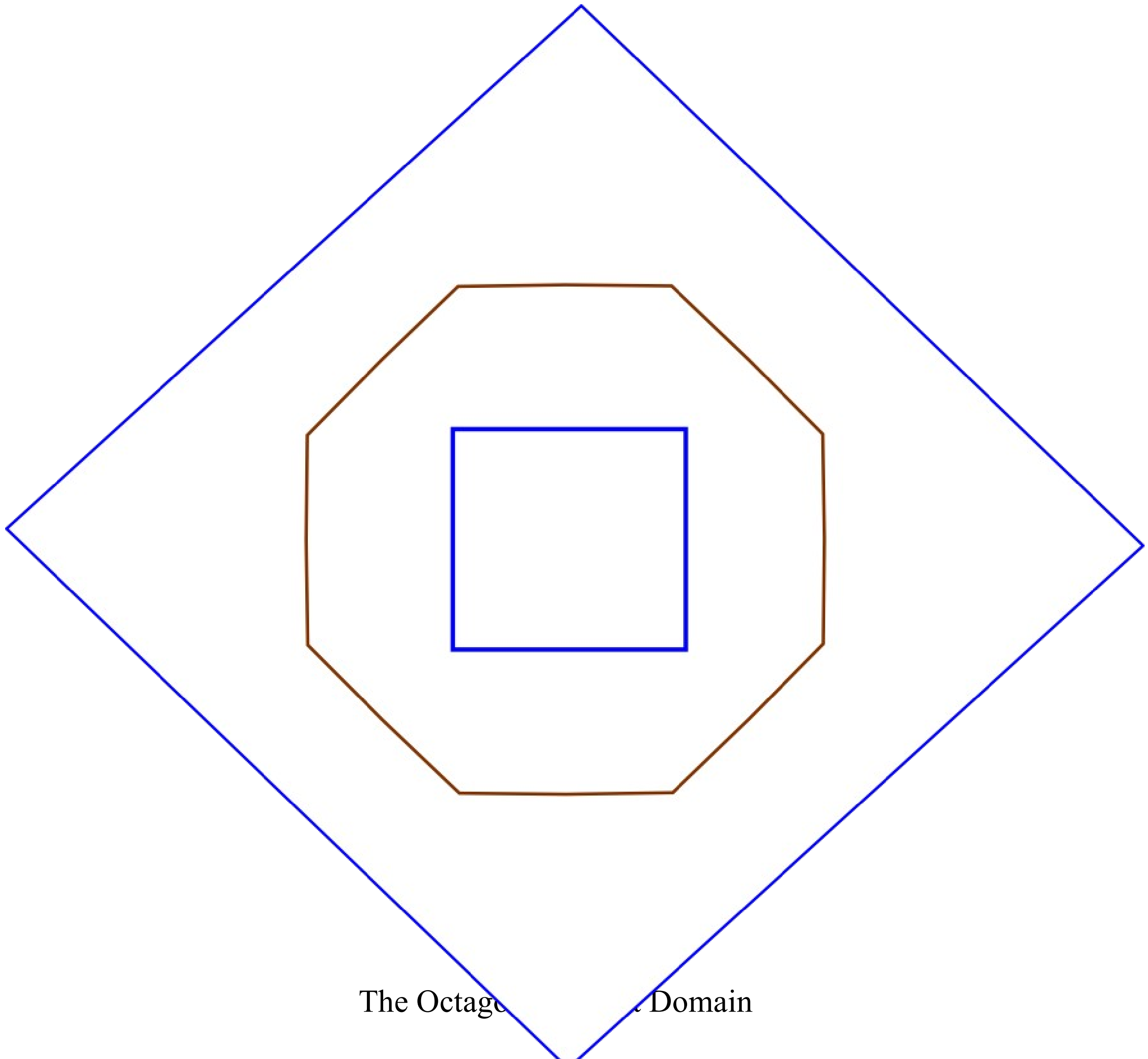
1. $\mathcal{D}^+(\mathbf{m}^+ \wedge \mathbf{n}^+) = \mathcal{D}^+(\mathbf{m}^+) \cap \mathcal{D}^+(\mathbf{n}^+)$.
2. $\mathcal{D}^+(\mathbf{m}^+ \vee \mathbf{n}^+) \supseteq \mathcal{D}^+(\mathbf{m}^+) \cup \mathcal{D}^+(\mathbf{n}^+)$.
3. If $\mathbf{m}^+$ and $\mathbf{n}^+$ represent non-empty octagons, then:

