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Not necessarily closed convex polyhedra and the double description method[1]

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Abstract. Since the seminal work of Cousot and Halbwachs, the domain of convex polyhedra has been employed in several systems for the analysis and verification of hardware and software components. Although most implementations of the polyhedral operations assume that the polyhedra are topologically closed (i.e., all the constraints defining them are non-strict), several analyzers and verifiers need to compute on a domain of convex polyhedra that are not necessarily closed (NNC). The usual approach to implementing NNC polyhedra is to embed them into closed polyhedra in a higher dimensional vector space and reuse the tools and techniques already available for closed polyhedra. In this work we highlight and discuss the issues underlying such an embedding for those implementations that are based on the *double description* method, where a polyhedron may be described by a system of linear constraints or by a system of generating rays and points. Two major achievements are the definition of a theoretically clean, high-level user interface and the specification of an efficient procedure for removing redundancies from the descriptions of NNC polyhedra.

Keywords: Convex polyhedra; Double description; Strict linear inequalities; Abstract interpretation; Data-flow analysis

1. Introduction

Convex polyhedra are regions of some n-dimensional space that are bounded by a finite set of hyperplanes. A convex polyhedron in $\mathbb{R}^n$ describes a relation between n real-valued quantities. The class of all such relations turns out to be useful for the representation of the abstract properties of various kinds of complex systems.

The seminal work of Cousot and Halbwachs [CH78] introduced the use of convex polyhedra as a domain of descriptions to solve, by *abstract interpretation* [CC77], a number of important data-flow analysis problems such as array bound checking, compile-time overflow detection, loop invariant computations and loop induction variables. Convex polyhedra are also used, among many other applications, for the analysis and verification of synchronous languages [BJT99, Hal93] and of linear hybrid automata (an extension of finite-state machines that models time requirements) [HHWT97, HPR94], for the computer-aided formal verification of concurrent and reactive systems based on temporal specifications [MBB$^+$99], for inferring argument size relationships in logic languages [BK97], and for the automatic parallelization of imperative programs [Fea91, Pug92]. Since the work of Cousot and Halbwachs, convex polyhedra have thus played an important role in the formal methods community and new uses continue to emerge (see, e.g., [CS01, DRS01]). As a consequence, several critical tasks, such

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as checking the correctness of synchronization protocols or verifying the absence of run-time errors of systems whose failure can cause serious damage, rely on the software implementations of convex polyhedra.

Traditionally, convex polyhedra are assumed to be topologically closed. With the *double description* (DD) method [MRTT53], a closed convex polyhedron can be specified in two ways, using a *constraint system* or a *generator system*: the constraint system contains a finite set of linear non-strict inequality constraints; the generator system contains two finite sets of vectors, collectively called *generators*, which are rays and points of the polyhedron.

However, some applications of static analysis and verification, including recent proposals such as [CS01], need to compute on the domain of *not necessarily closed* (NNC) convex polyhedra.[2] By definition, any NNC polyhedron can be represented by a so-called *mixed constraint system*, that is, a constraint system where a further finite set of linear *strict* inequality constraints is allowed to occur. No similar generalization of the concept of generator system is available, so that the DD method cannot be directly applied to the domain of NNC polyhedra. In contrast, the usual approach for implementing NNC polyhedra is to embed them into closed polyhedra in a vector space with one extra dimension. While this idea, originally proposed in [HPR94] and also described in [HPR97], proved to be quite effective, its direct application in the development of software libraries for the manipulation of convex polyhedra results in a low-level user interface, where most of the geometric intuition of the DD method gets lost under the "implementation details." This has a direct, negative impact on the usability of the resulting software (on this subject, see the discussion in Sect. 7, which also includes quotations taken from [HKP95, Sect. 4.5, pp. 10–11] and [Jea02, Sect. 1.1.4, p. 10]).

In this paper, we propose a much cleaner approach, where the concept of generator of an NNC polyhedron is extended to also account for the *closure points* of the polyhedron. In particular, we show that any NNC polyhedron can be defined directly by means of an *extended generator system*, namely, a triple of finite sets containing rays, points and closure points of the polyhedron. By combining the mixed constraint systems with these extended generator systems for describing NNC polyhedra we can obtain a two-fold improvement over the proposal in [HPR94, HPR97]: easier generalizations and a natural, implementation-independent interface.

Easier generalizations: Several operators, whose definition is in terms of the rays and points of the standard generator systems for closed polyhedra, need to be generalized to NNC polyhedra. Examples are given by the *time-elapse* operator of [HPR94, HPR97], the extrapolation operator '$\propto$' defined in [HH95], the generator-based extrapolation operators sketched in [BJT99], and the new widening operator proposed in [BHRZ03, BHRZ05]. The notion of extended generator system proves to be very effective in the definition and justification of these generalizations. As a remarkable example, in Sect. 4.2 it will be shown how the usual implementation of the inclusion test for closed polyhedra can be easily adapted to the case of NNC polyhedra. The elegance of this generalization is better appreciated by comparing it with the specification of the inclusion test for the low-level implementation of [HPR94], which appears to be much more tricky and obscure. The reason is that in [HPR94] the reader has no high-level interpretation of the generators occurring in the low-level encoding.

A natural, implementation-independent interface: The combination of mixed constraint systems and extended generator systems offers another improvement over the proposal in [HPR94, HPR97]: a high-level user interface that is completely separate from the implementation. On the one hand, an NNC polyhedron can be presented to the client application directly in terms of its defining strict and non-strict constraints or its generating rays, points and closure points; there is no need for the client to be aware of the use of an additional space dimension in the implementation and all issues related to its correct handling, such as the side constraints on this space dimension. On the other hand, by relying on the high-level specification only, the client application will be unaffected by the wider adoption of lazy and incremental computation techniques in the procedures implementing the operators on convex polyhedra. Moreover, if all the functionalities and invariants of the interface are maintained, it is then possible to change the low-level data structures without affecting the application.

In this paper we will also exploit the latter possibility by introducing two alternative classes of closed polyhedra for implementing the NNC polyhedra, both instances of the same basic class. The basis of this representation is a simple generalization of the class of polyhedra used in [HPR94, HPR97]. The new class continues to employ an additional dimension to encode whether or not each affine half-space defining the NNC polyhedron is closed and relies on the same semantic function for extracting the NNC polyhedron it embeds. We describe two alternative specializations of this class for representing the NNC polyhedra. One of these, shown to be biased for the use of

[2] NNC polyhedra have also been called *copolyhedra*, where "co" stands for closed/open[Kan92].

the constraint representation, corresponds to the embedding defined in [HPR94] while the other, which is biased for the use of the generator representation, is new to this paper.

The *Parma Polyhedra Library*,[3] a modern C++ library for the manipulation of convex polyhedra, has been extended so as to implement both approaches. One interesting and potentially useful consequence of having the option of these alternative encodings is that an improved implementation may choose to dynamically switch between them, depending on the particular descriptions needed to perform a given operation; for instance, the constraint-biased encodings may be used when computing intersections of polyhedra, whereas the generator-biased encodings would be preferred when computing convex polyhedral hulls.

Minimization procedures for the descriptions of NNC polyhedra: Another major contribution of this paper will be the identification of an important issue related to the above mentioned embedding of an NNC polyhedron into a closed polyhedron. It will be shown that the usual procedures for minimizing the constraint and generator descriptions of the topologically closed representation of an NNC polyhedron are not enough to obtain a non-redundant description of the NNC polyhedron itself. We will propose a solution for this problem, which affects both the constraint-biased and the generator-biased representations of NNC polyhedra, by providing procedures that are able to efficiently identify the semantically redundant constraints and generators, therefore allowing for the computation of truly minimal descriptions. A preliminary experimental evaluation will show how the use of the new minimization procedures may have a great impact on the efficiency of some of the most important operators on the domain of NNC polyhedra.

The paper is structured as follows: Section 2 recalls the required concepts and notations; Section 3 briefly presents the theoretical framework underlying the double description method for the representation and manipulation of closed convex polyhedra; Section 4 provides a generalization of this framework to also allow for the manipulation of convex polyhedra that are not necessarily closed; Section 5 presents a general class and two special subclasses of the set of closed polyhedra that are appropriate for the representation of NNC polyhedra; Section 6 identifies a problem related to the minimization of these NNC polyhedra representations and proposes a solution based on the notion of ϵ-minimal forms; Section 7 presents an implementation of the ideas contained in this paper, providing an informal comparison with other libraries offering some support for NNC polyhedra and discussing a preliminary experimental evaluation we have conducted. Section 8 concludes. The Appendix contains proofs of the formal results stated in the main part of the paper.

This paper is a combined, extended and improved version of [BHZ03] and [BRZH02a].

2. Preliminaries

The set of non-negative reals is denoted by $\mathbb{R}_+$. In the paper, all topological arguments refer to the Euclidean topological space $\mathbb{R}^n$, for any positive integer n. If $S \subseteq \mathbb{R}^n$, then the *topological closure* of S is defined as $\mathbb{C}(S) \stackrel{\mathrm{def}}{=} \bigcap\{ C \subseteq \mathbb{R}^n \mid S \subseteq C \text{ and } C \text{ is closed} \}$.

For each $i \in \{1, \dots, n\}$, v_i denotes the i-th component of the (column) vector $\boldsymbol{v} \in \mathbb{R}^n$. We denote by $\mathbf{0}$ the vector of $\mathbb{R}^n$ having all components equal to zero. A vector $\boldsymbol{v} \in \mathbb{R}^n$ can also be interpreted as a matrix in $\mathbb{R}^{n\times 1}$ and manipulated accordingly with the usual definitions for addition, multiplication (both by a scalar and by another matrix), and transposition, which is denoted by $\boldsymbol{v}^{\mathrm{T}}$. The *scalar product* of $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, denoted $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$, is the real number $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = \sum_{i=1}^{n} v_i w_i$.

For any relational operator $\bowtie \in \{=, \geqslant, \leqslant, <, >\}$, we write $\boldsymbol{v} \bowtie \boldsymbol{w}$ to denote the conjunctive proposition $\bigwedge_{i=1}^{n}(v_i \bowtie w_i)$. Moreover, $\boldsymbol{v} \neq \boldsymbol{w}$ will denote the proposition $\neg(\boldsymbol{v} = \boldsymbol{w})$. We will sometimes use the convenient notation $a \bowtie_1 b \bowtie_2 c$ to denote the conjunction $a \bowtie_1 b \wedge b \bowtie_2 c$ and we will not distinguish conjunctions of propositions from sets of propositions.

Let $S \subseteq \mathbb{R}^n$ be a set of vectors. The *orthogonal* of S is $S^{\perp} \stackrel{\mathrm{def}}{=} \bigl\{ \boldsymbol{w} \in \mathbb{R}^n \bigm| \forall \boldsymbol{v} \in S : \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \bigr\}$. If $S \subseteq \mathbb{R}^n$ has finite cardinality m, then $\mathrm{matrix}(S) \subseteq \mathbb{R}^{n\times m}$ denotes the set of all matrices having S as the set of their columns. In the following, we will abuse notation by letting S also denote a fixed arbitrary element of $\mathrm{matrix}(S)$. The context makes it clear when the symbol denotes a set or a matrix.

We assume some familiarity with the basic notions of lattice theory [Bir67].

[3] Freely available at URI http://www.cs.unipr.it/ppl/.

3. The double description method

For each vector $\boldsymbol{a} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$, where $\boldsymbol{a} \neq \boldsymbol{0}$, the linear non-strict inequality constraint $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr)$ defines a topologically closed affine half-space of $\mathbb{R}^n$. The linear equality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = b$ defines an affine hyperplane. A topologically closed convex polyhedron is usually described as a finite system of linear equality and non-strict inequality constraints. Theoretically speaking, it is simpler to express each equality constraint as the intersection of the two half-spaces $\beta^+ = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr)$ and $\beta^- = \bigl(\langle -\boldsymbol{a}, \boldsymbol{x} \rangle \geqslant -b\bigr)$; in such a case, we say that β^+ and β^- are *singular* constraints for polyhedron $\mathcal{P}$ and write $\{\beta^+, \beta^-\} \subseteq \mathrm{eq}(\mathcal{P})$. We do not distinguish between syntactically different constraints defining the same affine half-space so that, e.g., $x \geqslant 2$ and $2x \geqslant 4$ are considered to be the same constraint.

Definition 3.1 (Closed polyhedron). The set $\mathcal{P} \subseteq \mathbb{R}^n$ is a *closed polyhedron* if and only if $\mathcal{P}$ can be expressed as the intersection of a finite number of closed affine half-spaces of $\mathbb{R}^n$.

We write $\mathrm{con}(\mathcal{C})$ to denote the polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ described by the finite *constraint system* $\mathcal{C}$. Formally, we define

$$\mathrm{con}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \left\{ \boldsymbol{p} \in \mathbb{R}^n \;\middle|\; \forall \beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr) \in \mathcal{C} : \langle \boldsymbol{a}, \boldsymbol{p} \rangle \geqslant b \right\}.$$

The function 'con' enjoys an anti-monotonicity property, meaning that $\mathcal{C}_1 \subseteq \mathcal{C}_2$ implies $\mathrm{con}(\mathcal{C}_1) \supseteq \mathrm{con}(\mathcal{C}_2)$.

Alternatively, the definition of a topologically closed convex polyhedron can be based on some of its geometric features. A vector $\boldsymbol{r} \in \mathbb{R}^n$ such that $\boldsymbol{r} \neq \boldsymbol{0}$ is a *ray* (or *direction of infinity*) of a non-empty polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ if, for every point $\boldsymbol{p} \in \mathcal{P}$ and every non-negative scalar $\rho \in \mathbb{R}_+$, it holds $\boldsymbol{p} + \rho \boldsymbol{r} \in \mathcal{P}$; the set of all the rays of a polyhedron $\mathcal{P}$ is denoted by $\mathrm{rays}(\mathcal{P})$. A vector $\boldsymbol{l} \in \mathbb{R}^n$ is a *line* of $\mathcal{P}$ if both $\boldsymbol{l}$ and $-\boldsymbol{l}$ are rays of $\mathcal{P}$; in such a case, we say that $\boldsymbol{l}$ and $-\boldsymbol{l}$ are *singular* rays for polyhedron $\mathcal{P}$ and write $\{\boldsymbol{l}, -\boldsymbol{l}\} \subseteq \mathrm{lines}(\mathcal{P})$. The empty polyhedron has no rays and no lines. As was the case for equality constraints, the theory can dispense with the use of lines by using the corresponding pair of singular rays. Moreover, when vectors are used to denote rays, no distinction will be made between different vectors having the same direction so that, e.g., $\boldsymbol{r}_1 = (1, 3)^{\mathrm{T}}$ and $\boldsymbol{r}_2 = (2, 6)^{\mathrm{T}}$ are considered to be the same ray in $\mathbb{R}^2$. The following theorem is a simple consequence of well-known theorems by Minkowski and Weyl [SW70].

Theorem 3.2 The set $\mathcal{P} \subseteq \mathbb{R}^n$ is a closed polyhedron if and only if there exist finite sets $R, P \subseteq \mathbb{R}^n$ of cardinality r and p, respectively, such that $\boldsymbol{0} \notin R$ and

$$\mathcal{P} = \mathrm{gen}\bigl((R, P)\bigr) \stackrel{\mathrm{def}}{=} \left\{ R\boldsymbol{\rho} + P\boldsymbol{\pi} \in \mathbb{R}^n \;\middle|\; \boldsymbol{\rho} \in \mathbb{R}_+^r, \boldsymbol{\pi} \in \mathbb{R}_+^p, \sum_{i=1}^{p} \pi_i = 1 \right\}.$$

When $\mathcal{P} \neq \varnothing$, we say that $\mathcal{P}$ is described by the *generator system* $\mathcal{G} = (R, P)$. In particular, the vectors of R and P are rays and points of $\mathcal{P}$, respectively. Thus, each point of the generated polyhedron is obtained by adding a non-negative combination of the rays in R and a convex combination of the points in P. Informally speaking, if no "supporting point" is provided then an empty polyhedron is obtained; formally, $\mathcal{P} = \varnothing$ if and only if $P = \varnothing$. By convention, the empty system (i.e., the system with $R = \varnothing$ and $P = \varnothing$) is the only generator system for the empty polyhedron. We define a partial order relation '$\sqsubseteq$' on generator systems, which is the component-wise extension of set inclusion. Namely, for any generator systems $\mathcal{G}_1 = (R_1, P_1)$ and $\mathcal{G}_2 = (R_2, P_2)$, we have $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$ if and only if $R_1 \subseteq R_2$ and $P_1 \subseteq P_2$; if, in addition, $\mathcal{G}_1 \neq \mathcal{G}_2$, we write $\mathcal{G}_1 \sqsubset \mathcal{G}_2$. The function 'gen' enjoys a monotonicity property, as $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$ implies $\mathrm{gen}(\mathcal{G}_1) \subseteq \mathrm{gen}(\mathcal{G}_2)$.

The vector $\boldsymbol{v} \in \mathcal{P}$ is an *extreme point* (or *vertex*) of the polyhedron $\mathcal{P}$ if it cannot be expressed as a convex combination of some other points of $\mathcal{P}$. Similarly, $\boldsymbol{r} \in \mathrm{rays}(\mathcal{P})$ is an *extreme ray* of $\mathcal{P}$ if it cannot be expressed as a non-negative combination of some other rays of $\mathcal{P}$. It is worth stressing that, in general, the vectors in R and P are not the extreme rays and the vertices of the polyhedron: for instance, any half-space of $\mathbb{R}^2$ has two extreme rays and no vertices, but any generator system describing it will contain at least three rays and one point.

The combination of the two approaches outlined above is the basis of the double description method due to Motzkin et al. [MRTT53], which exploits the duality principle to compute each representation starting from the other one, possibly minimizing both descriptions. Clever implementations of this *conversion* procedure, such as those based on the extension by Le Verge [Le 92] of Chernikova's algorithms [Che64, Che65, Che68], are the starting point for the development of software libraries based on the DD method. While being characterized by a worst case computational cost which is exponential in the size of the input, these algorithms turn out to be practically useful for the purposes of many applications in the context of static analysis.

Definition 3.3 (DD pair and minimal forms). If $\mathrm{con}(\mathcal{C}) = \mathrm{gen}(\mathcal{G}) = \mathcal{P}$, then $(\mathcal{C}, \mathcal{G})$ is said to be a *DD pair* for $\mathcal{P}$, and we write $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. We say that

- $\mathcal{C}$ is in *minimal form* if there does not exist $\mathcal{C}' \subset \mathcal{C}$ such that $\mathrm{con}(\mathcal{C}') = \mathcal{P}$;
- $\mathcal{G}$ is in *minimal form* if there does not exist $\mathcal{G}' \sqsubset \mathcal{G}$ such that $\mathrm{gen}(\mathcal{G}') = \mathcal{P}$;
- The DD pair $(\mathcal{C}, \mathcal{G})$ is in *minimal form* if $\mathcal{C}$ and $\mathcal{G}$ are both in minimal form.

A polyhedron may be described by different constraint systems (respectively, generator systems) in minimal form. As a matter of fact, since we are expressing the equality constraints (respectively, lines) by using non-strict inequality constraints (respectively, rays), these equivalent descriptions in minimal form may also have a different number of constraints (resp., rays). For instance, the polyhedron $\mathcal{P} = \{\mathbf{0}\} \subseteq \mathbb{R}^n$ can be described by the constraint systems $\mathcal{C}_1 = \{0 \leqslant x \leqslant 0, 0 \leqslant y \leqslant 0\}$ and $\mathcal{C}_2 = \{x + y \geqslant 0, x - y \geqslant 0, x \leqslant 0\}$, which are both in minimal form and have different cardinalities.

Stronger characterizations for the descriptions of a polyhedron may be obtained by specifying further properties, besides the above minimality requirement. For any constraint $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr)$, the *slope* of β is defined as $\mathrm{slope}(\beta) = \boldsymbol{a}$; for a constraint system $\mathcal{C}$, let $\mathrm{slope}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \{\, \mathrm{slope}(\beta) \bigm| \beta \in \mathcal{C} \,\}$.

Definition 3.4 (Orthogonal forms). Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P} \neq \varnothing$ be a DD pair for the non-empty polyhedron $\mathcal{P}$. We say that

- $\mathcal{C}$ is in *orthogonal form* if $I \subseteq E^{\perp}$, where $I \stackrel{\mathrm{def}}{=} \mathrm{slope}\bigl(\mathcal{C} \setminus \mathrm{eq}(\mathcal{P})\bigr)$ and $E \stackrel{\mathrm{def}}{=} \mathrm{slope}\bigl(\mathcal{C} \cap \mathrm{eq}(\mathcal{P})\bigr)$;
- $\mathcal{G} = (R, P)$ is in *orthogonal form* if $(R \setminus L) \cup P \subseteq L^{\perp}$, where $L \stackrel{\mathrm{def}}{=} R \cap \mathrm{lines}(\mathcal{P})$.

For a topologically closed polyhedron, all descriptions in minimal form that are also in *orthogonal form* have the same set of non-singular inequality constraints, the same set of non-singular rays and the same set of points, whereas the sets of singular constraints and singular rays may still differ. Orthogonal forms can be computed by applying a simple variant of the well-known Gram-Shmidt orthogonalization procedure [Sch99] to a description such that, if the original description was in minimal form, then the derived orthogonal description is still in minimal form.

In the following, minimal forms and orthogonality are not assumed unless explicitly stated.

3.1. Operations on closed polyhedra

In this section we show that the ability to switch from a constraint description to a generator description, or vice versa, can be usefully exploited to provide simple implementations for the basic operations on the domain of closed polyhedra.

The set of all closed polyhedra on the vector space $\mathbb{R}^n$, denoted $\mathbb{CP}_n$, can be partially ordered by set-inclusion to form a lattice having the empty set and $\mathbb{R}^n$ as the bottom and top element, respectively. The binary meet operation, returning the greatest closed polyhedron smaller than or equal to the two arguments, is easily seen to correspond to set-intersection. The binary join operation, returning the least closed polyhedron greater than or equal to the two arguments, is denoted '$\uplus$' and called *convex polyhedral hull* (poly-hull, for short); note that, in general, the poly-hull of two polyhedra is different from their convex hull [SW70].

