Omega Test

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Decision Procedures

Decision procedures for Presburger Arithmetic

- Quantifier Elimination
- Cooper's Algorithm
- Automata based
- *Omega Test*
- ...

Description

- Satisfiability of the conjunction of two type of constraints

$$\sum_{i=1}^{n} a_i b_i = b$$

- Variable elimination using a method based on Euclid's algorithm

$$\sum_{i=1}^{n} a_i b_i \leq b$$

- Variable elimination using the Fourier-Motzkin method

Euclid's Algorithm

$$\mathrm{GCD}(a,b) = \begin{cases} a & \text{if } b = 0 \\ \mathrm{GCD}(b, a \bmod b) & \text{otherwise} \end{cases}$$

Example

$\mathrm{GCD}(22{,}6) = \mathrm{GCD}(6{,}4) = \mathrm{GCD}(4{,}2) = \mathrm{GCD}(2{,}0) = 2$

Least Remainder Algorithm

$$\text{GCD}(a,b) = \begin{cases} a & \text{if } b = 0 \\ \text{GCD}(b, a \widehat{\text{mod}}\ b) & \text{otherwise} \end{cases}$$

- Idea: Instead of replacing b by (a mod b) during the division step, replace it by (a mod b) - b if $|a \text{ mod } b| > \frac{1}{2}b$

$$a \widehat{\text{mod}}\ b \stackrel{def}{=} a - b\lfloor \frac{a}{b} + \frac{1}{2} \rfloor$$

Example

GCD(22,6) = GCD(6,-2) = GCD(-2,0) = -2

Equality Constraints

We want to eliminate the variable x_1 in the system using the following equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

1. if $a_1 = 1$: move all the terms other than x_1 to the right hand side
2. if $a_1 > 1$:
   - if a_1 divides $a_2 \cdots a_n$ then there is a solution only if it divides also b
     - Divide both sides by a_1 and compute as the first case
   - if a_1 fails to divide $a_2 \cdots a_n$
     - Introduce a new variable x_1' by

$$\lfloor a_1/a_1 \rfloor x_1 + \lfloor a_2/a_1 \rfloor x_2 + \cdots + \lfloor a_n/a_1 \rfloor x_n = x_1'$$

     - Solve the equation for x_1 and replace the original equation with the following

$$a_1x_1' + (a_2 \bmod a_1)x_2 + \cdots + (a_n \bmod a_1)x_n = b$$

Equality Constraints

Exercise

Remove the equality constraints from the following system

$$\begin{cases} x_1 + 3x_2 + 5x_3 = 9 \\ 2x_1 + 3x_2 = 6 \\ x_1 + x_3 \leq 7 \end{cases}$$

The Omega Test method uses the least remainder algorithm instead of the Euclid's algorithm

Fourier-Motzkin Elimination

$$(\exists x . ax \leq \alpha \land \beta \leq bx) \equiv a\beta \leq b\alpha$$

$$\left\{ \begin{array}{r} ax \leq \alpha \\ \beta \leq bx \end{array} \right\}$$

$$\Updownarrow \quad (a, b > 0)$$

$$a\beta \leq b\alpha$$

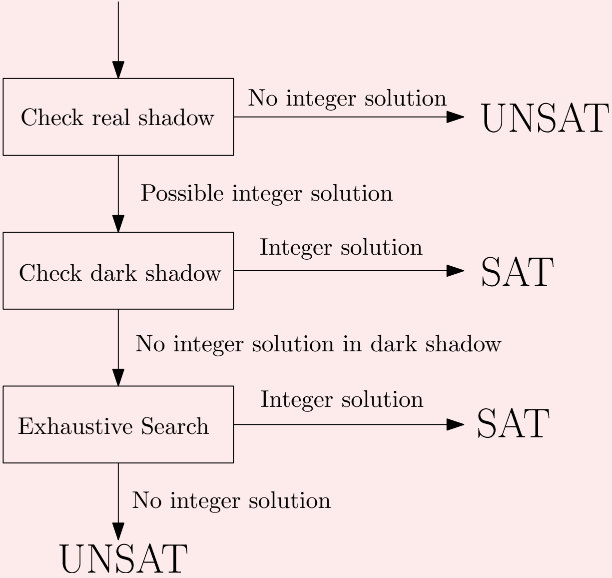
Does the equivalence hold for the integers?

$$(\exists x \in \mathbb{Z} . 3 \leq 2x \land 2x \leq 3) \not\equiv 6 \leq 6$$

Exercise

A sufficient (but not necessary) condition for existence of integer x in the equivalence is the following $b\alpha - a\beta \geq (a-1)(b-1)$

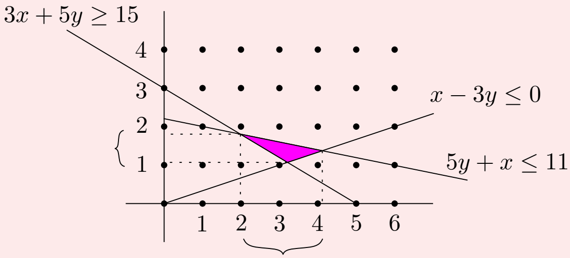
Inequality Constraints



Real Shadow

Check if there are integer solutions in the real solution space

$$\begin{cases} 3x + 5y \geq 15 \\ x - 3y \leq 0 \\ 5y + x \leq 11 \end{cases}$$



Dark Shadow

The subset of the real shadow that further satisfies the integer constraint

$$(b\alpha - a\beta \geq (a-1)(b-1)) \Rightarrow (\exists x. ax \leq \alpha \wedge \beta \leq bx)$$

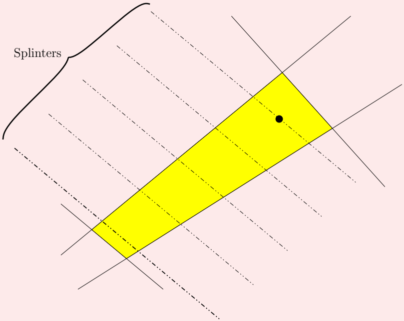
This can eliminate the false positive from the previous example

$$\begin{cases} 5y & \leq 11 - x \\ 15 - 3x & \leq 5y \end{cases}$$

3 does not satisfy the following constraint:
$5(11-x) - 5(15-3x) \geq (4)(4)$

Omega Nightmare

- No integer solution in the dark shadow is not enough
- If the real shadow is non-empty but the dark shadow is empty an exhaustive search is done $(\exists x.ax \leq \alpha \wedge \beta \leq bx)$
- Try all the integers i such that $0 \leq i \leq \frac{ab-a-b}{a}$
- See if any of the equations of the form $bx = \beta + i$ has an integer solution



References

- "Decision procedures: an algorithmic point of view"; Daniel Kroening, Ofer Strichman, EATCS Springer - 2008
- "Dependence Analysis"; Utpal K. Banerjee, Kluwer Academic Publishers - 1996