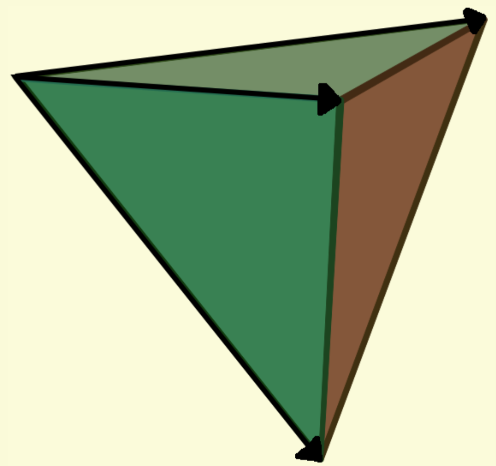
Seminar on Automated Reasoning 2010

Lecture 8: Decomposition Theorem for Polyhedra

$A\overline{x} \leq \overline{b} \; =$  $+$ 

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12. November 2010

Cones

A **cone** is a set $C \subseteq \mathbb{Q}^n$ with $\overline{x}, \overline{y} \in C, \lambda, \mu \leq 1 \in \mathbb{Q} \Rightarrow \lambda\overline{x} + \mu\overline{y} \in C$

A **polyhedral cone** is an intersection of finitely many linear halfspaces, defined by an $m \times n$-matrix A:

$$C = \{\overline{x} \mid A\overline{x} \leq 0\}$$

A **finitely generated cone** is defined by a finite number of vectors $\overline{b}_i$:

$$cone\{\overline{b}_1, \ldots, \overline{b}_n\} = \{\sum_{i=1}^{n} \lambda_i \overline{b}_i \mid \lambda_i \geq 0\} = \{B\overline{\lambda} \mid \overline{\lambda} \geq \overline{0}\}$$

Weyl's Theorem

Theorem: A non-empty finitely generated cone is polyhedral.

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$$\begin{aligned} C &= \{\overline{x} \in \mathbb{Q}^n \mid \exists \overline{\lambda} \in \mathbb{Q}^r : \overline{x} = B\overline{\lambda}, \overline{\lambda} \geq \overline{0}\} \\ &= \{\overline{x} \in \mathbb{Q}^n \mid \exists \overline{\lambda} \in \mathbb{Q}^r : \overline{x} - B\overline{\lambda} \leq \overline{0}, -\overline{x} + B\overline{\lambda} \leq \overline{0}, -\overline{\lambda} \leq \overline{0}\}. \end{aligned}$$

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This is the projection onto the first n coordinates of

$$C' = \left\{ \begin{pmatrix} \overline{x} \\ \overline{\lambda} \end{pmatrix} \in \mathbb{Q}^{n+r} \mid \overline{x} - B\overline{\lambda} \leq \overline{0}, -\overline{x} + B\overline{\lambda} \leq \overline{0}, -\overline{\lambda} \leq \overline{0} \right\}.$$

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We can use Fourier-Motzkin elimination to eliminate variables $\lambda_1, \ldots, \lambda_r$ from the system of inequations defining C', obtaining a new system of inequations defined by matrix A. Then we can write C as

$$C = \{\overline{x} \in \mathbb{Q}^n \mid A\overline{x} \leq 0\}.$$

Thus, C is polyhedral.

Dual Cones

The (polar) **dual** C^* of a cone C is defined as

$$C^* = \{\overline{a} \in \mathbb{Q}^n \mid \overline{a}^T \overline{x} \leq 0 \text{ for all } \overline{x} \in C\}.$$

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Since $C^* = D = \{A^T\overline{\lambda} \mid \overline{\lambda} \ge \overline{0}\}$, C^* is finitely generated.

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Let $D = \{\overline{\lambda}^T A \mid \overline{\lambda} \geq 0\}$. By Weyl's Theorem, D is polyhedral.
Then, by Prop. 2, $D^* = C$ and D^* is finitely generated.

Decomposition Theorem for Polyhedra

A **polyhedron** is a finite intersection of affine half-spaces, defined by

$$P = \{\overline{x} \in \mathbb{Q}^n \mid A\overline{x} \leq \overline{b}\}$$

A **polytope** is the convex hull of a finite set of vectors, i.e.

$$Q = hull\{\overline{x}_1, \ldots, \overline{x}_n\}$$

Theorem: A set $P \subseteq \mathbb{Q}^n$ is a polyhedron if and only if there is a polytope Q and a polyhedral cone C such that $P = Q + C$.

Decomposition Theorem for Polyhedra

Proof: $\Rightarrow$: Let $P = \{\overline{x} \in \mathbb{Q}^n \mid A\overline{x} \leq \overline{b}\}$. Consider the polyhedral cone

$$C_1 = \left\{ \begin{pmatrix} \overline{x} \\ \lambda \end{pmatrix} \in \mathbb{Q}^{n+1} \mid \lambda \geq 0, A\overline{x} - \lambda\overline{b} \leq \overline{0} \right\}.$$

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By Minkowski's Theorem, C_1 is generated by finitely many vectors, say $\begin{pmatrix} \overline{x}_1 \\ \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} \overline{x}_m \\ \lambda_m \end{pmatrix}$. Assume that all λ_i are either 0 or 1 (otherwise scale vectors).

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$\Leftarrow$: Let $P = Q + C$ for some polytope $P = hull\{\overline{x}_1, \ldots, \overline{x}_n\}$ and polyhedral cone $C = cone\{\overline{y}_1, \ldots, \overline{y}_m\}$.

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By Weyl's Theorem, this cone is polyhedral, i.e. it is equal to

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for some A and $\overline{b}$.

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for some A and $\overline{b}$. Then, $\overline{x}_0 \in P$ iff $A\overline{x}_0 \le -\overline{b}$, and therefore P is a polyhedron.