With the double description method, set-intersection is easily implemented by taking the union of the constraint systems representing the two arguments, whereas the poly-hull is implemented by taking the component-wise union of the generator systems representing the two arguments; as said above, the test for emptiness can be implemented by checking whether the generator system contains no points at all.

The elegance of this formalization is probably most appreciated in the implementation of the lattice partial order relation, i.e., subset inclusion. We say that point $\boldsymbol{p} \in \mathbb{R}^n$ *satisfies* the constraint $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr)$ if $\boldsymbol{p}$ belongs to the closed affine half-space defined by β, i.e., if $\langle \boldsymbol{a}, \boldsymbol{p} \rangle \geqslant b$ holds. Similarly, a ray $\boldsymbol{r} \in \mathbb{R}^n$ *satisfies* β if the corresponding direction of infinity is included in the half-space defined by β, i.e., if $\langle \boldsymbol{a}, \boldsymbol{r} \rangle \geqslant 0$ holds. Now, if $\mathcal{P}_1 = \mathrm{gen}(\mathcal{G}_1)$ and $\mathcal{P}_2 = \mathrm{con}(\mathcal{C}_2)$, the inclusion $\mathcal{P}_1 \subseteq \mathcal{P}_2$ holds if and only if each generator in $\mathcal{G}_1$ satisfies all the constraints in $\mathcal{C}_2$.

Static analysis and verification applications adopting the domain of convex polyhedra need to provide correct approximations to other concrete semantics operators, besides the lattice theory operators mentioned above. For instance, in the context of imperative languages, one of the most frequent operations is the assignment of an expression to a variable. Suppose that the set of all possible current values of the program variables is approximated by a convex polyhedron; then, if the considered assignment expression is a linear function of the variables' values, its effect will be correctly modeled by computing the image of the polyhedron under the affine transformation corresponding to the considered assignment expression. With the double description method, the result of such an *affine image* operator will be the polyhedron described by the generator system obtained by applying the affine transformation to the generators of the argument polyhedron. A similar approach, but using the constraint description of the polyhedron, allows for the computation of the *affine pre-image* of a polyhedron; this is of particular interest for a backward semantic construction, where the initial values of program variables are approximated starting from their final values.

4. Not necessarily closed polyhedra

For each vector $\boldsymbol{a} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$, where $\boldsymbol{a} \neq \mathbf{0}$, the linear strict inequality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b$ defines an open affine half-space. By allowing strict inequalities to occur in the system of constraints, it is possible to define convex polyhedra that are not necessarily closed (NNC polyhedra, for short).

Definition 4.1 (NNC polyhedron). The set $\mathcal{P} \subseteq \mathbb{R}^n$ is an *NNC polyhedron* if and only if $\mathcal{P}$ can be expressed as the intersection of a finite number of (not necessarily closed) affine half-spaces of $\mathbb{R}^n$.

Formally, we overload the function 'con' so that, for any *mixed constraint system* $\mathcal{C}$, that is, a constraint system possibly containing both strict and non-strict inequality constraints, we have

$$\operatorname{con}(\mathcal{C}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{p} \in \mathbb{R}^n \;\middle|\; \forall \beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr) \in \mathcal{C} : \langle \boldsymbol{a}, \boldsymbol{p} \rangle \bowtie b \right\},$$

where $\bowtie \in \{\geqslant, >\}$. Note that 'con' still satisfies the anti-monotonicity property.

4.1. The generators of NNC polyhedra

One of the fundamental features of the double description method, and the very reason for its name, is the possibility of representing a closed polyhedron using a system of constraints or a system of generators. As we have already explained in Sect. 3.1, there are contexts where each of these equivalent descriptions is the most appropriate.

Any NNC polyhedron can be easily described by using mixed constraint systems, but a similar generalization of the concept of generator system seems to be missing. Since by using lines, rays and points we can only represent closed polyhedra, the key step for the parametric description of NNC polyhedra is the introduction of a new kind of generator.

Definition 4.2 (Closure point). A vector $\boldsymbol{c} \in \mathbb{R}^n$ is a *closure point* of $S \subseteq \mathbb{R}^n$ if and only if $\boldsymbol{c} \in \mathbb{C}(S)$.

For NNC polyhedra, closure points can be characterized by the following property.

Proposition 4.3 A vector $\boldsymbol{c} \in \mathbb{R}^n$ is a closure point of the NNC polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ if and only if $\mathcal{P} \neq \varnothing$ and $\sigma \boldsymbol{p} + (1 - \sigma)\boldsymbol{c} \in \mathcal{P}$ for every point $\boldsymbol{p} \in \mathcal{P}$ and every $\sigma \in \mathbb{R}$ such that $0 < \sigma < 1$.

In the above proposition, it should be observed that not all of the possible convex combinations of $\boldsymbol{p}$ and $\boldsymbol{c}$ are considered. In particular, by neglecting the case when $\sigma = 0$ we do not force $\boldsymbol{c}$ to belong to $\mathcal{P}$. The case when $\sigma = 1$ would be harmless, but it is left out for the sake of symmetry.

We are now able to provide a parametric description for any NNC polyhedron.

Theorem 4.4 The set $\mathcal{P} \subseteq \mathbb{R}^n$ is an NNC polyhedron if and only if there exist finite sets $R, P, C \subseteq \mathbb{R}^n$ of cardinality r, p and c, respectively, such that $\mathbf{0} \notin R$ and

$$\mathcal{P} = \operatorname{gen}\bigl((R, P, C)\bigr) \stackrel{\text{def}}{=} \left\{ R\boldsymbol{\rho} + P\boldsymbol{\pi} + C\boldsymbol{\gamma} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \boldsymbol{\rho} \in \mathbb{R}^r_+, \boldsymbol{\pi} \in \mathbb{R}^p_+, \boldsymbol{\pi} \neq \mathbf{0}, \boldsymbol{\gamma} \in \mathbb{R}^c_+ \\ \sum_{i=1}^p \pi_i + \sum_{i=1}^c \gamma_i = 1 \end{array} \right\}.$$

When $\mathcal{P} \neq \varnothing$, we say that $\mathcal{P}$ is described by the *extended generator system* $\mathcal{G} = (R, P, C)$. As was the case for closed polyhedra, the vectors in R and P are rays and points of $\mathcal{P}$, respectively. The condition $\boldsymbol{\pi} \neq \mathbf{0}$ ensures that at least one of the points of P plays an active role in any convex combination of the vectors of P and C. It follows from Proposition 4.3 that the vectors of C are closure points of $\mathcal{P}$. Since both rays and closure points

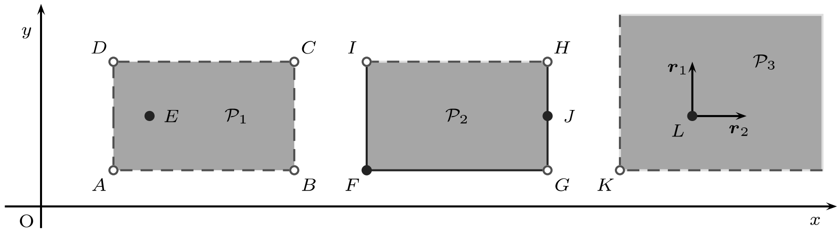

Fig. 1. Using closure points to define NNC polyhedra on $\mathbb{R}^2$

need a supporting point, we have $\mathcal{P} = \varnothing$ if and only if $P = \varnothing$. The partial order relation '$\sqsubseteq$' on generator systems is easily extended to also take into account the closure points component, so that the overloading of the function 'gen' still satisfies the monotonicity property. It is also worth stressing that, once we consider both mixed constraint systems and extended generator systems, then the notions of DD pair and minimal forms, exactly as stated in Definition 3.3, also apply to the representations of NNC polyhedra. The same observation holds for the notion of orthogonal form stated in Definition 3.4; however, in the case of an NNC polyhedron, the orthogonality requirement is less useful, as it will soon become clear that minimal orthogonal forms are insufficient to uniquely identify the non-singular components of a description.

In Fig. 1, we provide a few examples of the use of extended generator systems for the description of NNC polyhedra in $\mathbb{R}^2$: (closure) points are represented by small (un-)filled circles, whereas rays are represented by vectors that, for notational convenience, are applied to points. The NNC polyhedron $\mathcal{P}_1$ is an open rectangle and is described by the closure points A, B, C, D and the point E. Note that all the four closure points have to be included in any generator system for $\mathcal{P}_1$, whereas E could have been replaced by any other point of $\mathcal{P}_1$; moreover, since $\mathcal{P}_1$ has no rays, all generator systems for $\mathcal{P}_1$ in minimal form are also in orthogonal form, so that, as said above, minimality with orthogonality are not enough to uniquely identify the points in a generator system for $\mathcal{P}_1$. The NNC polyhedron $\mathcal{P}_2$ is another rectangle that is neither closed nor open: since F is a point, the open segments $]F, G[$ and $]F, I[$ are included in $\mathcal{P}_2$; similarly, the open segment $]G, H[$ is included in $\mathcal{P}_2$ because J is a point of the generator system (note that J is needed, since both G and H are not in $\mathcal{P}_2$, but it could have been replaced by any other point lying on this open segment); in contrast, the closed segment $[H, I]$ is disjoint from $\mathcal{P}_2$, because neither H nor I are points of $\mathcal{P}_2$. Finally, the NNC polyhedron $\mathcal{P}_3$ can be regarded as the translation by K of the open positive orthant. Thus the generator system includes the closure point K, the rays $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and the point L; again, the latter could have been replaced by any other point of $\mathcal{P}_3$.

4.2. Operations on NNC polyhedra

We denote by $\mathbb{P}_n$ the set of all NNC polyhedra on the vector space $\mathbb{R}^n$. As was the case for the domain $\mathbb{CP}_n$, when partially ordered by set-inclusion, $\mathbb{P}_n$ is a lattice having the empty set and $\mathbb{R}^n$ as the bottom and top element, respectively; the set-intersection and poly-hull operators are the binary meet and join of the lattice, respectively. Obviously, we have $\mathbb{CP}_n \subseteq \mathbb{P}_n$ and, in particular, $\mathbb{CP}_n$ is a sublattice of $\mathbb{P}_n$.

With the double description method as generalized above for NNC polyhedra, all the lattice operations on $\mathbb{P}_n$ can be implemented by following *the same approach* adopted for the domain $\mathbb{CP}_n$. Thus, a mixed constraint system representing the set-intersection of two NNC polyhedra is obtained by taking the union of the mixed constraint systems representing the two arguments; an extended generator system representing the poly-hull of two NNC polyhedra is obtained by taking the component-wise union of the extended generator systems representing the two arguments; the emptiness test is implemented, as before, by checking whether the extended generator system has no points at all (disregarding the closure points).

Even the implementation of the inclusion test $\mathcal{P}_1 \subseteq \mathcal{P}_2$ is still based on checking that each generator in an extended generator system for $\mathcal{P}_1$ satisfies all the constraints in a mixed constraint system for $\mathcal{P}_2$. Clearly, a suitable extension is needed for these satisfaction tests, covering the new combinations provided by the additional constraint and generator types, i.e., strict inequalities and closure points. All the possible cases are shown in Table 4.2. It can be seen that such an extension is fairly intuitive. With non-strict inequalities closure points behave the same as points. In contrast, only the points of the polyhedron are required to respect the strict inequalities; both closure points and rays just have to satisfy the corresponding non-strict inequalities. This is because closure points are limit points of the polyhedron, and do not necessarily belong to it.

Table 1. Testing if the constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b$ is satisfied by the generator $\boldsymbol{g}$

	Generator type		
Constraint type	Ray	Point	Closure point
Non-strict inequality	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geqslant 0$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geqslant b$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geqslant b$
Strict inequality	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geqslant 0$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle > b$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geqslant b$

Similar generalizations are easily obtained for all the usual semantic operators. For instance, an extended generator system representing the affine image of an NNC polyhedron can be computed by applying the affine transformation to all the elements of the extended generator system representing the argument.

4.3. Alternative approaches

The one we are following is not the only possible approach to the description and manipulation of NNC polyhedra. As an alternative, one could use the classical DD method to describe the topological closure of the NNC polyhedron together with a description of all the missing faces.[4] This would amount to replacing some metric information (the strict inequalities or some of the generating points) by combinatorial information (for each missing face, the subset of constraints or generators describing it). Even though such an approach is feasible from a theoretical point of view, to the best of our knowledge it has never been properly formalized or implemented. As a consequence, it is unclear whether it would lead to a specification of the required semantic operators which is as clean as the one outlined in the previous section. It is moreover our opinion that the use of strict inequalities and closure points results in a much more elegant and user-friendly interface. Such a claim is supported, as far as strict inequalities are concerned, by the similar choice made in the user interfaces of several constraint programming systems. The use of closure points in generator systems is going to be as intuitive as the use of strict inequalities in constraint systems, because these two notions are dual.

5. Implementing NNC polyhedra using closed polyhedra

In the previous section, we have shown how the representation of closed polyhedra in terms of constraint and generator systems can be suitably generalized to also allow for the case of NNC polyhedra. Moreover, we have shown that the availability of these representations naturally leads to corresponding generalizations of the operations defined on the domain of polyhedra. However, in our path from the original problem toward the solution, an intermediate, very important step is still missing.

The most critical operation in the DD framework is the so-called *conversion* algorithm. This comes into play whenever one of the two possible representations is needed, but only the other one is available. Such a change of representation is typically needed as a pre-processing step in most of the discussed operations on both closed and NNC polyhedra, to ensure that the most appropriate representation is available for each of the arguments of the operation. Thus, a direct implementation of the DD method for the domain of NNC polyhedra requires the generalization of this conversion algorithm. Even though this would be a really interesting line of research, the few existing software libraries (based on the DD method) providing support for the domain of NNC polyhedra all adopt an alternative, indirect approach that, to the best of our knowledge, was originally proposed in [HPR94] and also described in [HPR97].[5] In this section we present a generalization and formal justification of this approach, which has the very important advantage of allowing an implementation of NNC polyhedra to reuse almost all the code written, debugged and optimized for the support of closed polyhedra. We will also show the exact correspondence between this "low-level implementation" and the "high-level interface" proposed in the previous section.

The basic idea is to encode each NNC polyhedron of $\mathbb{P}_n$ into a closed polyhedron of $\mathbb{CP}_{n+1}$. In the following, we denote by ϵ the variable corresponding to the $(n+1)$-st Cartesian axis of $\mathbb{R}^{n+1}$. The interpretation function $[\![\cdot]\!]\colon \mathbb{CP}_{n+1} \to \mathbb{P}_n$ maps any closed polyhedron in $\mathbb{CP}_{n+1}$ to an NNC polyhedron in $\mathbb{P}_n$; in particular, points in the closed polyhedron with a positive ϵ coordinate correspond to points in the NNC polyhedron.

[4] A *face* of a polyhedron is the intersection of the polyhedron with one of its bounding hyperplanes.
[5] According to [HPR97], the idea has to be credited to P. Raymond.

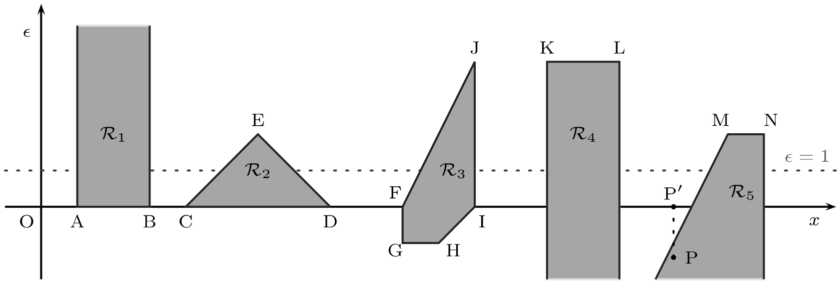

Fig. 2. Only $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ are ϵ-polyhedra

Definition 5.1 (Represented NNC polyhedron). A polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ is said to *represent* the NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ if and only if

$$\mathcal{P} = [\![\mathcal{R}]\!] \stackrel{\text{def}}{=} \left\{ \boldsymbol{v} \in \mathbb{R}^n \;\middle|\; \exists e \in \mathbb{R} \,.\, \bigl(e > 0 \land (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}\bigr) \right\}. \tag{1}$$

Note that any closed polyhedron that is included in the half-space defined by the constraint $\epsilon \leqslant 0$ actually represents the empty NNC polyhedron.

Not all the polyhedra in $\mathbb{CP}_{n+1}$ are good candidates for representing an NNC polyhedron in $\mathbb{P}_n$. The rationale driving the choice of an appropriate subclass of $\mathbb{CP}_{n+1}$ is that most of the operators defined on the domain of closed polyhedra could be used, with no more than minor modifications, to implement the corresponding operators on the represented domain $\mathbb{P}_n$ of NNC polyhedra. For instance, one would like to implement the intersection and the poly-hull of two NNC polyhedra by computing the intersection and the poly-hull of their closed representations, respectively. Under such a requirement, we will define two alternative representations for NNC polyhedra. The two classes of closed polyhedra used for these representations are instances of a more general class of closed polyhedra.

Definition 5.2 (ϵ-polyhedron). A closed polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ is said to be an *ϵ-polyhedron* if and only if

$$\exists \delta \in \mathbb{R} \,.\, \Bigl(\delta > 0 \land \mathcal{R} \subseteq \operatorname{con}\bigl(\{\epsilon \leqslant \delta\}\bigr)\Bigr); \tag{2}$$

$$\forall \boldsymbol{v} \in \mathbb{R}^n, e \in \mathbb{R} : (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R} \implies (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}. \tag{3}$$

The polyhedron $\mathcal{R}$ is said to be an ϵ-polyhedron *for* $\mathcal{P} \in \mathbb{P}_n$, denoted $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, if $\mathcal{R}$ is an ϵ-polyhedron and $\mathcal{P} = [\![\mathcal{R}]\!]$.

Condition (3) that every point in the ϵ-polyhedron $\mathcal{R}$ has a projection on the hyperplane defined by the constraint $(\epsilon = 0)$ corresponds to the following dual property concerning the constraints for $\mathcal{R}$.

Proposition 5.3 Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \subseteq \operatorname{con}\bigl(\{\epsilon \leqslant \delta\}\bigr)$, where $\delta > 0$. Then $\mathcal{R}$ is an ϵ-polyhedron if and only if

$$\mathcal{R} \subseteq \operatorname{con}\Bigl(\bigl\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr\}\Bigr) \implies \mathcal{R} \subseteq \operatorname{con}\Bigl(\bigl\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b\bigr\}\Bigr). \tag{4}$$

An intuitive reading for the dual conditions (3) and (4) will be provided at the end of this section, after showing the correspondence between ϵ-polyhedra and the high-level representation of NNC polyhedra as presented in Sect. 4.

In Fig. 2 we show several examples of polyhedra in $\mathbb{CP}_2$ (representing NNC polyhedra in $\mathbb{P}_1$), some of which happen to be ϵ-polyhedra. In particular, the semi-column polyhedron $\mathcal{R}_1$, which according to Definition 5.1 represents the closed interval $\mathcal{P}_1 = \operatorname{con}\bigl(\{1 \leqslant x \leqslant 3\}\bigr)$, is not an ϵ-polyhedron, because it is not provided with a finite upper-bound on the ϵ coordinate, therefore violating condition (2) of Definition 5.2. The triangle $\mathcal{R}_2$ is an ϵ-polyhedron for the open segment $\mathcal{P}_2 = \operatorname{con}\bigl(\{4 < x < 8\}\bigr)$. Polyhedron $\mathcal{R}_3$ is an ϵ-polyhedron for the segment $\mathcal{P}_3 = \operatorname{con}\bigl(\{10 < x \leqslant 12\}\bigr)$, which is neither closed nor open. Similarly, $\mathcal{R}_4$ is an ϵ-polyhedron for the closed segment $\mathcal{P}_4 = \operatorname{con}\bigl(\{14 \leqslant x \leqslant 16\}\bigr)$. Finally, polyhedron $\mathcal{R}_5$ represents the NNC polyhedron $\mathcal{P}_5 = \operatorname{con}\bigl(\{18 \leqslant x \leqslant 20\}\bigr)$, but it is not an ϵ-polyhedron because it violates condition (3) of Definition 5.2. For instance, even though $\mathrm{P} \in \mathcal{R}_5$, we have $\mathrm{P}' \notin \mathcal{R}_5$.

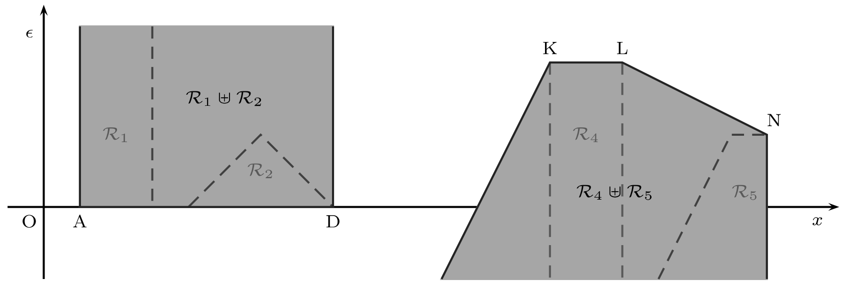

Fig. 3. $\mathcal{R}_1 \uplus \mathcal{R}_2$ (resp., $\mathcal{R}_4 \uplus \mathcal{R}_5$) does not represent the NNC polyhedron $\mathcal{P}_1 \uplus \mathcal{P}_2$ (respectively, $\mathcal{P}_4 \uplus \mathcal{P}_5$)

Figure 3, which shows the poly-hulls of some of the polyhedra in Fig. 2, provides a graphical and informal justification for the two conditions stated in Definition 5.2. Let us suppose we do not enforce condition (2) of Definition 5.2, thus admitting polyhedra such as $\mathcal{R}_1$, and consider the convex polyhedral hull $\mathcal{P}_1 \uplus \mathcal{P}_2 = \mathrm{con}\bigl(\{1 \leqslant x < 8\}\bigr)$. The poly-hull $\mathcal{R}_1 \uplus \mathcal{R}_2$ of the two encodings for $\mathcal{P}_1$ and $\mathcal{P}_2$ represents a wrong result, since $[\![\mathcal{R}_1 \uplus \mathcal{R}_2]\!] = \mathrm{con}\bigl(\{1 \leqslant x \leqslant 8\}\bigr)$. Suppose now we do not enforce condition (3) of Definition 5.2, thus allowing for polyhedra such as $\mathcal{R}_5$, and consider the poly-hull $\mathcal{P}_4 \uplus \mathcal{P}_5 = \mathrm{con}\bigl(\{14 \leqslant x \leqslant 20\}\bigr)$. Again, the computation of this poly-hull using the closed encodings of its arguments provides a wrong result, since we have $[\![\mathcal{R}_4 \uplus \mathcal{R}_5]\!] = \mathrm{con}\bigl(\{12 < x \leqslant 20\}\bigr)$.