$$((\mathbf{m}^+)^\bullet) \vee ((\mathbf{n}^+)^\bullet) = \inf_{\trianglelefteq}\{\mathbf{o}^+ \mid \mathcal{D}^+(\mathbf{o}^+) \supseteq \mathcal{D}^+(\mathbf{m}^+) \cup \mathcal{D}^+(\mathbf{n}^+)\}.$$

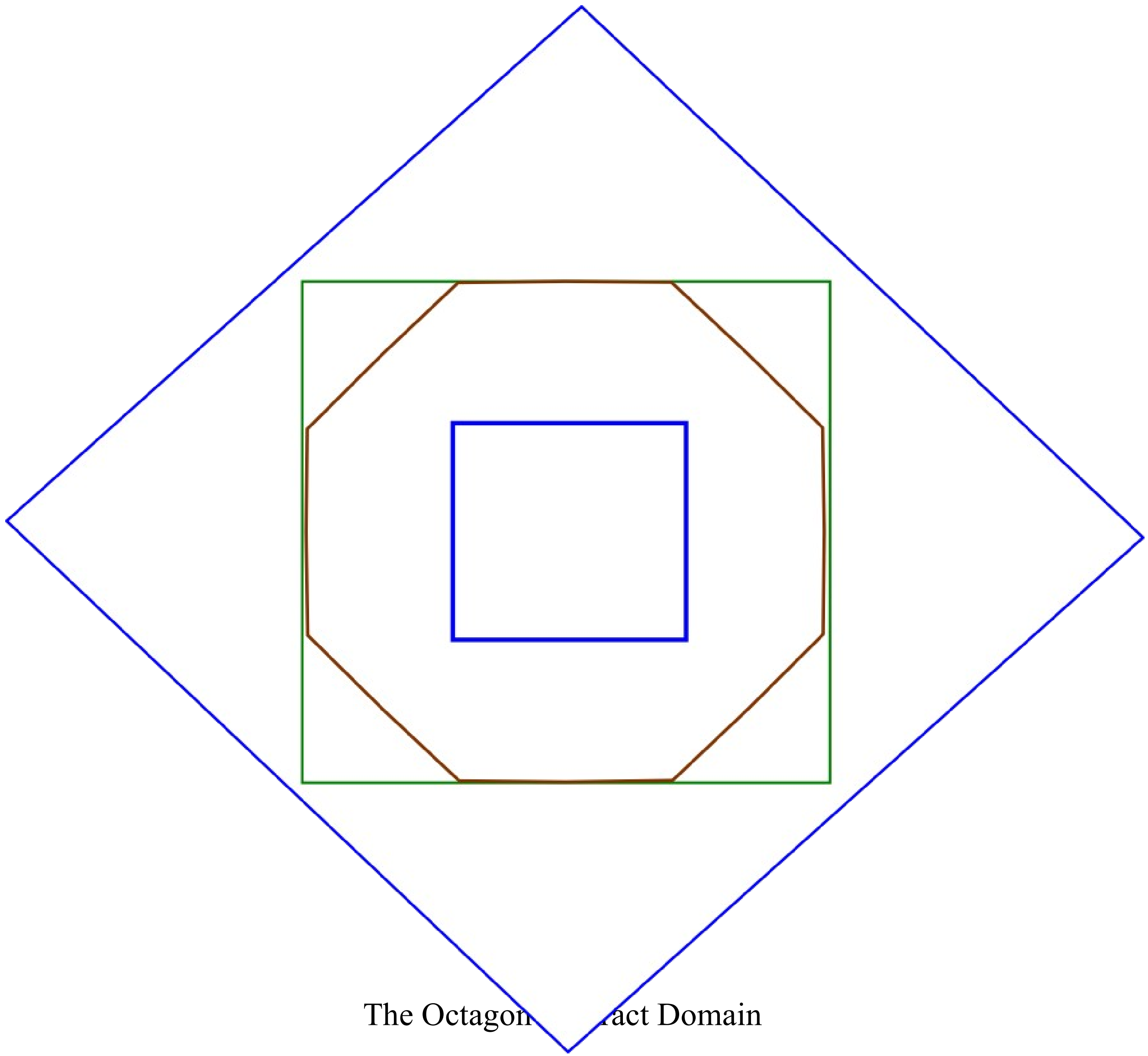
Union



Union



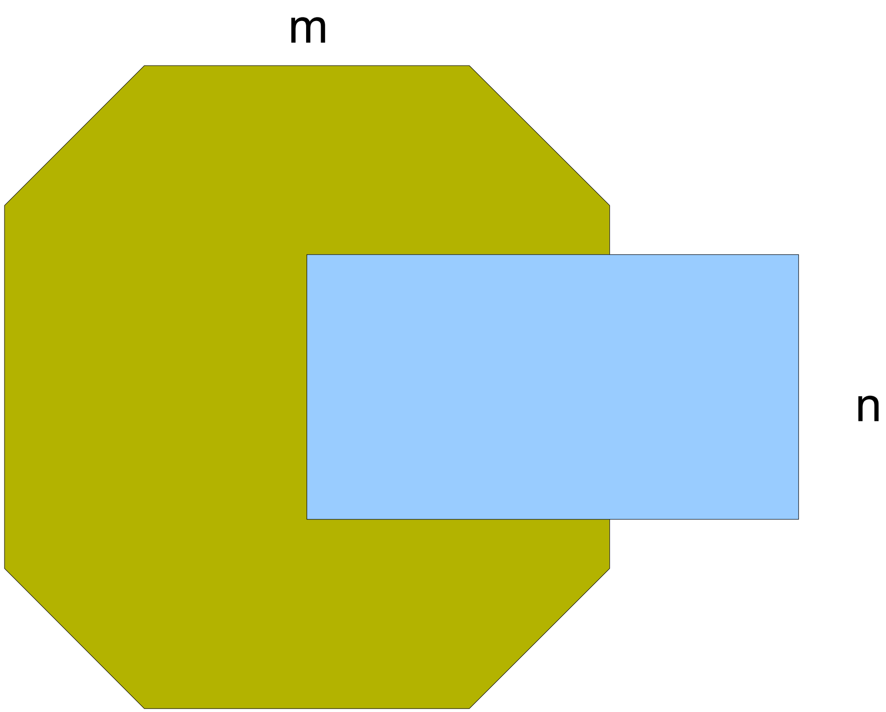
Union over approximation



Widening

$$[\mathbf{m}^+ \triangledown \mathbf{n}^+]_{ij} \triangleq \begin{cases} \mathbf{m}^+_{ij} & \text{if } \mathbf{n}^+_{ij} \leq \mathbf{m}^+_{ij}, \\ +\infty & \text{elsewhere} \quad . \end{cases}$$

Widening



Widening

Widening - 2

Theorem 8:

1. $\mathcal{D}^+(\mathbf{m}^+ \triangledown \mathbf{n}^+) \supseteq \mathcal{D}^+(\mathbf{m}^+) \cup \mathcal{D}^+(\mathbf{n}^+)$.
2. For all chains $(\mathbf{n}_i^+)_{i \in \mathbb{N}}$, the chain defined by induction:

$$\mathbf{m}_i^+ \triangleq \begin{cases} (\mathbf{n}_0^+)^\bullet & \text{if } i = 0, \\ \mathbf{m}_{i-1}^+ \triangledown ((\mathbf{n}_i^+)^\bullet) & \text{elsewhere,} \end{cases}$$

is increasing, ultimately stationary, and with a limit greater than $\bigvee_{i \in \mathbb{N}} (\mathbf{n}_i^+)^\bullet$.

