Ganzinger-Bachmaier Model Existence Theorem for Propositional Logic

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The Resolution Calculus Res

Definition

- Resolution inference rule

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

- (positive) factorisation

$$\frac{C \vee A \vee A}{C \vee A}$$

Refutational Completeness of Resolution

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  or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.

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- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.

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- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
- The limit interpretation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e.g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)

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2. Extend $\succ$ to an ordering $\succ_L$ on ground literals:

$$\begin{array}{ccc} [\neg]A & \succ_L & [\neg]B \quad \text{, if } A \succ B \\ \neg A & \succ_L & A \end{array}$$

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3. Extend $\succ_L$ to an ordering $\succ_C$ on ground clauses:
$\succ_C = (\succ_L)_{\mathrm{mul}}$, the multiset extension of $\succ_L$.
Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Multisets

Definition

Let E be a set. A multiset M over E is a mapping $M : E \to \mathbb{N}$. Hereby $M(e)$ specifies the number of occurrences of elements e of the base set E within the multiset M.

Let $(M, \succ)$ be a partial ordering. The multiset extension of $\succ$ to multisets over E is defined by

$$\begin{aligned} M_1 \succ_{mul} M_2 \Leftrightarrow M_1 \neq M_2 & \\ & \wedge \forall e \in E : [M_2(e) > M_1(e) \\ & \qquad \Rightarrow \exists e' \in E : (e' \succ e \wedge M_1(e') > M_2(e'))] \end{aligned}$$

Clause Orderings

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$.
Order the following clauses:

$\neg A_1 \vee \neg A_4 \vee A_3$

$\neg A_1 \vee A_2$

$\neg A_1 \vee A_4 \vee A_3$

$A_0 \vee A_1$

$\neg A_5 \vee A_5$

$A_1 \vee A_2$

Clause Orderings

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$.
Then:

$$\begin{array}{rc} & A_0 \vee A_1 \\ \prec & A_1 \vee A_2 \\ \prec & \neg A_1 \vee A_2 \\ \prec & \neg A_1 \vee A_4 \vee A_3 \\ \prec & \neg A_1 \vee \neg A_4 \vee A_3 \\ \prec & \neg A_5 \vee A_5 \end{array}$$

Properties of the Clause Ordering

Theorem

1. *The orderings on literals and clauses are total and well-founded.*
2. *Let* C *and* D *be clauses with* $A = \max(C)$, $B = \max(D)$, *where* $\max(C)$ *denotes the maximal atom in* C.
   (i) *If* $A \succ B$ *then* $C \succ D$.
   (ii) *If* $A = B$, A *occurs negatively in* C *but only positively in* D, *then* $C \succ D$.

Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:

$$\prec \downarrow \quad \begin{array}{ll|c|l}
 & & & \\
\hline
B & \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} \dots \vee B \\ \dots \vee B \vee B \\ \neg B \vee \dots \\ \dots \end{array} & \text{all } D \text{ where } \max(D) = B \\
\hline
\vdots & & \vdots & \\
\hline
A & \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} \dots \vee A \\ \dots \vee A \vee A \\ \neg A \vee \dots \\ \dots \end{array} & \text{all } C \text{ where } \max(C) = A \\
\hline
 & & & \\
\end{array}$$

Closure of Clause Sets under Res

Definition

$$
\begin{aligned}
Res(N) &= \{C \mid C \text{ is concl. of a rule in } Res \text{ w/ premises in } N\} \\
Res^0(N) &= N \\
Res^{n+1}(N) &= Res(Res^n(N)) \cup Res^n(N), \text{ for } n \geq 0 \\
Res^*(N) &= \textstyle\bigcup_{n \geq 0} Res^n(N)
\end{aligned}
$$

N is called saturated (wrt. resolution), if $Res(N) \subseteq N$.

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Construction according to $\succ$, starting with the minimal clause.

Construction of Interpretations

Example

Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

	clauses C	I_C	Δ_C	Remarks
1	$\neg A_0$	$\emptyset$	$\emptyset$	true in I_C
2	$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	A_1 maximal
3	$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	true in I_C
4	$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	A_2 maximal
5	$\neg A_1 \vee A_4 \vee A_3 \vee A_0$	$\{A_1, A_2\}$	$\{A_4\}$	A_4 maximal
6	$\neg A_1 \vee \neg A_4 \vee A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	A_3 not maximal; *min. counter-ex.*
7	$\neg A_1 \vee A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set
$\Rightarrow$ there exists a counterexample.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.

Main Ideas of the Construction

- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should be maintained the way it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding A will make C become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of A has no effect on smaller clauses*).

Resolution Reduces Counterexamples

Example

$$\frac{\neg A_1 \lor A_4 \lor A_3 \lor A_0 \qquad \neg A_1 \lor \neg A_4 \lor A_3}{\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_0$	$\emptyset$	$\emptyset$	
$A_0 \lor A_1$	$\emptyset$	$\{A_1\}$	
$A_1 \lor A_2$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \lor A_2$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\emptyset$	A_3 occurs twice *minimal counter-ex.*
$\neg A_1 \lor A_4 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\{A_4\}$	
$\neg A_1 \lor \neg A_4 \lor A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	counterexample
$\neg A_1 \lor A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

The same I, but smaller counterexample, hence some progress was made.

Factorization Reduces Counterexamples

Example

$$\frac{\neg A_1 \vee \neg A_1 \vee A_3 \vee A_3 \vee A_0}{\neg A_1 \vee \neg A_1 \vee A_3 \vee A_0}$$

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$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \vee \neg A_1 \vee A_3 \vee A_0$	$\{A_1, A_2\}$	$\{A_3\}$	
$\neg A_1 \vee \neg A_1 \vee A_3 \vee A_3 \vee A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_1 \vee A_4 \vee A_3 \vee A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	
$\neg A_1 \vee \neg A_4 \vee A_3$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_3 \vee A_5$	$\{A_1, A_2, A_3\}$	$\{A_5\}$	

The resulting $I = \{A_1, A_2, A_3, A_5\}$ is a model of the clause set.

Construction of Candidate Models Formally

Definition

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

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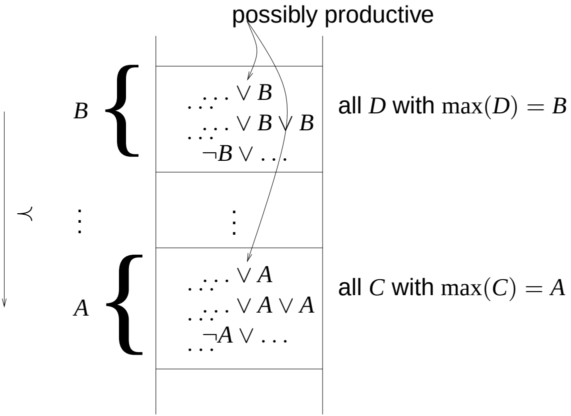
The candidate model for N (wrt. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$.
We also simply write I_N, or I, for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.

Structure of $N, \succ$

Let $A \succ B$; producing a new atom does not affect smaller clauses.



Model Existence Theorem

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(Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let N be saturated wrt. Res, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.

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Proof

Easy exercise! :-)