If we are to provide an implementation-independent interface for the user, we need to be able to extract from the constraint and generator systems describing an ϵ-polyhedron, the corresponding mixed constraint system and extended generator system describing the NNC polyhedron it represents. Reasoning at the intuitive level, consider an arbitrary ϵ-polyhedron, such as $\mathcal{R}_3$ in Fig. 2. Then, it is worth noting that any facet[6] that is parallel to the ϵ axis, such as the segment [I, J], corresponding to an inequality constraint having a zero coefficient for the ϵ variable, will encode a *non-strict* inequality constraint of the represented NNC polyhedron $\mathcal{P}_3$ (in this case, the constraint $x \leqslant 12$). On the other hand, any facet such as the segment [J, F], corresponding to an inequality constraint having a negative coefficient for the ϵ variable, will encode a *strict* inequality constraint of the represented NNC polyhedron $\mathcal{P}_3$ (in this case, the constraint $x > 10$). Equivalently, we could have noted that in polyhedron $\mathcal{R}_3$ the points having a strictly positive ϵ coordinate can be chosen arbitrarily close to vertex $\mathrm{F} = (10, 0)^{\mathrm{T}}$, but all the points having value 10 for their x coordinate happen to have a non-positive ϵ coordinate. Thus, the vector $\mathrm{F}' = (10) \in \mathbb{R}^1$ represented by F is not a point of the NNC polyhedron $\mathcal{P}_3$, but it is one of its closure points. All of the above observations can be formalized as follows.

Definition 5.4 (Encoded descriptions). Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be a DD pair for a closed polyhedron. Then, if $[\![\mathcal{R}]\!] \neq \emptyset$, the *mixed constraint system encoded by* $\mathcal{C}$ is defined as $\mathrm{con\_enc}(\mathcal{C}) = \mathcal{C}_{\mathrm{s}} \cup \mathcal{C}_{\mathrm{ns}}$, where

$$\mathcal{C}_{\mathrm{s}} \stackrel{\mathrm{def}}{=} \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle > b \Bigm| \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}, \boldsymbol{a} \neq \boldsymbol{0}, s < 0 \Bigr\},$$

$$\mathcal{C}_{\mathrm{ns}} \stackrel{\mathrm{def}}{=} \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b \Bigm| \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}, \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\bigr) \notin \mathcal{C}_{\mathrm{s}} \Bigr\}.$$

If $[\![\mathcal{R}]\!] = \emptyset$, then we define $\mathrm{con\_enc}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \{x_1 > 0, -x_1 > 0\}$. Also, the *extended generator system encoded by* $\mathcal{G} = (R, P)$ is defined as $\mathrm{gen\_enc}(\mathcal{G}) = (R', P', C')$, where

$$R' \stackrel{\mathrm{def}}{=} \bigl\{ \boldsymbol{r} \bigm| (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R \bigr\},$$

$$P' \stackrel{\mathrm{def}}{=} \bigl\{ \boldsymbol{p} \bigm| (\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in P, e > 0 \bigr\},$$

$$C' \stackrel{\mathrm{def}}{=} \bigl\{ \boldsymbol{c} \bigm| (\boldsymbol{c}^{\mathrm{T}}, 0)^{\mathrm{T}} \in P, \boldsymbol{c} \notin P' \bigr\}.$$

The following proposition states the correctness of the two mappings introduced above.

[6] A face is *proper* if it is non-empty and different from the polyhedron itself. A *facet* is a face which is different from the polyhedron and not contained in other proper faces.

Proposition 5.5 Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. Then

$$[\![\mathcal{R}]\!] = \mathrm{con}\bigl(\mathrm{con\_enc}(\mathcal{C})\bigr) = \mathrm{gen}\bigl(\mathrm{gen\_enc}(\mathcal{G})\bigr). \tag{5}$$

Hence, Definition 5.4 provides a high-level, more user-friendly, interpretation of the constraint and generator systems describing an ϵ-polyhedron. Thus the condition (3) of Definition 5.2, can be interpreted as saying "*every point is also a closure point.*" Similarly, condition (4) stated in Proposition 5.3 may be interpreted as "*every valid strict inequality is also a valid non-strict inequality.*" Even though these two assertions are trivially true in the encoded domain $\mathbb{P}_n$, the corresponding conditions on the encoding domain $\mathbb{CP}_{n+1}$ are essential if we are to avoid the problems such as those illustrated in Fig. 3 by the computation of the poly-hull $\mathcal{R}_4 \uplus \mathcal{R}_5$.

5.1. Constraint- and generator-biased representations

We now consider two special subclasses of the class of ϵ-polyhedra. The first of these requires the value zero to be a lower bound for the ϵ dimension.

Definition 5.6 (C-ϵ-polyhedron). An ϵ-polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ is said to be *constraint-biased* and called a *C-ϵ-polyhedron* if and only if $\mathcal{R} \subseteq \mathrm{con}\bigl(\{\epsilon \geqslant 0\}\bigr)$. We write $\mathcal{R} \Rrightarrow_C \mathcal{P}$ if $\mathcal{R}$ is a C-ϵ-polyhedron and $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$.

The set of constraint-biased ϵ-polyhedra corresponds, essentially, to the class of polyhedra originally proposed in [HPR94, HPR97]. (This is also the same class that was considered in [BRZH02a], where these polyhedra were called ϵ-representations.)

A C-ϵ-polyhedron for an NNC polyhedron $\mathcal{P}$ can be easily constructed starting from either a mixed constraint system or an extended generator system for $\mathcal{P}$.

Definition 5.7 ('con_repr$_C$' and 'gen_repr$_C$'). Let $\mathcal{P} \in \mathbb{P}_n$ be an NNC polyhedron such that $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. The *constraint-biased representation of* $\mathcal{C}$ is the constraint system $\mathrm{con\_repr}_C(\mathcal{C})$ on the vector space $\mathbb{R}^{n+1}$ where

$$\begin{aligned} \mathrm{con\_repr}_C(\mathcal{C}) &\stackrel{\mathrm{def}}{=} \bigl\{0 \leqslant \epsilon \leqslant 1\bigr\} \\ &\quad\cup \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x}\rangle - 1 \cdot \epsilon \geqslant b \Bigm| \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle > b\bigr) \in \mathcal{C} \Bigr\} \\ &\quad\cup \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b \Bigm| \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \geqslant b\bigr) \in \mathcal{C} \Bigr\}. \end{aligned}$$

The *constraint-biased representation of* $\mathcal{G} = (R, P, C)$ is the generator system $\mathrm{gen\_repr}_C(\mathcal{G}) = (R', P')$ on the vector space $\mathbb{R}^{n+1}$ where

$$\begin{aligned} R' &\stackrel{\mathrm{def}}{=} \bigl\{ (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \bigm| \boldsymbol{r} \in R \bigr\}, \\ P' &\stackrel{\mathrm{def}}{=} \bigl\{ (\boldsymbol{p}^{\mathrm{T}}, 1)^{\mathrm{T}} \bigm| \boldsymbol{p} \in P \bigr\} \cup \bigl\{ (\boldsymbol{q}^{\mathrm{T}}, 0)^{\mathrm{T}} \bigm| \boldsymbol{q} \in P \cup C \bigr\}. \end{aligned}$$

Observe that, in the mapping defined by the representation function 'gen_repr$_C$' and using the notation in Definition 5.7, each point in P corresponds to two distinct points in P', having a positive and a zero ϵ coordinate, respectively. This ensures that condition (3) of Definition 5.2 is met. In general, the above encodings require a constant number of additional constraints versus a linear number of additional generators: this is the reason why ϵ-polyhedra in this subclass are called "constraint-biased."

The following proposition states the correctness of the mappings introduced in Definition 5.7.

Proposition 5.8 Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P} \in \mathbb{P}_n$. Then $\mathrm{con}\bigl(\mathrm{con\_repr}_C(\mathcal{C})\bigr) \Rrightarrow_C \mathcal{P}$ and $\mathrm{gen}\bigl(\mathrm{gen\_repr}_C(\mathcal{G})\bigr) \Rrightarrow_C \mathcal{P}$.

The second special subclass of ϵ-polyhedra requires that $-\boldsymbol{e}_\epsilon \stackrel{\mathrm{def}}{=} (\boldsymbol{0}^{\mathrm{T}}, -1)^{\mathrm{T}}$ is a ray of all the non-empty ϵ-polyhedra, so that there is no lower bound for the ϵ dimension.

Definition 5.9 (G-ϵ-polyhedron). An ϵ-polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ is said to be *generator-biased* and called a *G-ϵ-polyhedron* if and only if $\mathcal{R} = \varnothing$ or $-\boldsymbol{e}_\epsilon \in \mathrm{rays}(\mathcal{R})$. We write $\mathcal{R} \Rrightarrow_G \mathcal{P}$ if $\mathcal{R}$ is a G-ϵ-polyhedron and $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$.

As for the constraint-biased case, generator-biased ϵ-polyhedra can also be used for representing any NNC polyhedron. In particular, a G-ϵ-polyhedron for an NNC polyhedron $\mathcal{P}$ may be constructed directly from any mixed constraint system or extended generator system describing $\mathcal{P}$.

Definition 5.10 ('$\mathrm{con\_repr}_G$' and '$\mathrm{gen\_repr}_G$'). Let $\mathcal{P} \in \mathbb{P}_n$ be an NNC polyhedron such that $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. The *generator-biased representation of* $\mathcal{C}$ is the constraint system $\mathrm{con\_repr}_G(\mathcal{C})$ on the vector space $\mathbb{R}^{n+1}$ where

$$\begin{aligned}
\mathrm{con_repr}_G(\mathcal{C}) \stackrel{\text{def}}{=} \{\epsilon \leqslant 1\} & \\
\cup \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle - 1 \cdot \epsilon \geqslant b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\big) \in \mathcal{C} \Big\} & \\
\cup \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geqslant b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\big) \in \mathcal{C} \Big\} & \\
\cup \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geqslant b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\big) \in \mathcal{C} \Big\}. &
\end{aligned}$$

The *generator-biased representation of* $\mathcal{G} = (R, P, C)$ is the generator system $\mathrm{gen\_repr}_G(\mathcal{G}) = (R', P')$ on the vector space $\mathbb{R}^{n+1}$ where

$$\begin{aligned}
R' &\stackrel{\text{def}}{=} \{-\boldsymbol{e}_\epsilon\} \cup \big\{ (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\big|\; \boldsymbol{r} \in R \big\}, \\
P' &\stackrel{\text{def}}{=} \big\{ (\boldsymbol{p}^{\mathrm{T}}, 1)^{\mathrm{T}} \;\big|\; \boldsymbol{p} \in P \big\} \cup \big\{ (\boldsymbol{q}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\big|\; \boldsymbol{q} \in C \big\}.
\end{aligned}$$

It can be seen that, for each strict inequality contained in $\mathcal{C}$, the representation function '$\mathrm{con\_repr}_G$' adds both the strict and the non-strict inequality encodings. This is similar to what is done for points in Definition 5.7 and, by virtue of Proposition 5.3, ensures that condition (3) of Definition 5.2 is met. In contrast, for each point in the generator system, the function '$\mathrm{gen\_repr}_G$' does not add the corresponding closure point. In fact, these closure points are not needed, because they can be generated by combining the corresponding point with the ray $-\boldsymbol{e}_\epsilon$, which is always added. Since the encodings for ϵ-polyhedra in this subclass require a linear number of additional constraints versus a constant number of additional generators, they are called "generator-biased."

The following proposition states the correctness of the mappings introduced in Definition 5.10.

Proposition 5.11 Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P} \in \mathbb{P}_n$. Then $\mathrm{con}\big(\mathrm{con\_repr}_G(\mathcal{C})\big) \Rrightarrow_G \mathcal{P}$ and $\mathrm{gen}\big(\mathrm{gen\_repr}_G(\mathcal{G})\big) \Rrightarrow_G \mathcal{P}$.

For both the constraint-biased and generator-biased representations, it should be noted that the choice of the value -1 for the ϵ coefficients in the constraints representing strict inequalities is arbitrary: any other negative value will do. Also, the side constraint $\epsilon \leqslant 1$ could be replaced, as stated in condition (2) of Definition 5.2, by any other constraint $\epsilon \leqslant \delta$ such that $\delta > 0$. Dually, the choice of the value 1 for the ϵ coordinate of the points of P' encoding the points of P could be replaced by any other positive value.

Returning to Fig. 2, it can be observed that $\mathcal{R}_2$ is a constraint-biased ϵ-polyhedron, $\mathcal{R}_4$ is a generator-biased ϵ-polyhedron, whereas the ϵ-polyhedron $\mathcal{R}_3$ is neither constraint-biased nor generator-biased. By comparing $\mathcal{R}_3$ with $\mathcal{R}_2$ and $\mathcal{R}_4$ it can be seen that those ϵ-polyhedra that are not members of one of the two subclasses can require *both* a linear number of additional constraints and a linear number of additional generators (with respect to the original NNC descriptions), resulting in a significant waste of both memory space and computation time.

By suitably combining the previous definitions and formal results, we are now able to systematically convert a mixed constraint system $\mathcal{C}$ for the NNC polyhedron $\mathcal{P} = \mathrm{con}(\mathcal{C}) \in \mathbb{P}_n$ into a corresponding extended generator system $\mathcal{G}$ such that $\mathrm{gen}(\mathcal{G}) = \mathcal{P}$; or, vice versa, we can start from any extended generator system for $\mathcal{P}$ to obtain a corresponding mixed constraint system.

The first of these conversions is obtained as follows, where we assume that the constraint-biased representation is adopted. We first apply Definition 5.7 to obtain the constraint system representation of a C-ϵ-polyhedron for $\mathcal{P}$, i.e., we compute $\mathcal{C}' = \mathrm{con\_repr}_C(\mathcal{C})$; then, by letting $\mathcal{R} = \mathrm{con}(\mathcal{C}') \in \mathbb{CP}_{n+1}$, we apply to $\mathcal{C}'$ the usual conversion algorithm for closed polyhedra to obtain a (standard) generator system $\mathcal{G}'$ such that $\mathrm{gen}(\mathcal{G}') = \mathcal{R}$; finally, we apply Definition 5.4 to extract from $\mathcal{G}'$ the extended generator system $\mathcal{G} = \mathrm{gen\_enc}(\mathcal{G}')$. By virtue of Propositions 5.5 and 5.8, we obtain $\mathrm{gen}(\mathcal{G}) = \mathcal{P}$. The dual conversion can be obtained similarly. By combining the above conversion procedure with the specifications provided in Sect. 4.2, we obtain a complete implementation of all the operations defined on the domain of NNC polyhedra.

5.2. Operations on ϵ-polyhedra

It should be noted that the encoding of operations' arguments from $\mathbb{P}_n$ into corresponding arguments of $\mathbb{CP}_{n+1}$ and the decoding of the corresponding results are only needed when performing input-output operations, to provide the end-user with the high-level view presented in Sect. 4. In all the other cases (and, in particular, for all the intermediate results obtained during the computation of a sequence of operations) these translations can be easily and efficiently filtered away. Namely, the next proposition shows that most of the operators defined on the domain of NNC polyhedra $\mathbb{P}_n$ can be easily mapped into the corresponding operators on the class of ϵ-polyhedra defined on $\mathbb{CP}_{n+1}$.

Proposition 5.12 Letting $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$, suppose that $\mathcal{R} \Rrightarrow_Y \mathcal{P}$, $\mathcal{R}_1 \Rrightarrow_Y \mathcal{P}_1$ and $\mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_2$. Then

1. $\mathcal{R}_1 \cap \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \cap \mathcal{P}_2$;
2. $(\mathcal{P}_1 \neq \varnothing \wedge \mathcal{P}_2 \neq \varnothing) \implies (\mathcal{R}_1 \uplus \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \uplus \mathcal{P}_2)$;
3. Let $f \stackrel{\text{def}}{=} \lambda \boldsymbol{x} \in \mathbb{R}^n . A\boldsymbol{x} + \boldsymbol{b}$ be any affine transformation defined on $\mathbb{P}_n$; then $g(\mathcal{R}) \Rrightarrow_Y f(\mathcal{P})$, where

$$g \stackrel{\text{def}}{=} \lambda \begin{pmatrix} \boldsymbol{x} \\ \epsilon \end{pmatrix} \in \mathbb{R}^{n+1} . \begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \epsilon \end{pmatrix} + \begin{pmatrix} \boldsymbol{b} \\ 0 \end{pmatrix}$$

is the corresponding affine transformation on $\mathbb{CP}_{n+1}$.

Hence, operations such as the intersection of NNC polyhedra and the application of affine transformations can be safely performed on any ϵ-polyhedra for the arguments; the same is true for the poly-hull operation, provided neither of the arguments is empty. Moreover, both the constraint-biased and the generator-biased subclasses are closed under the application of these operators.

6. The issue of minimization

With the ϵ dimension approach proposed in [HPR94], no matter if constraint- or generator-biased, each NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ may be represented by different (actually, an infinite number of) closed polyhedra in $\mathbb{CP}_{n+1}$. In the previous section we have shown that all of these possible representations are equally good for computing many operations required by applications such as static analysis. However, the choice of a particular ϵ-polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ for representing an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ affects the encoded high-level description for $\mathcal{P}$. In particular, the computation of a minimal form for $\mathcal{R}$ is not enough to ensure that the encoded NNC description for $\mathcal{P}$ is in minimal form too. Namely, letting $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$, even though the DD pair $(\mathcal{C}, \mathcal{G})$ is in minimal form, it may well happen that $\mathrm{con\_enc}(\mathcal{C})$ or $\mathrm{gen\_enc}(\mathcal{G})$ are a mixed constraint system and an extended generator system containing redundant constraints and generators, respectively. The following example illustrates this point.

Consider the two NNC polyhedra $\mathcal{P}_1 = \mathrm{con}(\mathcal{C}_1)$ and $\mathcal{P}_2 = \mathrm{con}(\mathcal{C}_2)$ of $\mathbb{P}_1$, where

$$\mathcal{C}_1 = \{0 < x < 2\}, \quad \mathcal{C}_2 = \{2 < x < 3\}.$$

These polyhedra can be represented by the C-ϵ-polyhedra $\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{CP}_2$ such that

$$\mathcal{R}_1 = \mathrm{con}\bigl(\mathrm{con\_repr}_C(\mathcal{C}_1)\bigr) = \left\{ (x, \epsilon)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} 0 \leqslant \epsilon \leqslant 1 \\ x - \epsilon \geqslant 0 \\ -x - \epsilon \geqslant -2 \end{array} \right\},$$

$$\mathcal{R}_2 = \mathrm{con}\bigl(\mathrm{con\_repr}_C(\mathcal{C}_2)\bigr) = \left\{ (x, \epsilon)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} 0 \leqslant \epsilon \leqslant 1 \\ x - \epsilon \geqslant 2 \\ -x - \epsilon \geqslant -3 \end{array} \right\}.$$

In Fig. 4, these two polyhedra correspond to the triangles having vertices O, A, B and A, C, D, respectively. Note that, in both cases, the ϵ upper bound constraint $\epsilon \leqslant 1$ happens to be redundant.

Suppose now that the user wants to compute the poly-hull of the two original NNC polyhedra, therefore obtaining the NNC polyhedron

$$\mathcal{P}_3 = \mathcal{P}_1 \uplus \mathcal{P}_2 = \mathrm{con}\bigl(\{0 < x < 3\}\bigr).$$

At the representation level, as shown in Fig. 4, $\mathcal{P}_3$ will be described by the ϵ-polyhedron $\mathcal{R}_3$ generated by the four vertices O, C, D, and B, whereas point A is identified as redundant. Formally,

$$\mathcal{R}_3 = \mathcal{R}_1 \uplus \mathcal{R}_2 = \left\{ (x, \epsilon)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; \begin{array}{r} \epsilon \geqslant 0 \\ x - \epsilon \geqslant 0 \\ x + \epsilon \leqslant 3 \\ x + 3\epsilon \leqslant 4 \end{array} \right\}.$$

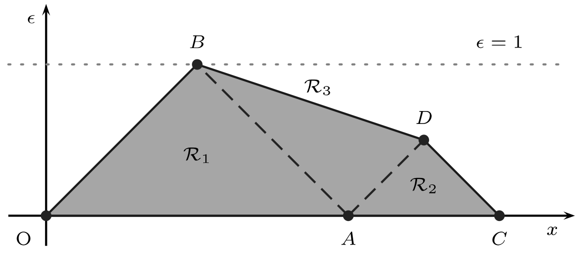

Fig. 4. The ϵ-representations of $\mathcal{P}_1$ and $\mathcal{P}_2$ and of their poly-hull

Note that the chosen constraint system

$$\mathcal{C}_3 = \{\epsilon \geqslant 0, x - \epsilon \geqslant 0, x + \epsilon \leqslant 3, x + 3\epsilon \leqslant 4\}$$

describing polyhedron $\mathcal{R}_3$ is in minimal form; in particular, the last non-strict inequality constraint $x + 3\epsilon \leqslant 4$, which corresponds to the segment $[B, D]$ in Fig. 4, is not redundant as far as the ϵ-polyhedron $\mathcal{R}_3$ is concerned. However, this non-strict inequality is actually encoding the strict inequality constraint $x < 4$, which is clearly redundant for the encoded polyhedron $\mathcal{P}_3 = [\![\mathcal{R}_3]\!]$. Formally, the mixed constraint system encoded by $\mathcal{C}_3$ is

$$\mathcal{C}_3' = \text{con\_enc}(\mathcal{C}_3) = \{0 < x < 3, x < 4\},$$

which, according to Definition 3.3, is not in minimal form. In this case, we say that $x + 3\epsilon \leqslant 4$ is an *ϵ-redundant constraint in $\mathcal{C}_3$*.