Equality and Assignment

Property 1:

1. $\mathcal{D}^+(\mathbf{m}^+_{(g)}) \supseteq \{s \in \mathcal{D}^+(\mathbf{m}^+) \mid s \text{ satisfies } g\}.$
2. $\mathcal{D}^+(\mathbf{m}^+_{(v_i \leftarrow e)}) \supseteq \{s[s_i \leftarrow e(s)] \mid s \in \mathcal{D}^+(\mathbf{m}^+)\}$

Example definition

Definition 2:

1. $$\left[\mathbf{m}^{+}_{(v_k+v_l\leq c)}\right]_{ij} \triangleq \begin{cases} \min(\mathbf{m}^{+}_{ij}, c) & \text{if } (j,i) \in \{(2k, 2l+1); (2l, 2k+1)\}, \\ \mathbf{m}^{+}_{ij} & \text{elsewhere,} \end{cases}$$

and similarly for $\mathbf{m}^{+}_{(v_k-v_l\leq c)}$ and $\mathbf{m}^{+}_{(-v_k-v_l\leq c)}$.

Example definition

2. $\mathbf{m}^{+}_{(v_k \leq c)} \stackrel{\triangle}{=} \mathbf{m}^{+}_{(v_k + v_k \leq 2c)}$, and
$\mathbf{m}^{+}_{(v_k \geq c)} \stackrel{\triangle}{=} \mathbf{m}^{+}_{(-v_k - v_k \leq -2c)}$.

3. $\mathbf{m}^{+}_{(v_k + v_l = c)} \stackrel{\triangle}{=} (\mathbf{m}^{+}_{(v_k + v_l \leq c)})_{(-v_k - v_l \leq -c)}$,
and similarly for $\mathbf{m}^{+}_{(v_k - v_l = c)}$.

Example definition

4. $\left[\mathbf{m}^{+}_{(v_k \leftarrow v_k + c)}\right]_{ij} \triangleq \mathbf{m}^{+}_{ij} + (\alpha_{ij} + \beta_{ij})c$, with

$$\alpha_{ij} \triangleq \begin{cases} +1 & \text{if } j = 2k, \\ -1 & \text{if } j = 2k+1, \\ 0 & \text{elsewhere }, \end{cases}$$

Example definition

and

$$\beta_{ij} \triangleq \begin{cases} -1 & \text{if } i = 2k, \\ +1 & \text{if } i = 2k+1, \\ 0 & \text{elsewhere} \quad . \end{cases}$$

Example definition

5. $$\left[\mathbf{m}^{+}_{(v_k \leftarrow v_l + c)}\right]_{ij} \triangleq \begin{cases} c & \text{if } (j,i) \in \{(2k,2l);(2l+1,2k+1)\}, \\ -c & \text{if } (j,i) \in \{(2l,2k);(2k+1,2l+1)\}, \\ (\mathbf{m}^{+})^{\bullet}_{ij} & \text{if } i,j \notin \{2k,2k+1\}, \\ +\infty & \text{elsewhere,} \end{cases}$$

for $k \neq l$.

Example definition

6. In all other cases, we simply choose:

$$\mathbf{m}^{+}_{(g)} \stackrel{\triangle}{=} \mathbf{m}^{+},$$

$$\left[\mathbf{m}^{+}_{(v_k \leftarrow e)}\right]_{ij} \stackrel{\triangle}{=} \begin{cases} (\mathbf{m}^{+})^{\bullet}_{ij} & \text{if } i, j \notin \{2k, 2k+1\}, \\ +\infty & \text{elsewhere} \end{cases} .$$

Coherent DBM's lattice

Theorem 9:

1. $(\mathcal{M}_\perp^+, \sqsubseteq, \sqcap, \sqcup, \perp, \top)$ is a lattice.
2. This lattice is complete if $(\mathbb{I}, \leq)$ is complete ($\mathbb{I} = \mathbb{Z}$ or $\mathbb{R}$, but not $\mathbb{Q}$).