The same problem as above can be observed for a generator system for the ϵ-polyhedron $\mathcal{R}_3$. Namely, we have $\mathcal{R}_3 = \text{gen}(\mathcal{G}_3)$, where $\mathcal{G}_3 = (\varnothing, P_3)$ and

$$P_3 = \{(0, 0)^{\mathrm{T}}, (3, 0)^{\mathrm{T}}, (2.5, 0.5)^{\mathrm{T}}, (1, 1)^{\mathrm{T}}\}.$$

Even though $\mathcal{G}_3$ is a generator system in minimal form, by Definition 5.4 we obtain

$$\mathcal{G}_3' = \text{gen\_enc}(\mathcal{G}_3) = (\varnothing, P_3', C_3'),$$

where $P_3' = \{1, 2.5\}$ and $C_3' = \{0, 3\}$. Since the point $1 \in P_3'$ (respectively, $2.5 \in P_3'$) is redundant in $\mathcal{G}_3'$, the encoded generator system is not in minimal form. In this case, we say that $(2.5, 0.5)^{\mathrm{T}} \in P_3$ (respectively, $(1, 1)^{\mathrm{T}} \in P_3$) is an *$\epsilon$-redundant generator in $\mathcal{G}_3$*.

The problem outlined above is even more critical for higher dimension vector spaces: it is straightforward to devise examples where more than half of the constraints or generators in any minimized description for an ϵ-polyhedron happen to be ϵ-redundant. Even when disregarding these pathological cases, this form of redundancy can have a serious negative impact on the efficiency of most of the operations computed on the ϵ-polyhedron; in particular, this is true when converting between constraint and generator systems. Moreover, it must be stressed that efficiency degradation is not the only issue. It turns out that the unnoticed presence of ϵ-redundant constraints may also cause headaches to the users of a software library computing on the domain of NNC polyhedra (and adopting the ϵ dimension approach). As an example, suppose one wants to know if a given NNC polyhedron is not topologically closed. Ordinary users of the software library (i.e., all the users but the experts) may be tempted to implement such a test by checking whether the constraint system in minimal form contains any strict inequality constraint. If the considered software library merely computes a minimal description for the ϵ-polyhedron representing the NNC polyhedron, then such an approach would be unsound, as illustrated by the scenario proposed in Fig. 5.

Here, the NNC polyhedron $\mathcal{P}_1$ defined previously is intersected with the NNC polyhedron

$$\mathcal{P}_4 = \text{con}\big(\{5 \leqslant 4x \leqslant 7\}\big) \in \mathbb{P}_1,$$

whose representation $\mathcal{R}_4 = \text{con}\big(\text{con\_repr}_C(\mathcal{P}_4)\big) \in \mathbb{CP}_2$ is the rectangle having vertices E, F, G and H. The resulting trapezium (having vertices E, F, I and J) is another ϵ-representation for the NNC polyhedron $\mathcal{P}_4$, which is clearly topologically closed. However, any constraint system describing the trapezium will also encode the (redundant) strict inequality constraint $x < 2$, corresponding to the closed segment $[I, J]$.

It is therefore meaningful to address the problem of providing a minimization operator that, starting from an arbitrary description of an ϵ-polyhedron $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, is able to compute a description of a (possibly different)

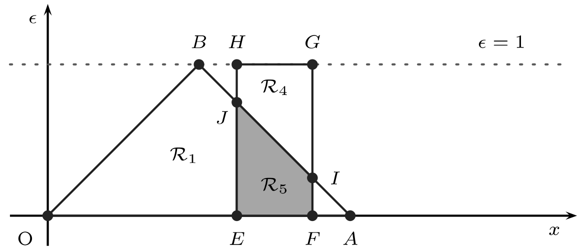

Fig. 5. The "minimized" trapezium $EFIJ$, obtained by intersecting $\mathcal{R}_1$ and $\mathcal{R}_4$ and still representing the topologically closed NNC polyhedron $\mathcal{P}_4$, also encodes the strict inequality $x > 0$

ϵ-polyhedron $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$ that encodes a non-redundant high-level description for the NNC polyhedron $\mathcal{P}$. The result of this computation is said to be a description in *ϵ-minimal form*.

Definition 6.1 (ϵ-minimal forms). Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$ and let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ be a DD pair for $\mathcal{R}$. Then

- $\mathcal{C}$ is in *ϵ-minimal form* if and only if $\mathrm{con\_enc}(\mathcal{C})$ is in minimal form;
- $\mathcal{G}$ is in *ϵ-minimal form* if and only if $\mathrm{gen\_enc}(\mathcal{G})$ is in minimal form.

The computation of ϵ-minimal forms will be based on the identification of all the ϵ-redundant constraints and generators. These will be either removed or replaced by other constraints and generators that are not ϵ-redundant in the resulting description.

While still reasoning at the informal level, it is worth stressing that this notion of redundancy has a double nature. In the examples just presented, ϵ-redundancy shows its *semantic* nature, meaning that the redundant information is encoded in the ϵ-polyhedron itself, rather than in one of its constraint or generator descriptions. Namely, all constraint systems and generator systems describing $\mathcal{R}_3$ necessarily contain the ϵ-redundant constraint and generators identified above. As a consequence, this kind of redundancy can only be eliminated by choosing a different ϵ-polyhedron $\mathcal{R}'_3$ representing the same NNC polyhedron $\mathcal{P}_3$.

There are also examples where ϵ-redundancy only has a *syntactic* nature, meaning that the redundancy can be eliminated by choosing a particular constraint or generator description for the same ϵ-polyhedron. Clearly, this happens when a description contains some constraints or generators that are redundant (in the classical sense) for the ϵ-polyhedron itself. However, this may also happen when a description is already in minimal form. For instance, let $\mathcal{R} = \mathrm{con}(\mathcal{C}) \in \mathbb{CP}_2$ be the ϵ-polyhedron defined by the constraint system

$$\mathcal{C} = \{0 \leqslant x \leqslant 0, \epsilon \geqslant 0, x + \epsilon \leqslant 1\}.$$

Even though $\mathcal{C}$ is in minimal form, it is not in ϵ-minimal form, because the mixed constraint system $\mathrm{con\_enc}(\mathcal{C}) = \{0 \leqslant x \leqslant 0, x < 1\}$ contains the redundant strict inequality constraint $x < 1$. However, the ϵ-polyhedron $\mathcal{R}$ can also be described by the constraint system

$$\mathcal{C}' = \{0 \leqslant x \leqslant 0, 0 \leqslant \epsilon \leqslant 1\},$$

where the ϵ-redundant constraint $x + \epsilon \leqslant 1$ has been replaced by the ϵ upper bound constraint $\epsilon \leqslant 1$, obtaining a description in ϵ-minimal form for the same ϵ-polyhedron $\mathcal{R}$. It will be shown that this syntactic kind of ϵ-redundancy cannot occur if we consider minimal descriptions in orthogonal form. As a matter of fact, for the particular example considered above, the constraint system $\mathcal{C}'$ is in orthogonal form, whereas this property does not hold for $\mathcal{C}$.

6.1. The computation of ϵ-minimal forms

The practicality of available conversion procedures for topologically closed polyhedra, such as the extension by Le Verge [Le 92] of Chernikova's algorithms [Che64, Che65, Che68], is mainly obtained thanks to the efficient detection (and removal) of redundancies in the computed representations. Redundant elements are usually identified by checking suitable *saturation conditions*.

We say that a point $\boldsymbol{p}$ (resp., a ray $\boldsymbol{r}$) *saturates* a constraint $\beta = \big(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \geqslant b\big)$ if and only if $\langle \boldsymbol{a}, \boldsymbol{p}\rangle = b$ (resp., $\langle \boldsymbol{a}, \boldsymbol{r}\rangle = 0$). For any point $\boldsymbol{p}$ and constraint system $\mathcal{C}$, we define

$$\mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}) \stackrel{\mathrm{def}}{=} \{\, \beta \in \mathcal{C} \mid \boldsymbol{p} \text{ saturates } \beta \,\};$$

and, for any constraint β and generator system $\mathcal{G} = (R, P)$, we define

$$\mathrm{sat\_gen}(\beta, \mathcal{G}) \stackrel{\mathrm{def}}{=} \big(\{\, \boldsymbol{r} \in R \mid \boldsymbol{r} \text{ saturates } \beta \,\}, \{\, \boldsymbol{p} \in P \mid \boldsymbol{p} \text{ saturates } \beta \,\}\big).$$

Intuitively, if $(\mathcal{C}, \mathcal{G})$ is a DD pair for a closed polyhedron and $\{\beta, \beta'\} \subseteq \mathcal{C}$, where $\beta \neq \beta'$, then the saturation condition $\mathrm{sat\_gen}(\beta, \mathcal{G}) \sqsubseteq \mathrm{sat\_gen}(\beta', \mathcal{G})$ implies that the constraint β' can be used in place of β without affecting the represented polyhedron: in other words, β is redundant in $\mathcal{C}$. By duality, other saturation conditions allow for the identification of redundant generators in $\mathcal{G}$.

In the case of an ϵ-polyhedron encoding an NNC polyhedron, the efficient detection of ϵ-redundant constraints and generators can be based on the checking of similar saturation conditions. The following notation is needed for their formal definition.

Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be a DD pair for an ϵ-polyhedron. The set of *strict* and *non-strict inequality encodings* $\mathcal{C}_{>}$ and $\mathcal{C}_{\geqslant}$ of the constraint system $\mathcal{C}$ are defined as

$$\mathcal{C}_{>} \stackrel{\mathrm{def}}{=} \Big\{\, \big(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\big) \in \mathcal{C} \;\Big|\; \boldsymbol{a} \neq \boldsymbol{0}, s < 0 \,\Big\};$$

$$\mathcal{C}_{\geqslant} \stackrel{\mathrm{def}}{=} \Big\{\, \big(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\big) \in \mathcal{C} \;\Big|\; \boldsymbol{a} \neq \boldsymbol{0}, s = 0 \,\Big\}.$$

The sets of *ray encodings* $\mathcal{G}_R \subseteq R$, *point encodings* $\mathcal{G}_P \subseteq P$ and *closure point encodings* $\mathcal{G}_C \subseteq P$ of the generator system $\mathcal{G} = (R, P)$ are defined as

$$\mathcal{G}_R \stackrel{\mathrm{def}}{=} \big\{\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in R \;\big|\; e = 0 \,\big\};$$

$$\mathcal{G}_P \stackrel{\mathrm{def}}{=} \big\{\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \;\big|\; e > 0 \,\big\};$$

$$\mathcal{G}_C \stackrel{\mathrm{def}}{=} \big\{\, (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \;\big|\; e = 0 \,\big\}.$$

We are now ready to provide the formal definition of *strong ϵ-redundancy*. As the name suggests, this notion gives *sufficient* conditions for the identification of an ϵ-redundant constraint or generator in a description of an ϵ-polyhedron.

Definition 6.2 (Strong ϵ-redundancy). Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$. A constraint β is *strongly ϵ-redundant* in $\mathcal{C}$ if $\beta \in \mathcal{C}_{>}$ and at least one of the following conditions holds:

$$\mathrm{sat\_gen}\big(\beta, (\mathcal{G}_R, \mathcal{G}_C)\big) \sqsubseteq (\mathcal{G}_R, \varnothing);$$

$$\exists \beta' \in \mathcal{C}_{>} \setminus \{\beta\} \,.\, \mathrm{sat\_gen}\big(\beta, (\mathcal{G}_R, \mathcal{G}_C)\big) \sqsubseteq \mathrm{sat\_gen}(\beta', \mathcal{G}).$$

A generator $\boldsymbol{p}$ is *strongly ϵ-redundant* in $\mathcal{G}$ if $\boldsymbol{p} \in \mathcal{G}_P$ and

$$\exists \boldsymbol{p}' \in \mathcal{G}_P \setminus \{\boldsymbol{p}\} \,.\, \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant}) \subseteq \mathrm{sat\_con}(\boldsymbol{p}', \mathcal{C}).$$

An intuitive reading of the above conditions can be obtained by viewing them under the perspective of the encoded NNC polyhedron: namely, we disregard the modulus of the slack variable ϵ and use just its sign in order to distinguish (non-) strict constraints and (closure) points. Therefore, consider a constraint β encoding a valid strict inequality constraint γ for the NNC polyhedron $\mathcal{P}$. First note that, since the points of $\mathcal{P}$ can not saturate a strict constraint such as γ, only the rays and closure points need be considered. Thus, if γ is saturated by none of the closure points of $\mathcal{P}$ then it is completely useless in the representation of $\mathcal{P}$; otherwise, if the constraint representation contains another strict inequality (γ' encoded by β') that is saturated by all the rays and closure points saturating γ, then the constraint γ is again useless, because its role can be played by the other constraint without affecting the represented polyhedron. Hence, in both cases, the encoding β is ϵ-redundant in the constraint system representation. Similarly, let $\boldsymbol{p}$ encode the generating point $\boldsymbol{v}$ for the NNC polyhedron $\mathcal{P}$. For the same reason as in the previous case, we can disregard (the encodings of) the strict inequality constraints. Hence, if the generator representation contains another point of $\mathcal{P}$ ($\boldsymbol{v}'$ encoded by $\boldsymbol{p}'$) that saturates all the non-strict inequalities saturated by $\boldsymbol{v}$, it means that $\boldsymbol{v}$ is useless for representing $\mathcal{P}$, so that $\boldsymbol{p}$ is ϵ-redundant in the generator system representation.

The next proposition formalizes the above reasoning by showing that strongly ϵ-redundant constraints and generators can be safely removed from (or replaced in) the descriptions of an ϵ-polyhedron without affecting the represented NNC polyhedron. If the ϵ-polyhedron was constraint- or generator-biased, it remains constraint- or generator-biased, respectively.

Proposition 6.3 Let $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$. Assume $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ are such that $\mathcal{R} \Rrightarrow_Y \mathcal{P} \neq \varnothing$. Suppose that β is a strongly ϵ-redundant constraint in $\mathcal{C}$ and $\boldsymbol{p}$ is a strongly ϵ-redundant generator in $\mathcal{G} = (R, P)$. Then the following hold:

$$\mathrm{con}\Bigl(\bigl(\mathcal{C} \setminus \{\beta\}\bigr) \cup \{\epsilon \leqslant 1\}\Bigr) \Rrightarrow_Y \mathcal{P}; \tag{6}$$

$$\mathrm{gen}\Bigl(\bigl(R, P \setminus \{\boldsymbol{p}\}\bigr)\Bigr) \Rrightarrow_Y \mathcal{P}. \tag{7}$$

According to Definition 6.2, only the strict inequality encodings and the point encodings of an ϵ-polyhedron can be identified as strongly ϵ-redundant constraints and generators, respectively. The following result shows that such a restriction is inconsequential; intuitively, all the non-strict inequality encodings, the ray encodings and the closure point encodings can only give rise to the syntactic kind of ϵ-redundancy.

Proposition 6.4 Let $\mathcal{R}, \mathcal{R}' \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ and $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$; let also $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$. Then

$$\forall \beta \in \mathcal{C}_{\geqslant} : \mathcal{R}' \subseteq \mathrm{con}\bigl(\{\beta\}\bigr); \tag{8}$$

$$\forall \boldsymbol{r} \in \mathcal{G}_R : \boldsymbol{r} \in \mathrm{rays}(\mathcal{R}'); \tag{9}$$

$$\forall \boldsymbol{p} \in \mathcal{G}_C : \boldsymbol{p} \in \mathcal{R}'. \tag{10}$$

When all the strongly ϵ-redundant constraints or generators have been filtered away, the descriptions of an ϵ-polyhedron no longer contain ϵ-redundancies of the semantic kind. Thus descriptions in ϵ-minimal form can be obtained by the computation of minimal orthogonal forms.

Proposition 6.5 Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ and let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ be a DD pair in minimal orthogonal form. Then the following hold:

1. If $\mathcal{C}$ contains no strongly ϵ-redundant constraint, then it is in ϵ-minimal form;
2. If $\mathcal{G}$ contains no strongly ϵ-redundant generator, then it is in ϵ-minimal form.

Note that, in general, removing the strongly ϵ-redundant elements from a constraint (respectively, generator) system describing an ϵ-polyhedron does not guarantee that the dual generator (respectively, constraint) system is also in ϵ-minimal form. Nonetheless, the computation of the ϵ-minimal form for one of the two descriptions is going to automatically remove most of the ϵ-redundancies in the dual description, so that even though one element of the DD pair may not be in ϵ-minimal form, the effect of any residual redundancy on the efficiency of the computations is likely to be small. Regarding the correctness of the computation, it should be stressed that a DD pair having both descriptions in ϵ-minimal form is rarely needed, if ever. In fact, for those NNC polyhedral operations that do use both descriptions, usually only one is actually required to be in ϵ-minimal form so that being able to ϵ-minimize the description of interest is enough; for instance, this is the case in the computation of the standard widening operator [CH78] (for closed polyhedra) adapted for NNC polyhedra. In the rare cases when minimality is required for both descriptions, (for instance, when adapting the widening operator defined in [BHRZ03, BHRZ05] to the domain of NNC polyhedra), it turns out that the descriptions do not need to be dual at the implementation level: it is enough that they are dual with respect to the encoded NNC polyhedron. Thus, one can find, independently, ϵ-minimal forms for the constraint and generator descriptions and still obtain correct results. To summarize, having DD pairs in ϵ-minimal form is not essential for correctness and not a major issue with respect to efficiency; on the other hand, it would be really interesting to find a procedure computing the ϵ-minimal form of both descriptions at the same time.

As an example, we now compute the ϵ-minimal forms for the polyhedron $\mathcal{R}_3$ represented in Fig. 4. Let us first consider the constraint system. The two strict inequality encodings $x - \epsilon \geqslant 0$ and $-x - \epsilon \geqslant -3$, which correspond to segments $[O, B]$ and $[C, D]$, are not strongly ϵ-redundant, because they are saturated by the closure point encodings O and C, respectively. In contrast, the constraint $x - 3\epsilon \geqslant 4$, corresponding to segment $[B, D]$, is identified as strongly ϵ-redundant (no closure point encoding saturates it) and can be replaced by the ϵ upper bound constraint $\epsilon \leqslant 1$. The resulting constraint system, which is in ϵ-minimal form, defines the trapezium

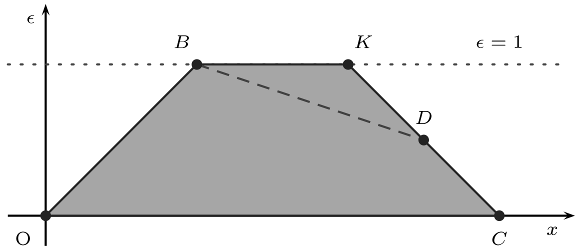

Fig. 6. The trapezium $OCKB$ is an ϵ-representations for $\mathcal{P}_3$, obtained by applying the ϵ-minimization process to the constraint system describing $\mathcal{R}_3$

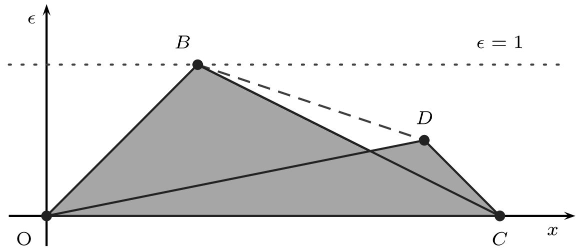

Fig. 7. The triangles OCB and OCD are other two different ϵ-representations for $\mathcal{P}_3$, which can be obtained by applying the ϵ-minimization process to the generator system describing $\mathcal{R}_3$

of vertices O, C, K, and B represented in Fig. 6. Note that the generator system for this trapezium is not in ϵ-minimal form. It is worth noting that, after removing an ϵ-redundant constraint, the addition of the ϵ upper bound constraint $\epsilon \leqslant 1$ is in general required to obtain another ϵ-representation. For instance, this happens when computing the ϵ-minimal form of the constraint system describing the trapezium $EFIJ$ of Fig. 5: the removal of the constraint corresponding to segment $[I, J]$ would yield a strip which is unbounded from above, so that it would not satisfy condition 2 of Definition 5.2; the addition of the ϵ upper bound constraint results in the rectangle $EFGH$ (i.e., the ϵ-representation $\mathcal{R}_4$ of $\mathcal{P}_4$).

Starting again from polyhedron $\mathcal{R}_3$, let us now apply the ϵ-minimization process to its generator system, which is made up of the four points O, C, D, and B. It is easy to observe that each one of the two point encodings is made strongly ϵ-redundant by the other one (they both saturate the empty set of non-strict inequality encodings); as a consequence, one of them can be removed, obtaining either one of the triangles OCB and OCD represented in Fig. 7, whose corresponding generator systems are both in ϵ-minimal form. In this particular case, the dual description corresponding to any of the two generator systems happens to be in ϵ-minimal form; however, as already observed, in general such a property does not hold.

7. Implementation and evaluation

All the ideas presented in this paper have been implemented and incorporated into the *Parma Polyhedra Library* (PPL, http://www.cs.unipr.it/ppl/). The PPL is a collaborative project started in January 2001 at the Department of Mathematics of the University of Parma and it aims at becoming a truly professional library for the handling of numeric approximations targeted at abstract interpretation and computer-aided verification. In particular, the library implements both the abstract domain of topologically closed convex polyhedra and the abstract domain of not necessarily closed convex polyhedra. In both cases, the coefficients of constraints and generators are expressed by using unbounded precision rational numbers.[7] A comparison of the PPL with other freely available polyhedra libraries shows that the core implementation for closed polyhedra is very efficient and compares favorably with these other libraries.[8]

[7] The correctness requirements of the static analysis research field prevent the adoption of floating-point coefficients, since any rounding error on the wrong side can invalidate the overall computation. For domains as complicated as that of polyhedra, the correct, precise and reasonably efficient handling of floating-point rounding errors is an open issue.

[8] See http://www.cs.unipr.it/ppl/performance for the libraries compared, the benchmarks, software and hardware used for the tests as well as the detailed results.

Apart from the PPL, the only two polyhedra libraries – among those based on the DD method that provide the services required by applications in static analysis and computer-aided verification – that support NNC polyhedra are "Polka" [HKP95], by N. Halbwachs, A. Kerbrat, and Y.-E. Proy, and "New Polka" [Jea02], by B. Jeannet. While the "Polka" polyhedra library is not available in source format and binaries are distributed under rather restrictive conditions (until about the year 1996 they could be freely downloaded), "New Polka" is free software. Both libraries can be compiled so as to work with strict inequalities although the support for these is incomplete, incurring avoidable inefficiencies and leaving the client application with the non-trivial task of a correct interpretation of the obtained results. As this paper concerns NNC polyhedra, their interface and implementation, we only compare the PPL support for NNC polyhedra with that for "Polka" and "New Polka".

First of all, the PPL allows us to demonstrate the real benefits of having an implementation-independent interface based on mixed constraint systems and extended generator systems. In particular, even though an NNC polyhedron can be easily described by using constraint systems containing strict inequalities, the "Polka" and "New Polka" libraries lack the corresponding extension for generator systems (i.e., the introduction of closure points), resulting in an asymmetric user interface. For instance, the following sentence comes from the documentation of "New Polka" [Jea02, Sect. 1.1.4, p. 10] (where s denotes the ϵ coefficient):

> Don't ask me the intuitive meaning of $s \neq 0$ in rays and vertices !

The problem is also present in "Polka" and discussed in more detail in [HKP95, Sect. 4.5, pp. 10–11]:

> While strict inequations handling is transparent for constraints (being displayed accurately), the extra dimension added to the variables space is apparent when it comes to generators : one extra coefficient, resp. extra vertices (as `epsilon` is bounded), materialize this dimension in every generator, resp. generators system.
>
> This makes more difficult to define polyhedra with the only help of generators : one should carefully study the extra vertices with non null `epsilon` coefficients added to constraints defined polyhedra, in the case of large inequations, and the case of strict inequations.

This kind of approach, which requires the user to be aware of so many implementation details, is far from being satisfactory. Finally, neither "Polka" nor "New Polka" provide support for the minimization of the descriptions of an NNC polyhedron, therefore suffering from the issues exposed in Sect. 6.

In order to asses the practical relevance of the minimization procedures proposed in this paper, we have conducted a preliminary experimental evaluation. Table 2 summarizes the results for four simple experiments, where a pair of rows corresponds to a single experiment. Each of the four experiments takes four NNC polyhedra $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, defined by extended generator systems $\mathcal{G}'_1$, $\mathcal{G}'_2$, $\mathcal{G}'_3$, $\mathcal{G}'_4$, respectively, and aims to compute a constraint description for the NNC polyhedron $\mathcal{P} = (\mathcal{P}_1 \cap \mathcal{P}_2) \uplus (\mathcal{P}_3 \cap \mathcal{P}_4)$. The four input polyhedra all have the same shape (they are equivalent up to translation and scaling transformations); in particular, if $\mathcal{G}'_i = (\varnothing, P'_i, C'_i)$, then the cardinalities $\# P'_i$ and $\# C'_i$ are invariant for $i = 1, \ldots, 4$ ($\# P'_i$ and $\# C'_i$ are given in the first column of the table). Each row of a pair of rows corresponds to one of two alternative evaluation strategies:

- The first row reports the measurements obtained using the *standard* evaluation strategy which does not compute the ϵ-minimal forms;
- The second row reports the measurements obtained using the *enhanced* evaluation strategy which computes the ϵ-minimal forms of the polyhedra descriptions just before the application of each operator (i.e., before computing the two intersections and before computing the poly-hull), as well as at the end of the overall computation.

All experiments start by computing the constraint-biased representation $\mathcal{G}_i$ of $\mathcal{G}'_i$, as given in Definition 5.7, which define the C-ϵ-polyhedra $\mathcal{R}_i = \mathrm{gen}(\mathcal{G}_i) \Rrightarrow_C \mathcal{P}_i$; they then compute a constraint system $\mathcal{C}$ for the C-ϵ-polyhedron $\mathcal{R} = (\mathcal{R}_1 \cap \mathcal{R}_2) \uplus (\mathcal{R}_3 \cap \mathcal{R}_4)$. To compute the two intersections $\mathcal{R}_{12} = \mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathcal{R}_{34} = \mathcal{R}_3 \cap \mathcal{R}_4$, the conversion algorithm is applied to obtain the constraint systems $\mathcal{C}_i$ such that $\mathcal{R}_i = \mathrm{con}(\mathcal{C}_i)$. In the second and third columns we give the cardinalities of $\mathcal{G}_i$ and $\mathcal{C}_i$, respectively. Note that, in the case of the enhanced evaluation strategy, these constraint systems are in ϵ-minimal form, so that $\# \mathcal{C}_i$ for the enhanced evaluation strategy is less than that for the standard evaluation strategy. The constraint systems for $\mathcal{R}_{12}$ and $\mathcal{R}_{34}$ are then computed and the conversion algorithm applied so as to obtain the generator systems $\mathcal{G}_{12}$ and $\mathcal{G}_{34}$ such that $\mathcal{R}_{12} = \mathrm{gen}(\mathcal{G}_{12})$ and $\mathcal{R}_{34} = \mathrm{gen}(\mathcal{G}_{34})$; the corresponding cardinalities are shown in the fourth and fifth columns where again, in the case of the enhanced evaluation strategy, the generator systems $\mathcal{G}_{12}$ and $\mathcal{G}_{34}$ are in ϵ-minimal form. Then, the poly-hull $\mathcal{R} = \mathcal{R}_{12} \uplus \mathcal{R}_{34}$ is computed and the conversion algorithm is again applied to obtain a constraint description $\mathcal{C}$ of $\mathcal{R}$; the cardinality of $\mathcal{C}$ is reported in the sixth column. Even though not necessary for the correctness of the computation, in the case of the enhanced evaluation strategy $\mathcal{C}$ is put into ϵ-minimal form. Note that, by Definitions 5.4 and 6.1, the cardinality of a minimal constraint system for $\mathcal{P}$ is just one or two less than the value of $\# \mathcal{C}$ reported for the enhanced evaluation strategy. The seventh column reports the speed-up ratio obtained for the considered experiment using the enhanced evaluation strategy (i.e., the ratio *standard evaluation*

Table 2. Exploiting ϵ-minimal forms to improve efficiency

$4 \times (\# P_i' + \# C_i')$	$4 \times \# \mathcal{G}_i$	$4 \times \# \mathcal{C}_i$	$\# \mathcal{G}_{12}$	$\# \mathcal{G}_{34}$	$\# \mathcal{C}$	Speed-up	ϵ-mf time
$4 \times (4 + 8)$	4×16	4×37	130	76	332		
	4×16	4×22	39	16	33	18	0.0%
$4 \times (8 + 8)$	4×24	4×55	208	124	520		
	4×24	4×30	49	20	43	15	0.0%
$4 \times (8 + 10)$	4×26	4×109	413	304	2,693		
	4×26	4×46	57	24	127	110	2.1%
$4 \times (16 + 10)$	4×42	4×163	696	656	4,994		
	4×42	4×66	77	28	152	231	2.3%

time/enhanced evaluation time). The last column reports the percentage of time spent for the computation of the ϵ-minimal form for the constraint system for $\mathcal{R}$.

Note that the results are explained in terms of the low-level implementation details of the two evaluation strategies; these inner steps are in fact transparent to an end-user of the library. Also, we adopted the constraint-biased implementation supported by the *Parma Polyhedra Library* [BHZ04, BRZH02a], but similar results have been obtained with the generator-biased implementation.

Even though the considered examples are not meant to provide a faithful representation of typical computation patterns, we can make a couple of observations based on these experiments. The application of even a few operators on the closed representations of NNC polyhedra may produce a huge number of constraints and/or generators that are strongly ϵ-redundant; these can slow-down subsequent computations considerably and are likely to confuse anyone who looks at the final output. Moreover, the computation of ϵ-minimal forms seems to have a negligible cost (see the last column in Table 2), so the adoption of the enhanced evaluation strategy is likely to result in significant efficiency gains even with more general computation patterns. On the other hand, the enhanced evaluation strategy does not come with an improved efficiency guarantee, because there can be cases where the representations in ϵ-minimal form, even though having fewer constraints or generators, happen to be more expensive. The conjecture, which is supported by all the experiments conducted so far, is that any efficiency losses will be both less frequent and less significant than efficiency gains.

As a final remark, it must be observed that the enhanced computation strategy can be made smarter than in the above example. It can be observed that the standard evaluation strategy allows for the application of the incremental version of the conversion algorithm: for instance, we start from the DD pair $(\mathcal{C}_1, \mathcal{G}_1)$ for the ϵ-polyhedron $\mathcal{R}_1$ and incrementally add the constraints $\mathcal{C}_2$ describing the ϵ-polyhedron $\mathcal{R}_2$, keeping the generator system of the result up-to-date so that, at the end of the computation of the intersection $\mathcal{R}_1 \cap \mathcal{R}_2$, we still have a DD pair (i.e., the generator system $\mathcal{G}_{12}$ will already be up-to-date). In contrast, the enhanced evaluation strategy does not fit very well with this incremental approach, because after computing the ϵ-minimal form for $\mathcal{C}_1$, we no longer have a DD pair; thus, the generator representation for the intersection $\mathcal{R}_1 \cap \mathcal{R}_2$ has to be recomputed from scratch. The same happens when computing the other intersection and the poly-hull operation. Since the number of strongly ϵ-redundant elements contained in a description can be efficiently computed, an improved evaluation strategy may heuristically predict and compare the gains coming from the computation of ϵ-minimal forms with respect to the losses coming from the lack of incrementality, therefore choosing the evaluation strategy that seems to be more appropriate for the considered context.

8. Conclusion

Convex polyhedra provide the basis for several abstractions used in static analysis and computer-aided verification of complex system. Some of these applications require the manipulation of convex polyhedra that are not necessarily closed. In this work we have proposed an elegant way of decoupling the essential geometric features of NNC polyhedra from their traditional implementation. This separation, besides providing a natural and easy to use interface, enables the search for new implementation techniques and makes their eventual integration into existing software libraries seamless (i.e., transparent to the client application). In fact, we have shown that the standard implementation of NNC polyhedra, which happens to be biased for constraint-intensive computations, has a dual that is biased for generator-intensive computations. For both kinds of implementations, we have provided new minimization procedures that allow to extract a non-redundant constraint or generator description of an NNC polyhedron from its low level encodings.

We have implemented all these ideas in the *Parma Polyhedra Library*, a modern C++ library for the manipulation of convex polyhedra. Since it is based on the high level interface for the specification of NNC polyhedra and it implements the new minimization procedures introduced in this work, the *Parma Polyhedra Library* can be regarded as the first software library (based on the DD method) providing full support for the domain of NNC polyhedra. Some very preliminary experiments on purely synthetic benchmarks have shown that a careful use of the new minimization procedures for NNC polyhedra can have a dramatic effect on the size of the representations and, thus, on the efficiency of the algorithms operating upon them.

The *Parma Polyhedra Library* has also been extended to experiment with the two alternative implementations of NNC polyhedra. In this respect, it seems likely that the performance of one encoding with respect to the other will depend on the particular application and, more specifically, on the kind of polyhedra and operations that are more common in that application. For future work, given the dual characteristics of the two representations, it would be interesting to investigate whether efficient techniques can be devised so as to use both constraint- and generator-biased encodings, switching dynamically from one to the other in an attempt to maximize performance.

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A. Proofs

As already observed, Theorem 3.2 is a simple consequence of well known theorems by Minkowski (stating the '*only if*' part) and Weyl (stating the '*if*' part). We provide here proofs of the other formal results stated in the main part of the paper.

A.1. Proofs of the results stated in Section 4

In order to simplify the proof of Proposition 4.3, we introduce the following lemma.

Lemma A.1 Let $\mathcal{P} = \mathrm{con}(\mathcal{C}) \in \mathbb{P}_n$ and $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$. Let also $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \bowtie b\bigr) \in \mathcal{C}$, where $\bowtie \in \{\geqslant, >\}$. Then $\langle \boldsymbol{a}, \boldsymbol{v}\rangle \geqslant b$.

Proof. Let $\mathcal{H}$ be the set of affine half-spaces corresponding to the set of constraints $\mathcal{C}$. Since $\mathbb{C}(\cdot)$ is an upper closure operator,

$$\mathbb{C}(\mathcal{P}) = \mathbb{C}\bigl(\bigcap \mathcal{H}\bigr) \subseteq \mathbb{C}\Bigl(\bigcap\bigl\{\, \mathbb{C}(H) \bigm| H \in \mathcal{H} \,\bigr\}\Bigr) = \bigcap\bigl\{\, \mathbb{C}(H) \bigm| H \in \mathcal{H} \,\bigr\}.$$

As $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$, we have $\boldsymbol{v} \in \mathbb{C}(H)$, for all $H \in \mathcal{H}$. Let $H_\beta \in \mathcal{H}$ denote the affine half-space corresponding to $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \bowtie b\bigr)$. Hence $\boldsymbol{v} \in \mathbb{C}(H_\beta)$. If $\bowtie \in \{\geqslant\}$, then $\mathbb{C}(H_\beta) = \mathrm{con}\bigl(\{\beta\}\bigr)$. On the other hand, if $\bowtie \in \{>\}$, then $\mathbb{C}(H_\beta) = \mathrm{con}\bigl(\{\beta'\}\bigr)$, where $\beta' = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \geqslant b\bigr)$. Thus, as $\boldsymbol{v} \in \mathbb{C}(H_\beta)$, we obtain the thesis $\langle \boldsymbol{a}, \boldsymbol{v}\rangle \geqslant b$. $\square$

Proof of Proposition 4.3. Let $\mathcal{P} = \mathrm{con}(\mathcal{C}) \in \mathbb{P}_n$ be an NNC polyhedron defined by the mixed constraint system $\mathcal{C} = \{\beta_1, \dots, \beta_m\}$.

To prove the '*only if*' branch, suppose that $\boldsymbol{c}$ is a closure point of $\mathcal{P}$, so that $\boldsymbol{c} \in \mathbb{C}(\mathcal{P})$. Then $\mathcal{P} \neq \varnothing$, because the topological closure of the empty set is still empty. Considering an arbitrary point $\boldsymbol{p} \in \mathcal{P}$ and a scalar σ such that $0 < \sigma < 1$, we have to prove that vector $\boldsymbol{v} = \sigma \boldsymbol{p} + (1 - \sigma)\boldsymbol{c}$ is such that $\boldsymbol{v} \in \mathcal{P}$. To this end, we show that $\boldsymbol{v}$ satisfies all constraints $\beta_i = \bigl(\langle \boldsymbol{a}_i, \boldsymbol{x}\rangle \bowtie_i b_i\bigr) \in \mathcal{C}$. Since $\boldsymbol{p} \in \mathcal{P}$, it holds $\langle \boldsymbol{a}_i, \boldsymbol{p}\rangle \bowtie_i b_i$; moreover, by applying Lemma A.1, we also have $\langle \boldsymbol{a}_i, \boldsymbol{c}\rangle \geqslant b_i$. Therefore, we obtain

$$\langle \boldsymbol{a}_i, \boldsymbol{v}\rangle = \bigl\langle \boldsymbol{a}_i, \sigma \boldsymbol{p} + (1 - \sigma)\boldsymbol{c}\bigr\rangle = \sigma\langle \boldsymbol{a}_i, \boldsymbol{p}\rangle + (1 - \sigma)\langle \boldsymbol{a}_i, \boldsymbol{c}\rangle \bowtie_i \sigma b_i + (1 - \sigma)\langle \boldsymbol{a}_i, \boldsymbol{c}\rangle \geqslant \sigma b_i + (1 - \sigma) b_i = b_i.$$

It follows that if $\bowtie_i \in \{\geqslant\}$, then $\langle \boldsymbol{a}_i, \boldsymbol{v}\rangle \geqslant b_i$ and, if $\bowtie \in \{>\}$, $\langle \boldsymbol{a}_i, \boldsymbol{v}\rangle > b_i$. Therefore, in both cases we obtain $\langle \boldsymbol{a}_i, \boldsymbol{v}\rangle \bowtie_i b_i$

To prove the '*if*' branch, suppose now that $\mathcal{P} \neq \varnothing$ and $\sigma \boldsymbol{p} + (1 - \sigma)\boldsymbol{c} \in \mathcal{P}$, for all $\boldsymbol{p} \in \mathcal{P}$ and $0 < \sigma < 1$. We have to show that $\boldsymbol{c} \in \mathbb{C}(\mathcal{P})$. To this end, for each $i \in \mathbb{N}$, let $\sigma_i = \frac{1}{i}$, B_i be the open ball centered in $\boldsymbol{c}$ with radius σ_i, and $\boldsymbol{v}_i = \sigma_{i+1}\boldsymbol{p} + (1 - \sigma_{i+1})\boldsymbol{c}$; then, as $0 < \sigma_{i+1} < \sigma_i < 1$, we have $\boldsymbol{v}_i \in \mathcal{P} \cap B_i$. As this holds for any $i \in \mathbb{N}$, $\boldsymbol{c} \in \mathbb{C}(\mathcal{P})$. $\square$

A direct proof of Theorem 4.4 would require a generalization of Minkowski's and Weyl's theorems. In contrast, we will provide an indirect proof: this will be based on the standard (i.e., non-generalized) version of Minkowski's and Weyl's theorems, stated as Theorem 3.2, as well as on Propositions 5.5, 5.8. and 5.11. Thus, the proof of Theorem 4.4 will appear at the end of the next section.

A.2. Proofs of the Results Stated in Section 5

Proof of Proposition 5.3. Let $\mathcal{R} = \mathrm{con}(\mathcal{C})$. We first assume that (4) holds for any constraint $\beta \in \mathcal{C}$ and show that $\mathcal{R}$ is an ϵ-polyhedron. Condition (2) of Definition 5.2 holds by hypothesis. We prove condition (3) holds. Let $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr)$. Then, by (4), $\mathcal{R} \subseteq \mathrm{con}\Bigl(\bigl\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b\bigr\}\Bigr)$. Thus, for any point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$ we have $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + 0 \cdot e \geqslant b$, so that also $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + s \cdot 0 \geqslant b$ and hence $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ satisfies β. As β was an arbitrary constraint in $\mathcal{C}$, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and condition (3) holds.

Second we assume that $\mathcal{R}$ is an ϵ-polyhedron and prove that (4) holds. Suppose $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr)$ and that $\mathcal{R} \subseteq \mathrm{con}\bigl(\{\beta\}\bigr)$. Then, any point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$ satisfies β. By condition (3) of Definition 5.2, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and therefore satisfies β. Thus we have $\langle \boldsymbol{a}, \boldsymbol{v}\rangle \geqslant b$. Hence, if $\beta_0 = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b\bigr)$, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ satisfies β_0. As $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ was an arbitrary point in $\mathcal{R}$, $\mathcal{R} \subseteq \mathrm{con}\bigl(\{\beta_0\}\bigr)$. $\square$

To prove Proposition 5.5, we need a few additional lemmas.

Lemma A.2 Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. If $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathrm{rays}(\mathcal{R})$ and $\boldsymbol{r} \neq \boldsymbol{0}$, then $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathrm{rays}(\mathcal{R})$.

Proof. Since $\mathcal{R}$ has a ray, it is not empty. Thus, let $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}$. By condition (3) of Definition 5.2, we have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Since $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}}$ is a ray of $\mathcal{R}$, we have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} = \bigl((\boldsymbol{v} + \rho\boldsymbol{r})^{\mathrm{T}}, \rho e\bigr)^{\mathrm{T}} \in \mathcal{R}$ for all $\rho \in \mathbb{R}_{+}$. From this, again by condition (3) of Definition 5.2, we obtain $\bigl((\boldsymbol{v} + \rho\boldsymbol{r})^{\mathrm{T}}, 0\bigr)^{\mathrm{T}} = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$, proving that also $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray of $\mathcal{R}$. $\square$

Lemma A.3 Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. If $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathrm{rays}(\mathcal{R})$, then $e \leqslant 0$. Moreover, if $\mathcal{R}$ is a C-ϵ-polyhedron, then $e = 0$.

Proof. Since $\mathcal{R}$ is an ϵ-polyhedron, condition (2) of Definition 5.2 holds so that, for some $\delta > 0$ every point in $\mathcal{R}$ satisfies the constraint $\epsilon \leqslant \delta$. Since $\mathcal{R}$ has a ray, it is non-empty, so that there exists a point $(\boldsymbol{v}^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}$ such that $e_0 \leqslant \delta$. Thus, for all $\rho \in \mathbb{R}_{+}$, $(\boldsymbol{v}_{\rho}^{\mathrm{T}}, e_{\rho})^{\mathrm{T}} = (\boldsymbol{v}^{\mathrm{T}}, e_0)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. By condition (2) of Definition 5.2, $e_{\rho} = e_0 + \rho e \leqslant \delta$. Therefore, as this holds for all $\rho \in \mathbb{R}_{+}$, we have $e \leqslant 0$.