Strongly Closed DBM's Lattice

$$\top^{\bullet}{}_{ij} \triangleq \begin{cases} 0 & \text{if } i = j, \\ +\infty & \text{elsewhere,} \end{cases}$$

$$\mathbf{m}^+ \sqsubseteq^{\bullet} \mathbf{n}^+ \stackrel{\triangle}{\Longleftrightarrow} \begin{cases} \text{either} & \mathbf{m}^+ = \bot^{\bullet}, \\ \text{or} & \mathbf{m}^+, \mathbf{n}^+ \neq \bot^{\bullet},\ \mathbf{m}^+ \trianglelefteq \mathbf{n}^+, \end{cases}$$

$$\mathbf{m}^+ \sqcup^{\bullet} \mathbf{n}^+ \triangleq \begin{cases} \mathbf{m}^+ & \text{if } \mathbf{n}^+ = \bot^{\bullet}, \\ \mathbf{n}^+ & \text{if } \mathbf{m}^+ = \bot^{\bullet}, \\ \mathbf{m}^+ \vee \mathbf{n}^+ & \text{elsewhere,} \end{cases}$$

$$\mathbf{m}^+ \sqcap^{\bullet} \mathbf{n}^+ \triangleq \begin{cases} \bot^{\bullet} & \text{if } \bot^{\bullet} \in \{\mathbf{m}^+, \mathbf{n}^+\} \text{ or} \\ & \quad \mathcal{D}^+(\mathbf{m}^+ \wedge \mathbf{n}^+) = \emptyset, \\ (\mathbf{m}^+ \wedge \mathbf{n}^+)^{\bullet} & \text{elsewhere} \ . \end{cases}$$

Meaning function

$$\gamma(\mathbf{m}^+) \stackrel{\triangle}{=} \begin{cases} \emptyset & \text{if } \mathbf{m}^+ = \bot^\bullet, \\ \mathcal{D}^+(\mathbf{m}^+) & \text{elsewhere} \end{cases}.$$

Galois Connection

Theorem 10:

1. $(\mathcal{M}_\perp^\bullet, \sqsubseteq^\bullet, \sqcap^\bullet, \sqcup^\bullet, \perp^\bullet, \top^\bullet)$ is a lattice and γ is one-to-one.
2. If $(\mathbb{I}, \leq)$ is complete, this lattice is complete and γ is meet-preserving: $\gamma(\bigsqcap^\bullet X) = \bigcap\{\gamma(x) \mid x \in X\}$. We can—according to Cousot and Cousot [18, Prop. 7]—build a canonical *Galois insertion*:

$$\mathcal{P}(\mathcal{V}^+ \mapsto \mathbb{I}) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{M}_\perp^\bullet$$

where the *abstraction function* α is defined by:
$\alpha(X) = \bigsqcap^\bullet \{\, x \in \mathcal{M}_\perp^\bullet \mid X \subseteq \gamma(x) \,\}$.

Program Interpretation

- For $[\![(l_i)\ v_i \leftarrow e\ (l_{i+1})]\!]$, we set $\mathbf{m}_{i+1}^+ = (\mathbf{m}_i^+)_{(v_i \leftarrow e)}$.
- For a test $[\![(l_i)\ \textbf{if}\ g\ \textbf{then}\ (l_{i+1})\ \cdots\ \textbf{else}\ (l_j)\ \cdots]\!]$, we set $\mathbf{m}_{i+1}^+ = (\mathbf{m}_i^+)_{(g)}$ and $\mathbf{m}_j^+ = (\mathbf{m}_i^+)_{(\neg g)}$.
- When the control flow merges after a test $[\![\textbf{then}\ \cdots\ (l_i)\ \textbf{else}\ \cdots\ (l_j)\ \textbf{fi}\ (l_{j+1})]\!]$, we set $\mathbf{m}_{j+1}^+ = ((\mathbf{m}_i^+)^\bullet) \sqcup ((\mathbf{m}_j^+)^\bullet)$.