By Definition 5.6, if $\mathcal{R}$ is a C-ϵ-polyhedron then it also satisfies the constraint $\epsilon \geqslant 0$. By repeating the above argument, we obtain $e_{\rho} = e_0 + \rho e \geqslant 0$. As this holds for all $\rho \in \mathbb{R}_{+}$, we also have $e \geqslant 0$, i.e., $e = 0$. $\square$

In the following lemmas, for any $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ such that $\mathcal{R} \Rrightarrow_{\epsilon} \mathcal{P} \neq \varnothing$, we assume the notations $\mathcal{C}_{>}$, $\mathcal{C}_{\geqslant}$, $\mathcal{G}_R$, $\mathcal{G}_P$, and $\mathcal{G}_C$ introduced in Sect. 6.1. Moreover, we will denote the set of ϵ upper bounds of the constraint system $\mathcal{C}$ as

$$\mathcal{C}_{\epsilon} \stackrel{\text{def}}{=} \Bigl\{ \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C} \Bigm| \boldsymbol{a} = \boldsymbol{0}, s < 0 \Bigr\}.$$

A constraint $\beta \in \mathcal{C}_{\epsilon}$ will be usually denoted as $\epsilon \leqslant \delta$. Since $\mathcal{P} \neq \varnothing$, we have $\delta > 0$.

Lemma A.4 Let $\mathcal{R} = \mathrm{gen}(\mathcal{G}) \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron, where $\mathcal{G} = (R, P)$. Let also $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}$ for some $e' \in \mathbb{R}$ and take $e_{\max} \in \mathbb{R}$ to be the maximal ϵ coordinate such that $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathcal{R}$. Then $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathrm{gen}\big((\mathcal{G}_R, \mathcal{G}_P \cup C')\big)$, where $C' = \big\{ (\boldsymbol{c}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{G}_C \bigm| \forall e' \in \mathbb{R} : (\boldsymbol{c}^{\mathrm{T}}, e')^{\mathrm{T}} \notin \mathcal{G}_P \big\}$.

Proof. By hypothesis, $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathrm{gen}\big((R, P)\big)$ so that

$$(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} = \big(\rho_1 \boldsymbol{r}_1 + \cdots + \rho_r \boldsymbol{r}_r\big) + \big(\pi_1 \boldsymbol{p}_1 + \cdots + \pi_p \boldsymbol{p}_p\big), \tag{11}$$

where $r \geqslant 0$, $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r\} \subseteq R$, $\rho_1, \ldots, \rho_r > 0$ and $p > 0$, $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_p\} \subseteq P$, $\pi_1, \ldots, \pi_p > 0$ and $\sum_{i=1}^{p} \pi_i = 1$.

For all $1 \leqslant i \leqslant r$, let $\boldsymbol{r}_i = (\boldsymbol{v}_i^{\mathrm{T}}, e_i)^{\mathrm{T}}$. Suppose that, for some $1 \leqslant j \leqslant r$, we have $e_j \neq 0$. Then, as $\mathcal{R}$ is an ϵ-polyhedron, by Lemma A.3, $e_j < 0$. If $\boldsymbol{v}_j = \boldsymbol{0}$ then, by removing the ray $\boldsymbol{r}_j$ in (11), we would obtain

$$(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} - \rho_j \boldsymbol{r}_j = (\boldsymbol{v}^{\mathrm{T}}, e_{\max} - \rho_j e_j)^{\mathrm{T}} \in \mathcal{R}.$$

On the other hand, if $\boldsymbol{v}_j \neq \boldsymbol{0}$, then, by Lemma A.2, $\boldsymbol{r}_0 = (\boldsymbol{v}_j^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathrm{rays}(\mathcal{R})$. Thus, in this case, by replacing the ray $\boldsymbol{r}_j$ by $\boldsymbol{r}_0$ in (11), we again obtain

$$(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} - \rho_j \boldsymbol{r}_j + \rho_j \boldsymbol{r}_0 = (\boldsymbol{v}^{\mathrm{T}}, e_{\max} - \rho_j e_j)^{\mathrm{T}} \in \mathcal{R}.$$

In both cases, as $\rho_j > 0$ and $e_j < 0$, $e_{\max} - \rho_j e_j > e_{\max}$, contradicting the maximality of $e_{\max}$. Thus we must have $e_j = 0$ so that $\boldsymbol{r}_j \in \mathcal{G}_R$. As the choice of $1 \leqslant j \leqslant r$ was arbitrary, $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r\} \subseteq \mathcal{G}_R$.

For all $1 \leqslant i \leqslant p$, let $\boldsymbol{p}_i = (\boldsymbol{v}_i^{\mathrm{T}}, e_i)^{\mathrm{T}}$. Suppose that, for some $1 \leqslant j \leqslant p$, we have $e_j < 0$. Then, as $\mathcal{R}$ is an ϵ-polyhedron, by condition (3) of Definition 5.2, $\boldsymbol{p}_0 = (\boldsymbol{v}_j^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Replacing the point $\boldsymbol{p}_j$ by $\boldsymbol{p}_0$ in (11), we would obtain

$$(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} - \pi_j \boldsymbol{p}_j + \pi_j \boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, e_{\max} - \pi_j e_j)^{\mathrm{T}} \in \mathcal{R},$$

where, since $\pi_j > 0$, $e_{\max} - \pi_j e_j > e_{\max}$, contradicting the maximality of $e_{\max}$. Hence, for all $1 \leqslant i \leqslant p$, we have $e_i \geqslant 0$. Now suppose that, for some $1 \leqslant j \leqslant p$, we have $e_j = 0$ and there exists $e_j^+ > 0$ such that $\boldsymbol{p}_j^+ = (\boldsymbol{v}_j^{\mathrm{T}}, e_j^+)^{\mathrm{T}} \in \mathcal{R}$. Replacing the point $\boldsymbol{p}_j$ by $\boldsymbol{p}_j^+$ in (11), we would obtain

$$(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} - \pi_j \boldsymbol{p}_j + \pi_j \boldsymbol{p}_j^+ = (\boldsymbol{v}^{\mathrm{T}}, e_{\max} + \pi_j e_j^+)^{\mathrm{T}} \in \mathcal{R},$$

where since $\pi_j > 0$, $e_{\max} + \pi_j e_j^+ > e_{\max}$, again contradicting the maximality of $e_{\max}$. It follows that, for all $1 \leqslant i \leqslant p$, the point $\boldsymbol{p}_i \in \mathcal{G}_P$, if $e_i > 0$, and $\boldsymbol{p}_i \in C'$, otherwise. Thus, $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathrm{gen}\big((\mathcal{G}_R, \mathcal{G}_P \cup C')\big)$. □

Proof of Proposition 5.5. Let $\mathcal{G} = (R, P)$ and $\mathrm{gen\_enc}(\mathcal{G}) = (R_1, P_1, C_1)$ be the corresponding extended generator system, where $R_1 = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r\}$, $P_1 = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_p\}$, and $C_1 = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_c\}$.

Suppose first that $[\![\mathcal{R}]\!] = \varnothing$. Then, by Definition 5.4, $\mathrm{con}\big(\mathrm{con\_enc}(\mathcal{C})\big) = \varnothing$. Also, by Definition 5.1, $\mathcal{R} \subseteq \mathrm{con}\big(\{\epsilon \leqslant 0\}\big)$ so that, by Definition 5.4, $P_1 = \varnothing$ and $\mathrm{gen}((R_1, P_1, C_1)) = \varnothing$. Thus (5) holds.

Suppose now that $[\![\mathcal{R}]\!] \neq \varnothing$. We will first prove that $\mathrm{con}\big(\mathrm{con\_enc}(\mathcal{C})\big) \subseteq [\![\mathcal{R}]\!]$ and $\mathrm{gen}\big(\mathrm{gen\_enc}(\mathcal{G})\big) \subseteq [\![\mathcal{R}]\!]$. To this end, we assume that one of the following holds:

$$\boldsymbol{v} \in \mathrm{con}\big(\mathrm{con\_enc}(\mathcal{C})\big), \tag{12}$$

$$\boldsymbol{v} \in \mathrm{gen}\big(\mathrm{gen\_enc}(\mathcal{G})\big), \tag{13}$$

and, in each case, we show that there exists $e_v > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}} \in \mathcal{R}$ so that, by Definition 5.1, we obtain $\boldsymbol{v} \in [\![\mathcal{R}]\!]$.

Suppose that (12) holds. By Definition 5.2, the set $\mathcal{C}_> \cup \mathcal{C}_\epsilon$ is non-empty so that, as the value of the ϵ coefficient in each constraint in the set is non-zero, the set

$$\Big\{ e \in \mathbb{R} \Bigm| e = -s^{-1}\big(\langle \boldsymbol{a}, \boldsymbol{v}\rangle - b\big), \big(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\big) \in \mathcal{C}_> \cup \mathcal{C}_\epsilon \Big\}$$

is also non-empty. Let e_v be the least element of this set. Suppose $\beta = \big(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\big) \in \mathcal{C}_> \cup \mathcal{C}_\epsilon$. By Definition 5.4, if $\beta \in \mathcal{C}_>$, then $\big(\langle \boldsymbol{a}, \boldsymbol{x}\rangle > b\big) \in \mathrm{con\_enc}(\mathcal{C})$ so that, as (12) holds, $\langle \boldsymbol{a}, \boldsymbol{v}\rangle > b$. On the other hand, if $\beta \in \mathcal{C}_\epsilon$ then $\boldsymbol{a} = \boldsymbol{0}$ and, as $[\![\mathcal{R}]\!]$ is non-empty, by Definition 5.2, $b < 0$. Thus, in both cases $\langle \boldsymbol{a}, \boldsymbol{v}\rangle - b > 0$ so that, as $s < 0$, $-s^{-1}\big(\langle \boldsymbol{a}, \boldsymbol{v}\rangle - b\big) > 0$. It follows that $e_v > 0$. For each $\big(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b\big) \in \mathcal{C}_\geqslant$, for some $\bowtie \in \{\geqslant, >\}$, the constraint $\big(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \bowtie b\big) \in \mathrm{con\_enc}(\mathcal{C})$ so that, by (12), $\langle \boldsymbol{a}, \boldsymbol{v}\rangle \geqslant b$ holds. Hence $(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}}$ satisfies all the constraints in $\mathcal{C}_\geqslant$. It follows that $(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}}$ satisfies every constraint in $\mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon$. Reasoning toward a contradiction, suppose that $(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}} \notin \mathcal{R}$. As $[\![\mathcal{R}]\!] \neq \emptyset$, by Definition 5.1, there exists $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}$ such that $e_w > 0$; in particular, $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}$ satisfies all constraints in $\mathcal{C}$. Thus there exists a point $(\boldsymbol{w}_0^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}$ which lies on the line segment joining $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}$ and $(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}}$ and saturates a constraint

$$\beta' = \bigl(\langle \boldsymbol{a}', \boldsymbol{x} \rangle + s' \cdot \epsilon \geqslant b'\bigr) \in \mathcal{C} \setminus (\mathcal{C}_{>} \cup \mathcal{C}_{\geqslant} \cup \mathcal{C}_{\epsilon}).$$

Thus $\langle \boldsymbol{a}', \boldsymbol{w}_0 \rangle + s' \cdot e_0 = b'$. However, since $s' > 0$ and $e_0 > 0$, $\langle \boldsymbol{a}', \boldsymbol{w}_0 \rangle < b'$ so that $(\boldsymbol{w}_0^{\mathrm{T}}, 0)^{\mathrm{T}} \notin \mathcal{R}$, contradicting condition (3) of Definition 5.2. Thus $(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}} \in \mathcal{R}$.

Suppose next that (13) holds. By definition of function ‘gen’ in Theorem 4.4,

$$\boldsymbol{v} = \sum_{i=1}^{r} \rho_i \boldsymbol{r}_i + \sum_{i=1}^{p} \pi_i \boldsymbol{p}_i + \sum_{i=1}^{c} \gamma_i \boldsymbol{c}_i$$

where $p > 0$, $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r\} \subseteq R_1$, $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_p\} \subseteq P_1$, $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_c\} \subseteq C_1$, $\boldsymbol{\rho} \in \mathbb{R}_+^r$, $\boldsymbol{\pi} \in \mathbb{R}_+^p \setminus \{\boldsymbol{0}\}$, $\boldsymbol{\gamma} \in \mathbb{R}_+^c$ and $\sum_{i=1}^{p} \pi_i + \sum_{i=1}^{c} \gamma_i = 1$. By Definition 5.4, for some $e_1, \ldots, e_p > 0$,

$$\bigl\{(\boldsymbol{r}_1^{\mathrm{T}}, 0)^{\mathrm{T}}, \ldots, (\boldsymbol{r}_r^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\} \subseteq R, \quad \bigl\{(\boldsymbol{p}_1^{\mathrm{T}}, e_1)^{\mathrm{T}}, \ldots, (\boldsymbol{p}_p^{\mathrm{T}}, e_p)^{\mathrm{T}}\bigr\} \cup \bigl\{(\boldsymbol{c}_1^{\mathrm{T}}, 0)^{\mathrm{T}}, \ldots, (\boldsymbol{c}_c^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\} \subseteq P.$$

Thus, letting

$$(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}} = \sum_{i=1}^{r} \rho_i (\boldsymbol{r}_i^{\mathrm{T}}, 0)^{\mathrm{T}} + \sum_{i=1}^{p} \pi_i (\boldsymbol{p}_i^{\mathrm{T}}, e_i)^{\mathrm{T}} + \sum_{i=1}^{c} \gamma_i (\boldsymbol{c}_i^{\mathrm{T}}, 0)^{\mathrm{T}}$$

we obtain $(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}} \in \mathrm{gen}(\mathcal{G}) = \mathcal{R}$. Since $p > 0$ and $\boldsymbol{\pi} \neq \boldsymbol{0}$, we also obtain $e_v > 0$.

We now prove that $[\![\mathcal{R}]\!] \subseteq \mathrm{con}\bigl(\mathrm{con\_enc}(\mathcal{C})\bigr)$ and $[\![\mathcal{R}]\!] \subseteq \mathrm{gen}\bigl(\mathrm{gen\_enc}(\mathcal{G})\bigr)$. To this end, let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$, where $e > 0$; since $\mathcal{R}$ is an ϵ-polyhedron, by condition (2) of Definition 5.2, the ϵ dimension is bounded from above and thus there exists a value $e_{\max} > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathcal{R}$ and, for all $e' > e_{\max}$, $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \notin \mathcal{R}$. We show that both (12) and (13) hold.

Suppose that $\beta' = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr) \in \mathrm{con\_enc}(\mathcal{C})$, where $\bowtie \in \{\geqslant, >\}$. Then, by Definition 5.4, there exists $s \leqslant 0$ such that $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}$. Since $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathcal{R}$, then $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + s \cdot e_{\max} \geqslant b$ so that, as $e_{\max} > 0$ holds, we obtain $\langle \boldsymbol{a}, \boldsymbol{v} \rangle \geqslant b$. Moreover, if $\bowtie \in \{>\}$, then $s < 0$ and we obtain $\langle \boldsymbol{a}, \boldsymbol{v} \rangle > b$. Thus, for any $\bowtie \in \{\geqslant, >\}$, $\boldsymbol{v}$ satisfies β'. As $\beta' \in \mathrm{con\_enc}(\mathcal{C})$ was chosen arbitrarily, (12) holds.

We next prove (13). Since $e_{\max}$ was chosen to be maximal for $\boldsymbol{v}$, we can apply Lemma A.4, so that $(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} \in \mathrm{gen}\bigl((\mathcal{G}_R, \mathcal{G}_P \cup C')\bigr)$, where

$$C' = \bigl\{\, (\boldsymbol{c}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{G}_C \bigm| \forall e' \in \mathbb{R} : (\boldsymbol{c}^{\mathrm{T}}, e')^{\mathrm{T}} \notin \mathcal{G}_P \,\bigr\}.$$

By definition of ‘gen’, we obtain

$$(\boldsymbol{v}^{\mathrm{T}}, e_{\max})^{\mathrm{T}} = \sum_{i=1}^{r} \rho_i (\boldsymbol{r}_i^{\mathrm{T}}, 0)^{\mathrm{T}} + \sum_{i=1}^{p} \pi_i (\boldsymbol{p}_i^{\mathrm{T}}, e_i)^{\mathrm{T}} + \sum_{i=1}^{c} \gamma_i (\boldsymbol{c}_i^{\mathrm{T}}, 0)^{\mathrm{T}};$$

where

$$\bigl\{(\boldsymbol{r}_1^{\mathrm{T}}, 0)^{\mathrm{T}}, \ldots, (\boldsymbol{r}_r^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\} \subseteq \mathcal{G}_R, \quad \bigl\{(\boldsymbol{p}_1^{\mathrm{T}}, e_1)^{\mathrm{T}}, \ldots, (\boldsymbol{p}_p^{\mathrm{T}}, e_p)^{\mathrm{T}}\bigr\} \subseteq \mathcal{G}_P, \quad \bigl\{(\boldsymbol{c}_1^{\mathrm{T}}, 0)^{\mathrm{T}}, \ldots, (\boldsymbol{c}_c^{\mathrm{T}}, 0)^{\mathrm{T}}\bigr\} \subseteq C',$$

$\boldsymbol{\rho} \in \mathbb{R}_+^r$, $\boldsymbol{\pi} \in \mathbb{R}_+^p$, $\boldsymbol{\gamma} \in \mathbb{R}_+^c$ and $\sum_{i=1}^{p} \pi_i + \sum_{i=1}^{c} \gamma_i = 1$. As $e_{\max} > 0$, we obtain $p > 0$ and $\boldsymbol{\pi} \neq \boldsymbol{0}$. By Definition 5.4, $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r\} \subseteq R_1$, $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_p\} \subseteq P_1$ and $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_c\} \subseteq C_1$ so that

$$\boldsymbol{v} = \sum_{i=1}^{r} \rho_i \boldsymbol{r}_i + \sum_{i=1}^{p} \pi_i \boldsymbol{p}_i + \sum_{i=1}^{c} \gamma_i \boldsymbol{c}_i$$

and hence $\boldsymbol{v} \in \mathrm{gen}\bigl((R_1, P_1, C_1)\bigr)$. Thus (13) holds. □

A.3. Proofs of the results stated in Section 5.1

Propositions 5.8 and 5.11 are corollaries of the following two lemmas which, for technical reasons, recombine the four relations to be proved.

Lemma A.5 Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P} \in \mathbb{P}_n$. Then $\mathrm{con}\bigl(\mathrm{con\_repr}_C(\mathcal{C})\bigr) \Rrightarrow_C \mathcal{P}$ and $\mathrm{con}\bigl(\mathrm{con\_repr}_G(\mathcal{C})\bigr) \Rrightarrow_G \mathcal{P}$.

Proof. We first show that, for any $Y \in \{C, G\}$,

$$\mathrm{con}(\mathcal{C}) = \mathrm{con}\Bigl(\mathrm{con\_enc}\bigl(\mathrm{con\_repr}_Y(\mathcal{C})\bigr)\Bigr). \tag{14}$$

Let $\mathcal{C}_1 = \mathrm{con\_repr}_Y(\mathcal{C})$ and $\mathcal{C}_2 = \mathrm{con\_enc}(\mathcal{C}_1)$. Let $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \bowtie b\bigr) \in \mathcal{C}$, for some $\bowtie \in \{\geqslant, >\}$. If $\bowtie \in \{>\}$, then, by Definitions 5.7 and 5.10, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle - 1 \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}_1$ and hence, by Definition 5.4, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle > b\bigr) \in \mathcal{C}_2$. If otherwise $\bowtie \in \{\geqslant\}$, then, by Definitions 5.7 and 5.10, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}_1$ and hence, by Definition 5.4, either $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \geqslant b\bigr) \in \mathcal{C}_2$ or $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle > b\bigr) \in \mathcal{C}_2$. Thus β is satisfied by all the points in $\mathrm{con}(\mathcal{C}_2)$. As β was an arbitrary constraint in $\mathcal{C}$, we obtain $\mathrm{con}(\mathcal{C}_2) \subseteq \mathrm{con}(\mathcal{C})$.

Now let $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \bowtie b\bigr) \in \mathcal{C}_2$, for some $\bowtie \in \{\geqslant, >\}$. If $\bowtie \in \{>\}$, then $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}_1$, where $s < 0$. By Definitions 5.7 and 5.10, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle > b\bigr) \in \mathcal{C}$. If $\bowtie \in \{\geqslant\}$, then, by Definition 5.4, $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}_1$. Thus, by Definitions 5.7 and 5.10, either $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle \geqslant b\bigr) \in \mathcal{C}$ or $\bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle > b\bigr) \in \mathcal{C}$. Thus β is satisfied by all the points in $\mathrm{con}(\mathcal{C})$. As β was an arbitrary constraint in $\mathcal{C}_2$, we obtain the other inclusion $\mathrm{con}(\mathcal{C}) \subseteq \mathrm{con}(\mathcal{C}_2)$. Thus (14) holds. As a consequence, by Proposition 5.5, we have $\mathrm{con}(\mathcal{C}_1) \Rrightarrow_\epsilon \mathrm{con}(\mathcal{C})$.

Suppose now that $Y = C$. Then, by Definition 5.7, $\mathrm{con}(\mathcal{C}_1) \subseteq \mathrm{con}\bigl(\{\epsilon \geqslant 0\}\bigr)$ so that, by Definition 5.6, $\mathrm{con}(\mathcal{C}_1)$ is a C-ϵ-polyhedron. Otherwise, suppose that $Y = G$. If $\mathrm{con}(\mathcal{C}_1) = \varnothing$ then, by Definition 5.9, it is a G-ϵ-polyhedron. Otherwise, let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathrm{con}(\mathcal{C}_1)$ and consider $\beta' \in \mathcal{C}_1$. By Definition 5.10, either $\beta' = (\epsilon \leqslant 1)$ or, for some $\boldsymbol{a} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, $b \in \mathbb{R}$ and $s \in \{0, -1\}$, $\beta' = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr)$. Thus, for all $\rho \in \mathbb{R}_+$, $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon)$ satisfies β', so that $-\boldsymbol{e}_\epsilon$ is a ray in $\mathrm{con}\bigl(\{\beta'\}\bigr)$. As the choice of β' was arbitrary, $-\boldsymbol{e}_\epsilon \in \mathrm{rays}\bigl(\mathrm{con}(\mathcal{C}_1)\bigr)$ so that, by Definition 5.9, $\mathrm{con}(\mathcal{C}_1)$ is a G-ϵ-polyhedron. □

Lemma A.6 Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P} \in \mathbb{P}_n$. Then $\mathrm{gen}\bigl(\mathrm{gen\_repr}_C(\mathcal{G})\bigr) \Rrightarrow_C \mathcal{P}$ and $\mathrm{gen}\bigl(\mathrm{gen\_repr}_G(\mathcal{G})\bigr) \Rrightarrow_G \mathcal{P}$.

Proof. We first show that for any $Y \in \{C, G\}$,

$$\mathrm{gen}(\mathcal{G}) = \mathrm{gen}\Bigl(\mathrm{gen\_enc}\bigl(\mathrm{gen\_repr}_Y(\mathcal{G})\bigr)\Bigr). \tag{15}$$

Let $\mathcal{G} = (R, P, C)$, $\mathcal{G}_1 = \mathrm{gen\_repr}_Y(\mathcal{G}) = (R_1, P_1)$ and $\mathcal{G}_2 = \mathrm{gen\_enc}(\mathcal{G}_1) = (R_2, P_2, C_2)$. Suppose first that $\boldsymbol{v} \in R \cup P \cup C$. If $\boldsymbol{v} \in R$, then, by Definitions 5.7 and 5.10, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R_1$ and hence, by Definition 5.4, $\boldsymbol{v} \in R_2$. If $\boldsymbol{v} \in P$, then, by Definitions 5.7 and 5.10, $(\boldsymbol{v}^{\mathrm{T}}, 1)^{\mathrm{T}} \in P_1$ and hence, by Definition 5.4, $\boldsymbol{v} \in P_2$. If $\boldsymbol{v} \in C$, then, by Definitions 5.7 and 5.10, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in P_1$ and hence, by Definition 5.4, $\boldsymbol{v} \in P_2 \cup C_2$. Therefore, by definition of 'gen', we obtain $\mathrm{gen}(\mathcal{G}) \subseteq \mathrm{gen}(\mathcal{G}_2)$.

Now suppose $\boldsymbol{v} \in R_2 \cup P_2 \cup C_2$. If $\boldsymbol{v} \in R_2$, then, by Definition 5.4, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R_1$. By Definitions 5.7 and 5.10, $\boldsymbol{v} \in R$. If $\boldsymbol{v} \in P_2$, then, by Definition 5.4, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P_1$, for some $e > 0$. Thus, by Definitions 5.7 and 5.10, $\boldsymbol{v} \in P$. If $\boldsymbol{v} \in C_2$, then, by Definition 5.4, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P_1$, for some $e \geqslant 0$. Thus, by Definitions 5.7 and 5.10, $\boldsymbol{v} \in P \cup C$. Thus, by definition of 'gen', we obtain the other inclusion $\mathrm{gen}(\mathcal{G}_2) \subseteq \mathrm{gen}(\mathcal{G})$ and (15) holds. As a consequence, by Proposition 5.5, we have $\mathrm{gen}(\mathcal{G}_1) \Rrightarrow_\epsilon \mathrm{gen}(\mathcal{G})$.

Suppose now that $Y = G$. Then, by Definition 5.10, $-\boldsymbol{e}_\epsilon \in R_1$ so that, by Definition 5.9, $\mathrm{gen}(\mathcal{G}_1)$ is a G-ϵ-polyhedron. Otherwise, suppose that $Y = C$. By Definition 5.7, for each vector $(v_1^{\mathrm{T}}, e_1)^{\mathrm{T}} \in R_1 \cup P_1$, we have $e_1 \geqslant 0$; this implies that, for any point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathrm{gen}(\mathcal{G}_1)$, we still have $e \geqslant 0$. Thus $\mathrm{gen}(\mathcal{G}_1) \subseteq \mathrm{con}\bigl(\{\epsilon \geqslant 0\}\bigr)$ so that, by Definition 5.6, $\mathrm{gen}(\mathcal{G}_1)$ is a C-ϵ-polyhedron. □

Proof of Proposition 5.8. The relation $\mathrm{con}\bigl(\mathrm{con\_repr}_C(\mathcal{C})\bigr) \Rrightarrow_C \mathcal{P}$ holds by Lemma A.5 whereas the relation $\mathrm{gen}\bigl(\mathrm{gen\_repr}_C(\mathcal{G})\bigr) \Rrightarrow_C \mathcal{P}$ holds by Lemma A.6. □

Proof of Proposition 5.11. The relation $\mathrm{con}\bigl(\mathrm{con\_repr}_G(\mathcal{C})\bigr) \Rrightarrow_G \mathcal{P}$ holds by Lemma A.5 whereas the relation $\mathrm{gen}\bigl(\mathrm{gen\_repr}_G(\mathcal{G})\bigr) \Rrightarrow_G \mathcal{P}$. holds by Lemma A.6. □

We are now ready to provide the formal proof of the generalization of Minkowski and Weyl's theorems for the domain of NNC polyhedra.

Proof of Theorem 4.4. To prove the '*only if*' branch, letting $\mathcal{P} \in \mathbb{P}_n$ we will prove that there exists an extended generator system $\mathcal{G} = (R, P, C)$ such that $\mathcal{P} = \mathrm{gen}(\mathcal{G})$. If $\mathcal{P} = \varnothing$, then we simply take $\mathcal{G} = (\varnothing, \varnothing, \varnothing)$. Otherwise, let $\mathcal{P} \neq \varnothing$. By definition of NNC polyhedron, there exists a mixed constraint system $\mathcal{C}$ such that $\mathcal{P} = \mathrm{con}(\mathcal{C})$. Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$. Note that we can always find such an ϵ-polyhedron for $\mathcal{P}$; e.g., by Proposition 5.8, we can consider $\mathrm{con}\bigl(\mathrm{con\_repr}_C(\mathcal{C})\bigr)$ or, by Proposition 5.11, we can consider $\mathrm{con}\bigl(\mathrm{con\_repr}_G(\mathcal{C})\bigr)$.

By Theorem 3.2, there exists a (standard) generator system $\mathcal{G}' = (R', P')$ such that $\mathcal{R} = \mathrm{gen}(\mathcal{G}')$. By defining $\mathcal{G} = \mathrm{gen\_enc}(\mathcal{G}')$, the thesis $\mathcal{P} = [\![\mathcal{R}]\!] = \mathrm{gen}(\mathcal{G})$ follows from Proposition 5.5.

To prove the '*if*' branch, letting $\mathcal{G} = (R, P, C)$ be an extended generator system, we will show that $\mathcal{P} = \mathrm{gen}(\mathcal{G})$ is an NNC polyhedron. If $P = \varnothing$, then we obtain $\mathcal{P} = \varnothing$ and the empty set is an NNC polyhedron. Otherwise, let $P \neq \varnothing$, so that $\mathcal{P} \neq \varnothing$. Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$. As said above, we can always find such an ϵ-polyhedron for $\mathcal{P}$; e.g., by Proposition 5.8, we can consider $\mathrm{gen}\bigl(\mathrm{gen\_repr}_C(\mathcal{G})\bigr)$ or, by Proposition 5.11, we can consider $\mathrm{gen}\bigl(\mathrm{gen\_repr}_G(\mathcal{G})\bigr)$. By Theorem 3.2, there exists a constraint system $\mathcal{C}'$, containing non-strict linear inequalities only, such that $\mathcal{R} = \mathrm{con}(\mathcal{C}')$. By defining $\mathcal{C} = \mathrm{con\_enc}(\mathcal{C}')$, the thesis $\mathcal{P} = [\![\mathcal{R}]\!] = \mathrm{con}(\mathcal{C})$ follows from Proposition 5.5. $\square$

A.4. Proofs of the results stated in Section 5.2

The proof of Proposition 5.12 requires a number of additional preliminary results.

For any ϵ-polyhedron, closure points in the NNC polyhedron are represented by points lying on the hyperplane defined by $\epsilon = 0$.

Lemma A.7 Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Then $\mathbb{C}(\mathcal{P}) = \bigl\{\, \boldsymbol{v} \in \mathbb{R}^n \bigm| (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R} \,\bigr\}$.

Proof. Letting $\mathcal{P}' = \bigl\{\, \boldsymbol{v} \in \mathbb{R}^n \bigm| (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R} \,\bigr\}$, we will prove $\mathcal{P}' = \mathbb{C}(\mathcal{P})$.

First, we show that $\mathcal{P}' \subseteq \mathbb{C}(\mathcal{P})$. Let $\boldsymbol{v} \in \mathcal{P}'$, so that $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$, and consider any point $\boldsymbol{p} \in \mathcal{P}$ (note that such a point exists by hypothesis). Then, since $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, there exists $e > 0$ such that $(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. Since $\mathcal{R}$ is a convex set, for all $\sigma \in \mathbb{R}$ such that $0 < \sigma < 1$ we have

$$\sigma(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} + (1-\sigma)(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} = \bigl(\sigma \boldsymbol{p}^{\mathrm{T}} + (1-\sigma)\boldsymbol{v}^{\mathrm{T}}, \sigma e\bigr)^{\mathrm{T}} \in \mathcal{R}.$$

Since $\sigma e > 0$, by Definition 5.1, we obtain $\sigma \boldsymbol{p} + (1-\sigma)\boldsymbol{v} \in \mathcal{P}$. As the choices of $\boldsymbol{p} \in \mathcal{P}$ and σ were both arbitrary, we can apply Proposition 4.3 and conclude $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$.

Now we show that $\mathbb{C}(\mathcal{P}) \subseteq \mathcal{P}'$. Let $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$ and, for all $i \in \mathbb{N}$ such that $i > 1$, define $\sigma_i = \frac{1}{i}$, so that $0 < \sigma_i < 1$. Then, by Proposition 4.3, for all $\boldsymbol{p} \in \mathcal{P}$ we have $\boldsymbol{v}_i = \sigma_i \boldsymbol{p} + (1-\sigma_i)\boldsymbol{v} \in \mathcal{P}$. Since $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, by applying the fact that $\mathcal{P} = [\![\mathcal{R}]\!]$ and then property (3) of Definition 5.2, we obtain $(\boldsymbol{v}_i^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. If $\boldsymbol{p} = \boldsymbol{v}$, then $\boldsymbol{v}_i = \boldsymbol{v}$, so that the thesis holds. Otherwise, let $\boldsymbol{p} \neq \boldsymbol{v}$. For any open ball of $\mathbb{R}^{n+1}$ centered in $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ and having radius $\delta > 0$, there exists $j \in \mathbb{N}$ such that $\sigma_j < \delta$; thus, $(\boldsymbol{v}_j^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ belongs to the ball and, as the choice of δ is arbitrary, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathbb{C}(\mathcal{R})$. However, $\mathcal{R} \in \mathbb{CP}_{n+1}$ is a topologically closed set, so that $\mathcal{R} = \mathbb{C}(\mathcal{R})$ and $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Hence, $\boldsymbol{v} \in \mathcal{P}'$, completing the proof. $\square$

Lemma A.8 Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Then $\boldsymbol{r} \in \mathrm{rays}(\mathcal{P})$ if and only if $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathrm{rays}(\mathcal{R})$.

Proof. Assuming that $\boldsymbol{r} \in \mathrm{rays}(\mathcal{P})$, let $\boldsymbol{v} \in \mathcal{P}$ and $\rho \in \mathbb{R}_+$. Then $\boldsymbol{v} + \rho\boldsymbol{r} \in \mathcal{P}$. By Definition 5.1, for some $e_1, e_2 > 0$, we have $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}$ and $\bigl((\boldsymbol{v} + \rho\boldsymbol{r})^{\mathrm{T}}, e_2\bigr)^{\mathrm{T}} \in \mathcal{R}$ and hence, by condition (3) of Definition 5.2, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and $\bigl((\boldsymbol{v} + \rho\boldsymbol{r})^{\mathrm{T}}, 0\bigr)^{\mathrm{T}} \in \mathcal{R}$. Thus $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. As this holds for all $\rho \in \mathbb{R}_+$, $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray of $\mathcal{R}$.

To prove the other direction, assume that $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is a ray of $\mathcal{R}$. Let $\boldsymbol{v} \in \mathcal{P}$ and $\rho \in \mathbb{R}_+$. By Definition 5.1, for some $e > 0$, we have $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$; by assumption, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} + \rho(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and hence, by Definition 5.1, $\boldsymbol{v} + \rho\boldsymbol{r} \in \mathcal{P}$, proving that $\boldsymbol{r} \in \mathrm{rays}(\mathcal{P})$. $\square$

Lemma A.9 Let $\mathcal{R} = \mathrm{gen}\bigl((R, P)\bigr) \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Let also

$$R' = \bigl\{\, (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \bigm| (\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R, \boldsymbol{r} \neq \boldsymbol{0} \,\bigr\} \cup \bigl\{\, -\boldsymbol{e}_\epsilon \bigm| (\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R, e < 0 \,\bigr\}$$

and $\mathcal{R}' = \mathrm{gen}\bigl((R', P)\bigr)$. Then $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$.

Proof. Suppose that for all $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R$ we have $e = 0$. Then, the result holds by observing that, in such a case, we obtain $R' = R$ and thus $\mathcal{R}' = \mathcal{R}$. Otherwise, suppose that there exists $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R$ such that $e \neq 0$. By Lemma A.3, it holds $e < 0$. It follows from the hypothesis that $(-\boldsymbol{e}_\epsilon) \in R'$.

We first show that $[\![\mathcal{R}']\!] = \mathcal{P}$ by proving the two inclusions separately.

Consider a ray $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R$. If $e = 0$, then $\boldsymbol{r} \neq \boldsymbol{0}$ so that, by hypothesis, $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R'$. If $e < 0$ and $\boldsymbol{r} = \boldsymbol{0}$, then we can write $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} = -e(-\boldsymbol{e}_\epsilon)$, where $(-\boldsymbol{e}_\epsilon) \in R'$ and $-e > 0$ is a positive factor. Otherwise, if $e < 0$ and $\boldsymbol{r} \neq \boldsymbol{0}$, then, by the hypothesis, $\bigl\{(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}, -\boldsymbol{e}_\epsilon\bigr\} \subseteq R'$ and we can write $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} = (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} - e(-\boldsymbol{e}_\epsilon)$. Thus, each element of R can be obtained as a non-negative combination of elements of $\mathcal{R}'$, therefore proving that $\mathcal{R} \subseteq \mathcal{R}'$ and, by monotonicity, $\mathcal{P} \subseteq [\![\mathcal{R}']\!]$.

To prove the other inclusion, let $R'' = R' \setminus \{-\boldsymbol{e}_\epsilon\}$ and $\mathcal{R}'' = \operatorname{gen}\bigl((R'', P)\bigr)$. For each ray $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R''$, by hypothesis, we have $(\boldsymbol{r}^{\mathrm{T}}, e)^{\mathrm{T}} \in R$ so that, by Lemma A.2, $(\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}}$ is also a ray of $\mathcal{R}$. Hence, $\mathcal{R}'' \subseteq \mathcal{R}$. By the above observations, we obtain

$$\forall (\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}' : \exists (\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}, \rho \in \mathbb{R}_+ \,.\, (\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} = (\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon). \tag{16}$$

Let now $\boldsymbol{p} \in [\![\mathcal{R}']\!]$, so that there exists $(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$ such that $e > 0$. By applying (16), we obtain that $(\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}$, where $e_0 = e + \rho > 0$, proving that $\boldsymbol{p} \in [\![\mathcal{R}]\!] = \mathcal{P}$. As the choice of $\boldsymbol{p}$ was arbitrary, $[\![\mathcal{R}']\!] \subseteq \mathcal{P}$.

To complete the proof, we have to show that $\mathcal{R}'$ is an ϵ-polyhedron. Condition (2) of Definition 5.2 easily follows from (16), because $\mathcal{R}$ is an ϵ-polyhedron: namely, we can consider the same ϵ upper bound constraint $\epsilon \leqslant \delta$ used for $\mathcal{R}$. To prove condition (3) of Definition 5.2, let $(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$. By (16), there exist $(\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} \in \mathcal{R}$ and $\rho \in \mathbb{R}_+$ such that $(\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} = (\boldsymbol{p}^{\mathrm{T}}, e_0)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon)$. As $\mathcal{R}$ is an ϵ-polyhedron, we also have $(\boldsymbol{p}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Since we already observed that $\mathcal{R} \subseteq \mathcal{R}'$, this completes the proof. □

Lemma A.10 Let $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$, $\mathcal{R}_1 \Rrightarrow_Y \mathcal{P}_1$ and $\mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_2$. Then $\mathcal{R}_1 \cap \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \cap \mathcal{P}_2$.

Proof. We first prove condition (2) of Definition 5.2. Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are ϵ-polyhedra there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $\mathcal{R}_1 \subseteq \operatorname{con}\bigl(\{\epsilon \leqslant \delta_1\}\bigr)$ and $\mathcal{R}_2 \subseteq \operatorname{con}\bigl(\{\epsilon \leqslant \delta_2\}\bigr)$. Letting $\delta = \min\{\delta_1, \delta_2\}$, we have $\mathcal{R}_1 \cap \mathcal{R}_2 \subseteq \operatorname{con}\bigl(\{\epsilon \leqslant \delta\}\bigr)$.

To prove condition (3) of Definition 5.2, let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. Then, as $\mathcal{R}_1$ and $\mathcal{R}_2$ are ϵ-polyhedra, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_1$ and $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_2$. Hence $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$.

Having shown that $\mathcal{R}_1 \cap \mathcal{R}_2$ is an ϵ-polyhedron, we next show that $[\![\mathcal{R}_1 \cap \mathcal{R}_2]\!] = \mathcal{P}_1 \cap \mathcal{P}_2$. By hypothesis, $\mathcal{R}_1 \Rrightarrow_\epsilon \mathcal{P}_1$ and $\mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_2$, so that, by Definition 5.2, $\mathcal{P}_1 = [\![\mathcal{R}_1]\!]$ and $\mathcal{P}_2 = [\![\mathcal{R}_2]\!]$. By Definition 5.1, we have to show that $\boldsymbol{v} \in \mathcal{P}_1 \cap \mathcal{P}_2$ if and only if there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. First, let $\boldsymbol{v} \in \mathcal{P}_1 \cap \mathcal{P}_2$. Hence, by Definition 5.1, there exist $e_1, e_2 > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1$ and $(\boldsymbol{v}^{\mathrm{T}}, e_2)^{\mathrm{T}} \in \mathcal{R}_2$. Suppose, without loss of generality, that $e_1 \leqslant e_2$. By condition (3) of Definition 5.2, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_2$. Thus, since $\mathcal{R}_2$ is a convex set, $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_2$. Hence $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. Secondly, suppose that there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. Then $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1$ and $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_2$. By Definition 5.1, $\boldsymbol{v} \in \mathcal{P}_1$ and $\boldsymbol{v} \in \mathcal{P}_2$, so that $\boldsymbol{v} \in \mathcal{P}_1 \cap \mathcal{P}_2$. It follows that $[\![\mathcal{R}_1 \cap \mathcal{R}_2]\!] = \mathcal{P}_1 \cap \mathcal{P}_2$. By Definition 5.2, $\mathcal{R}_1 \cap \mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_1 \cap \mathcal{P}_2$.

We now prove that $\mathcal{R}_1 \cap \mathcal{R}_2$ is a C-ϵ-polyhedron when $\mathcal{R}_1$ and $\mathcal{R}_2$ are C-ϵ-polyhedra. By Definition 5.6, for $j \in \{1, 2\}$ we have $\mathcal{R}_j \subseteq \operatorname{con}\bigl(\{\epsilon \geqslant 0\}\bigr)$. Thus $\mathcal{R}_1 \cap \mathcal{R}_2 \subseteq \operatorname{con}\bigl(\{\epsilon \geqslant 0\}\bigr)$ so that, by Definition 5.6, $\mathcal{R}_1 \cap \mathcal{R}_2$ is a C-ϵ-polyhedron.

To prove that $\mathcal{R}_1 \cap \mathcal{R}_2$ is a G-ϵ-polyhedron when $\mathcal{R}_1$ and $\mathcal{R}_2$ are G-ϵ-polyhedra, we consider two subcases. If $\mathcal{R}_1 \cap \mathcal{R}_2 = \varnothing$, then there is nothing to prove. Suppose instead that $\mathcal{R}_1 \cap \mathcal{R}_2 \neq \varnothing$. Then, by Definition 5.9, $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R}_1)$ and $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R}_2)$. Let $\boldsymbol{v}' \in \mathcal{R}_1 \cap \mathcal{R}_2$ and consider, for any $\rho \in \mathbb{R}_+$, $\boldsymbol{v}'_\rho = \boldsymbol{v}' + \rho(-\boldsymbol{e}_\epsilon)$. As $\boldsymbol{v}' \in \mathcal{R}_1$, $\boldsymbol{v}'_\rho \in \mathcal{R}_1$ and, as $\boldsymbol{v}' \in \mathcal{R}_2$, $\boldsymbol{v}'_\rho \in \mathcal{R}_2$; hence $\boldsymbol{v}'_\rho \in \mathcal{R}_1 \cap \mathcal{R}_2$. As this holds for any $\rho \in \mathbb{R}_+$, $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R}_1 \cap \mathcal{R}_2)$. Thus, by Definition 5.9, $\mathcal{R}_1 \cap \mathcal{R}_2$ is a G-ϵ-polyhedra. □

Lemma A.11 For $j \in \{1, 2\}$, let $\mathcal{P}_j = \operatorname{gen}\bigl((R_j, P_j, C_j)\bigr)$ be a non-empty NNC polyhedron. Then $\boldsymbol{x} \in \mathcal{P}_1 \uplus \mathcal{P}_2$ if and only if there exist $\boldsymbol{r}_1 \in \{\boldsymbol{0}\} \cup \operatorname{rays}(\mathcal{P}_1)$, $\boldsymbol{r}_2 \in \{\boldsymbol{0}\} \cup \operatorname{rays}(\mathcal{P}_2)$, $\boldsymbol{x}_1 \in \mathbb{C}(\mathcal{P}_1)$, $\boldsymbol{x}_2 \in \mathbb{C}(\mathcal{P}_2)$, and $0 \leqslant \sigma \leqslant 1$ such that $\boldsymbol{x} = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \sigma \boldsymbol{x}_1 + (1 - \sigma)\boldsymbol{x}_2$, where $(\boldsymbol{x}_1 \in \mathcal{P}_1 \wedge \sigma > 0) \vee (\boldsymbol{x}_2 \in \mathcal{P}_2 \wedge \sigma < 1)$.