While Loop Interpretation

- For a loop $[\![\ (l_i)\ \textbf{while}\ g\ \textbf{do}\ (l_j) \cdots\ (l_k)\ \textbf{done}\ (l_{k+1})]\!]$, we must solve the relation $\mathbf{m}_j^+ = (\mathbf{m}_i^+ \sqcup \mathbf{m}_k^+)_{(g)}$. We solve it iteratively using the widening: suppose $\mathbf{m}_i^+$ is known and we can deduce a $\mathbf{m}_k^+$ from any $\mathbf{m}_j^+$ by propagation; we compute the limit $\mathbf{m}_j^+$ of

$$\begin{cases} \mathbf{m}_{j,0}^+ = (\mathbf{m}_i^+)_{(g)} \\ \mathbf{m}_{j,n+1}^+ = \mathbf{m}_{j,n}^+ \nabla ((\mathbf{m}_{k,n}^+)^\bullet_{(g)}) \end{cases}$$

then $\mathbf{m}_k^+$ is computed by propagation of $\mathbf{m}_j^+$ and we set $\mathbf{m}_{k+1}^+ = ((\mathbf{m}_i^+)^\bullet_{(\neg g)}) \sqcup ((\mathbf{m}_k^+)^\bullet_{(\neg g)})$

At the end of this process, each $\mathbf{m}_i^+$ is a valid invariant that holds at program location l_i. This method is called *abstract execution.*

Example Program

$(l_0)\ a \leftarrow 0;\ i \leftarrow 1\ (l_1)$
$\textbf{while}\ i \leq m\ \textbf{do}\ (l_2)$
$\quad \textbf{if}\ ?$
$\quad\quad \textbf{then}\ (l_3)\ a \leftarrow a + 1\ (l_4)$
$\quad\quad \textbf{else}\ (l_5)\ a \leftarrow a - 1\ (l_6)$
$\quad \textbf{fi}\ (l_7)$
$\quad i \leftarrow i + 1\ (l_8)$
$\textbf{done}\ (l_9)$

Initial State

$$\mathbf{m}_0^+ = \top$$
$$\mathbf{m}_1^+ = \{i = 1;\ a = 0;\ 1 - i \leq a \leq i - 1\}$$

First Iteration

$$
\begin{aligned}
&\textit{First iteration of the loop}\\
&\mathbf{m}^+_{2.0} = \{i = 1;\ a = 0;\ 1 - i \le a \le i - 1;\ i \le m\}\\
&\mathbf{m}^+_{3,0} = \mathbf{m}^+_{5,0} = \mathbf{m}^+_{2.0}\\
&\mathbf{m}^+_{4,0} = \{i = 1;\ a = 1;\ 2 - i \le a \le i;\ i \le m\}\\
&\mathbf{m}^+_{6,0} = \{i = 1;\ a = -1;\ -i \le a \le i - 2;\ i \le m\}\\
&\mathbf{m}^+_{7,0} = \{i = 1;\ a \in [-1, 1];\ -i \le a \le i;\ i \le m\}\\
&\mathbf{m}^+_{8,0} = \{i = 2;\ a \in [-1, 1];\ 1 - i \le a \le i - 1;\ i \le m + 1\}
\end{aligned}
$$

Second Iteration

Second iteration of the loop

$$\begin{aligned}
\mathbf{m}^{+}_{2,1} = \mathbf{m}^{+}_{3,1} = \mathbf{m}^{+}_{5,1} &= \mathbf{m}^{+}_{2,0} \,\nabla\, (\mathbf{m}^{+}_{8,0})_{(i \leq m)} \\
&= \{1 \leq i \leq m;\ 1 - i \leq a \leq i - 1\} \\
\mathbf{m}^{+}_{4,1} &= \{1 \leq i \leq m;\ 2 - i \leq a \leq i\} \\
\mathbf{m}^{+}_{6,1} &= \{1 \leq i \leq m;\ -i \leq a \leq i - 2\} \\
\mathbf{m}^{+}_{7,1} &= \{1 \leq i \leq m;\ -i \leq a \leq i\} \\
\mathbf{m}^{+}_{8,1} &= \{2 \leq i \leq m + 1;\ 1 - i \leq a \leq i - 1\}
\end{aligned}$$

Third Iteration

Third iteration of the loop

$\mathbf{m}_{2,2}^{+} = \mathbf{m}_{2,1}^{+}$ *(fixpoint reached)*

$\mathbf{m}_{2}^{+} = \mathbf{m}_{2,1}^{+} \qquad \mathbf{m}_{8}^{+} = \mathbf{m}_{8,1}^{+}$

$\mathbf{m}_{9}^{+} = \{i = m + 1;\ 1 - i \leq a \leq i - 1\}$