Proof. For $j \in \{1, 2\}$, let r_j, p_j and c_j be the cardinalities of R_j, P_j and C_j, respectively. By definition of the poly-hull operation, $\mathcal{P}_1 \uplus \mathcal{P}_2 = \operatorname{gen}\bigl((R, P, C)\bigr)$, where $R = R_1 \cup R_2$, $P = P_1 \cup P_2$ and $C = C_1 \cup C_2$, having cardinalities r, p and c, respectively. (In general, we have $r \leqslant r_1 + r_2$, $p \leqslant p_1 + p_2$ and $c \leqslant c_1 + c_2$, since there may be generators that occur in both generator systems.)

Suppose first that $\boldsymbol{x} \in \mathcal{P}_1 \uplus \mathcal{P}_2$. Then, by the definition of function 'gen',

$$\boldsymbol{x} = R\boldsymbol{\rho} + P\boldsymbol{\pi} + C\boldsymbol{\gamma},$$

where $\boldsymbol{\rho} \in \mathbb{R}_+^r$, $\boldsymbol{\pi} \in \mathbb{R}_+^p$, $\boldsymbol{\gamma} \in \mathbb{R}_+^c$, $\sum \boldsymbol{\pi} + \sum \boldsymbol{\gamma} = 1$ and $\boldsymbol{\pi} \neq \boldsymbol{0}$. Therefore, we can also rewrite it as

$$\begin{aligned} \boldsymbol{x} &= R_1\boldsymbol{\rho}_1 + R_2\boldsymbol{\rho}_2 + P_1\boldsymbol{\pi}_1 + P_2\boldsymbol{\pi}_2 + C_1\boldsymbol{\gamma}_1 + C_2\boldsymbol{\gamma}_2 \\ &= R_1\boldsymbol{\rho}_1 + R_2\boldsymbol{\rho}_2 + (P_1\boldsymbol{\pi}_1 + C_1\boldsymbol{\gamma}_1) + (P_2\boldsymbol{\pi}_2 + C_2\boldsymbol{\gamma}_2), \end{aligned}$$

where $\boldsymbol{\rho}_j \in \mathbb{R}_+^{r_j}$, $\boldsymbol{\pi}_j \in \mathbb{R}_+^{p_j}$ and $\boldsymbol{\gamma}_j \in \mathbb{R}_+^{c_j}$, for $j \in \{1, 2\}$, $\sum \boldsymbol{\pi}_1 + \sum \boldsymbol{\pi}_2 + \sum \boldsymbol{\gamma}_1 + \sum \boldsymbol{\gamma}_2 = 1$ and $\sum \boldsymbol{\pi}_1 + \sum \boldsymbol{\pi}_2 > 0$. It follows that either $\boldsymbol{\pi}_1 \neq \boldsymbol{0}$ or $\boldsymbol{\pi}_2 \neq \boldsymbol{0}$.

For each $j \in \{1, 2\}$, let $\boldsymbol{r}_j = R_j\boldsymbol{\rho}_j$, so that $\boldsymbol{r}_j$ is a non-negative combination of the rays of $\mathcal{P}_j$. Note that either $\boldsymbol{r}_j = \mathbf{0}$ (e.g., if $\boldsymbol{\rho}_j = \mathbf{0}$) or $\boldsymbol{r}_j \in \operatorname{rays}(\mathcal{P}_j)$. If $\sum \boldsymbol{\pi}_1 + \sum \boldsymbol{\gamma}_1 = 1$ (so that $\boldsymbol{\pi}_2 = \boldsymbol{\gamma}_2 = \mathbf{0}$ and $\boldsymbol{\pi}_1 \neq \mathbf{0}$), then, by the definition of function 'gen', we obtain $\boldsymbol{x}_1 = P_1\boldsymbol{\pi}_1 + C_1\boldsymbol{\gamma}_1 \in \mathcal{P}_1$. Taking $\sigma = 1$, we have $1 - \sigma = 0$, so that we can take an arbitrary $\boldsymbol{x}_2 \in \mathbb{C}(\mathcal{P}_2)$ (there must exist one, since $\mathcal{P}_2 \neq \varnothing$). Similarly, if $\sum \boldsymbol{\pi}_2 + \sum \boldsymbol{\gamma}_2 = 1$ (so that $\boldsymbol{\pi}_1 = \boldsymbol{\gamma}_1 = \mathbf{0}$ and $\boldsymbol{\pi}_2 \neq \mathbf{0}$), we obtain $\boldsymbol{x}_2 = P_2\boldsymbol{\pi}_2 + C_2\boldsymbol{\gamma}_2 \in \mathcal{P}_2$, so that we can take $\sigma = 0$ and an arbitrary $\boldsymbol{x}_1 \in \mathbb{C}(\mathcal{P}_1)$. Otherwise, let both $\sum \boldsymbol{\pi}_1 + \sum \boldsymbol{\gamma}_1 \neq 0$ and $\sum \boldsymbol{\pi}_2 + \sum \boldsymbol{\gamma}_2 \neq 0$. Then, by taking $\sigma = \sum \boldsymbol{\pi}_1 + \sum \boldsymbol{\gamma}_1$ we have $\sigma > 0$ and $1 - \sigma = \sum \boldsymbol{\pi}_2 + \sum \boldsymbol{\gamma}_2 > 0$. Therefore we can define $\boldsymbol{x}_1 = \frac{1}{\sigma}(P_1\boldsymbol{\pi}_1 + C_1\boldsymbol{\gamma}_1)$ and $\boldsymbol{x}_2 = \frac{1}{1-\sigma}(P_2\boldsymbol{\pi}_2 + C_2\boldsymbol{\gamma}_2)$. Thus $\boldsymbol{x}_1 \in \mathbb{C}(\mathcal{P}_1)$ and $\boldsymbol{x}_2 \in \mathbb{C}(\mathcal{P}_2)$. Moreover, by the definition of function 'gen', as $\boldsymbol{\pi}_1 \neq \mathbf{0}$ or $\boldsymbol{\pi}_2 \neq \mathbf{0}$, either $\boldsymbol{x}_1 \in \mathcal{P}_1$ or $\boldsymbol{x}_2 \in \mathcal{P}_2$. Thus, in all cases we obtain

$$\boldsymbol{x} = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \sigma\boldsymbol{x}_1 + (1 - \sigma)\boldsymbol{x}_2,$$

with $\boldsymbol{r}_j \in \{\mathbf{0}\} \cup \operatorname{rays}(\mathcal{P}_j)$ and $\boldsymbol{x}_j \in \mathbb{C}(\mathcal{P}_j)$ for $j = \{1, 2\}$, where either $\boldsymbol{x}_1 \in \mathcal{P}_1$ and $\sigma > 0$ or $\boldsymbol{x}_2 \in \mathcal{P}_2$ and $\sigma < 1$, as required.

To prove the other direction, suppose that $\boldsymbol{r}_j \in \{\mathbf{0}\} \cup \operatorname{rays}(\mathcal{P}_j)$ and $\boldsymbol{x}_j \in \mathbb{C}(\mathcal{P}_j)$, for $j \in \{1, 2\}$, and there exists $0 \leqslant \sigma \leqslant 1$ such that

$$\boldsymbol{x} = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \sigma\boldsymbol{x}_1 + (1 - \sigma)\boldsymbol{x}_2,$$

where either $\boldsymbol{x}_1 \in \mathcal{P}_1$ and $\sigma > 0$ or $\boldsymbol{x}_2 \in \mathcal{P}_2$ and $\sigma < 1$.

For $j \in \{1, 2\}$, there exists $\varrho_j \in \mathbb{R}_+^{r_j}$ such that $\boldsymbol{r}_j = R_j\varrho_j$; moreover, since $\boldsymbol{x}_j \in \mathbb{C}(\mathcal{P}_j)$, there exist $\boldsymbol{\rho}_j \in \mathbb{R}_+^{r_j}$, $\boldsymbol{\pi}_j \in \mathbb{R}_+^{p_j}$ and $\boldsymbol{\gamma}_j \in \mathbb{R}_+^{c_j}$ such that $\boldsymbol{x}_j = R_j\boldsymbol{\rho}_j + P_j\boldsymbol{\pi}_j + C_j\boldsymbol{\gamma}_j$, where $\sum \boldsymbol{\pi}_j + \sum \boldsymbol{\gamma}_j = 1$. Thus,

$$\begin{aligned}
\boldsymbol{x} &= R_1\varrho_1 + R_2\varrho_2 + \sigma(R_1\boldsymbol{\rho}_1 + P_1\boldsymbol{\pi}_1 + C_1\boldsymbol{\gamma}_1) + (1 - \sigma)(R_2\boldsymbol{\rho}_2 + P_2\boldsymbol{\pi}_2 + C_2\boldsymbol{\gamma}_2)\\
&= R_1(\varrho_1 + \sigma\boldsymbol{\rho}_1) + R_2\big(\varrho_2 + (1 - \sigma)\boldsymbol{\rho}_2\big) + P_1\sigma\boldsymbol{\pi}_1 + P_2(1 - \sigma)\boldsymbol{\pi}_2 + C_1\sigma\boldsymbol{\gamma}_1 + C_2(1 - \sigma)\boldsymbol{\gamma}_2.
\end{aligned}$$

Note that $\varrho_1 + \sigma\boldsymbol{\rho}_1 \in \mathbb{R}_+^{r_1}$, $\varrho_2 + (1-\sigma)\boldsymbol{\rho}_2 \in \mathbb{R}_+^{r_2}$, and $\sigma\sum \boldsymbol{\pi}_1 + (1-\sigma)\sum \boldsymbol{\pi}_2 + \sigma\sum \boldsymbol{\gamma}_1 + (1-\sigma)\sum \boldsymbol{\gamma}_2 = 1$. Moreover, as either $\boldsymbol{x}_1 \in \mathcal{P}_1$ and $\sigma > 0$ or $\boldsymbol{x}_2 \in \mathcal{P}_2$ and $\sigma < 1$, we obtain $\sigma\boldsymbol{\pi}_1 \neq \mathbf{0}$ or $(1 - \sigma)\boldsymbol{\pi}_2 \neq \mathbf{0}$. Thus, there exist $\boldsymbol{\rho} \in \mathbb{R}_+^r$, $\boldsymbol{\pi} \in \mathbb{R}_+^p$ and $\boldsymbol{\gamma} \in \mathbb{R}_+^c$ such that $\sum \boldsymbol{\pi} + \sum \boldsymbol{\gamma} = 1$, $\boldsymbol{\pi} \neq \mathbf{0}$, and $\boldsymbol{x} = R\boldsymbol{\rho} + P\boldsymbol{\pi} + C\boldsymbol{\gamma}$. Then, by the definition of function 'gen', $\boldsymbol{x} \in \mathcal{P}_1 \uplus \mathcal{P}_2$, completing the proof. □

Lemma A.12 Letting $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$, suppose that $\mathcal{R}_1 \Rrightarrow_Y \mathcal{P}_1 \neq \varnothing$ and $\mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_2 \neq \varnothing$. Then $\mathcal{R}_1 \uplus \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \uplus \mathcal{P}_2$.

Proof. For $j \in \{1, 2\}$, let $\mathcal{P}_j = \operatorname{gen}\big((R_j, P_j, C_j)\big)$, where the three components of the extended generator system have cardinalities r_j, p_j and c_j, respectively; similarly, let $\mathcal{R}_j = \operatorname{gen}\big((R'_j, P'_j)\big)$, with cardinalities r'_j and p'_j, respectively. Note that each $\mathcal{R}_j \in \mathbb{CP}_{n+1}$ is also an NNC polyhedron in $\mathbb{P}_{n+1}$. Thus, we can write $\mathcal{R}_j = \operatorname{gen}\big((R'_j, P'_j, \varnothing)\big)$ and observe that $\mathcal{R}_j = \mathbb{C}(\mathcal{R}_j)$. By applying Lemma A.11, if $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, then for some $\boldsymbol{r}'_1 = (\boldsymbol{r}_1^{\mathrm{T}}, e'_1)^{\mathrm{T}} \in \{\mathbf{0}\} \cup \operatorname{rays}(\mathcal{R}_1)$, $\boldsymbol{r}'_2 = (\boldsymbol{r}_2^{\mathrm{T}}, e'_2)^{\mathrm{T}} \in \{\mathbf{0}\} \cup \operatorname{rays}(\mathcal{R}_2)$, $\boldsymbol{v}'_1 = (\boldsymbol{v}_{\boldsymbol{\Delta}}{}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1$, $\boldsymbol{v}'_2 = (\boldsymbol{v}_{\boldsymbol{\Theta}}{}^{\mathrm{T}}, e_2)^{\mathrm{T}} \in \mathcal{R}_2$, and $0 \leqslant \sigma \leqslant 1$, we have

$$\begin{pmatrix}\boldsymbol{v}\\ e\end{pmatrix} = \boldsymbol{r}'_1 + \boldsymbol{r}'_2 + \sigma\boldsymbol{v}'_1 + (1 - \sigma)\boldsymbol{v}'_2 = \begin{pmatrix}\boldsymbol{r}_1\\ e'_1\end{pmatrix} + \begin{pmatrix}\boldsymbol{r}_2\\ e'_2\end{pmatrix} + \begin{pmatrix}\sigma\boldsymbol{v}_1 + (1 - \sigma)\boldsymbol{v}_2\\ \sigma e_1 + (1 - \sigma)e_2\end{pmatrix}. \tag{17}$$

For each $j \in \{1, 2\}$, since $\boldsymbol{r}'_j \in \{\mathbf{0}\} \cup \operatorname{rays}(\mathcal{R}_j)$, by Lemma A.3 we obtain $e'_j \leqslant 0$.

We first show that $\mathcal{R}_1 \uplus \mathcal{R}_2$ is an ϵ-polyhedron. As $\mathcal{R}_1$, $\mathcal{R}_2$ are ϵ-polyhedra, by condition (2) of Definition 5.2 there exist $\delta_1, \delta_2 > 0$ such that $\mathcal{R}_1 \subseteq \operatorname{con}\big(\{\epsilon \leqslant \delta_1\}\big)$ and $\mathcal{R}_2 \subseteq \operatorname{con}\big(\{\epsilon \leqslant \delta_2\}\big)$. Suppose that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$ and rewrite it according to (17). By letting $\delta = \max\{\delta_1, \delta_2\}$, we obtain $e_1 \leqslant \delta$ and $e_2 \leqslant \delta$. Since $e'_1 \leqslant 0$ and $e'_2 \leqslant 0$, we obtain $e = \sigma e_1 + (1 - \sigma)e_2 + e'_1 + e'_2 \leqslant \delta$, so that $\mathcal{R}_1 \uplus \mathcal{R}_2$ satisfies condition (2) of Definition 5.2.

To prove condition (3) of Definition 5.2, suppose that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, so that we can rewrite it according to (17). For each $j \in \{1, 2\}$, as $\mathcal{R}_j$ is an ϵ-polyhedron, by Lemma A.2, $(\boldsymbol{r}_j^{\mathrm{T}}, 0)^{\mathrm{T}} \in \{\mathbf{0}\} \cup \operatorname{rays}(\mathcal{R}_j)$; also, by condition (3) of Definition 5.2, $(\boldsymbol{v}_j^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_j$. Thus, by applying again Lemma A.11, we obtain

$$\begin{pmatrix}\boldsymbol{v}\\ 0\end{pmatrix} = \begin{pmatrix}\boldsymbol{r}_1\\ 0\end{pmatrix} + \begin{pmatrix}\boldsymbol{r}_2\\ 0\end{pmatrix} + \sigma\begin{pmatrix}\boldsymbol{v}_1\\ 0\end{pmatrix} + (1 - \sigma)\begin{pmatrix}\boldsymbol{v}_2\\ 0\end{pmatrix} \in \mathcal{R}_1 \uplus \mathcal{R}_2.$$

Having shown that $\mathcal{R}_1 \uplus \mathcal{R}_2$ is an ϵ-polyhedron, we next show that it represents $\mathcal{P}_1 \uplus \mathcal{P}_2$. By hypothesis, $\mathcal{R}_1 \Rrightarrow_\epsilon \mathcal{P}_1$ and $\mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_2$, so that, by Definition 5.2, $\mathcal{P}_1 = [\![\mathcal{R}_1]\!]$ and $\mathcal{P}_2 = [\![\mathcal{R}_2]\!]$. By Definition 5.1, we have to prove that $\boldsymbol{v} \in \mathcal{P}_1 \uplus \mathcal{P}_2$ if and only if there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$. First suppose that $\boldsymbol{v} \in \mathcal{P}_1 \uplus \mathcal{P}_2$. Then, by Lemma A.11, there exist $\boldsymbol{r}_j \in \{\mathbf{0}\} \cup \operatorname{rays}(\mathcal{P}_j)$ and $\boldsymbol{v}_j \in \mathbb{C}(\mathcal{P}_j)$, for each $j \in \{1, 2\}$, and $0 \leqslant \sigma \leqslant 1$ such that $\boldsymbol{v} = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \sigma \boldsymbol{v}_1 + (1 - \sigma)\boldsymbol{v}_2$, where $\boldsymbol{v}_1 \in \mathcal{P}_1$ and $\sigma > 0$ or $\boldsymbol{v}_2 \in \mathcal{P}_2$ and $\sigma < 1$. Suppose, without loss of generality, that $\boldsymbol{v}_1 \in \mathcal{P}_1$ and $\sigma > 0$. By Definition 5.1, there exists $e_1 > 0$ such that $(\boldsymbol{v}_1^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1$. As $\mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_2$ and $\boldsymbol{v}_2 \in \mathbb{C}(\mathcal{P}_2)$, by Lemma A.7, we obtain $(\boldsymbol{v}_2^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_2$. Moreover, by Lemma A.8, for $j \in \{1, 2\}$ we have $(\boldsymbol{r}_j^{\mathrm{T}}, 0)^{\mathrm{T}} \in \{\mathbf{0}\} \cup \operatorname{rays}(\mathcal{R}_j)$. Therefore, by letting

$$\begin{pmatrix} \boldsymbol{v} \\ e_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{r}_2 \\ 0 \end{pmatrix} + \sigma \begin{pmatrix} \boldsymbol{v}_1 \\ e_1 \end{pmatrix} + (1 - \sigma) \begin{pmatrix} \boldsymbol{v}_2 \\ 0 \end{pmatrix},$$

we obtain, again by Lemma A.11, $(\boldsymbol{v}^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, where $e_1 > 0$ as required. Secondly, suppose that there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, so that we can rewrite it according to (17). As $e > 0$ and $\sigma \geqslant 0$, either $e_1 > 0$ and $\sigma > 0$ or $e_2 > 0$ and $\sigma < 1$. Without loss of generality, we assume that $e_1 > 0$ and $\sigma > 0$. By Definition 5.1, we have $\boldsymbol{v}_1 \in \mathcal{P}_1$. By hypothesis, $\mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_2$. Thus, as $(\boldsymbol{v}_2^{\mathrm{T}}, e_2)^{\mathrm{T}} \in \mathcal{R}_2$ for some $e_2 \in \mathbb{R}$, by condition (3) of Definition 5.2, $(\boldsymbol{v}_2^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}_2$. Therefore, by Lemma A.7, $\boldsymbol{v}_2 \in \mathbb{C}(\mathcal{P}_2)$. Thus, by Lemma A.11, $\boldsymbol{v} = \sigma \boldsymbol{v}_1 + (1 - \sigma)\boldsymbol{v}_2 + \boldsymbol{r}_1 + \boldsymbol{r}_2 \in \mathcal{P}_1 \uplus \mathcal{P}_2$. Therefore $\mathcal{R}_1 \uplus \mathcal{R}_2 \Rrightarrow_\epsilon \mathcal{P}_1 \uplus \mathcal{P}_2$.

To prove that $\mathcal{R}_1 \uplus \mathcal{R}_2$ is a C-ϵ-polyhedron when $\mathcal{R}_1$ and $\mathcal{R}_2$ are C-ϵ-polyhedra, we have to show that $\mathcal{R}_1 \uplus \mathcal{R}_2 \subseteq \operatorname{con}\bigl(\{\epsilon \geqslant 0\}\bigr)$. Let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}_1 \uplus \mathcal{R}_2$, so that we can rewrite it according to (17). As $\mathcal{R}_1$ and $\mathcal{R}_2$ are both C-ϵ-representations, we obtain $e_1 \geqslant 0$ and $e_2 \geqslant 0$; moreover, by Lemma A.3 we have $e_1' = 0$ and $e_2' = 0$. Thus $e = \sigma e_1 + (1 - \sigma)e_2 + e_1' + e_2'$ satisfies $e \geqslant 0$.

To prove that $\mathcal{R}_1 \uplus \mathcal{R}_2$ is a G-ϵ-polyhedron when $\mathcal{R}_1$ and $\mathcal{R}_2$ are G-ϵ-polyhedra, since $\mathcal{R}_1 \uplus \mathcal{R}_2 \neq \varnothing$, we have to show that $-\boldsymbol{e}_\epsilon$ is a ray in $\mathcal{R}_1 \uplus \mathcal{R}_2$. To this end, it is sufficient to observe that all the rays of $\mathcal{R}_1$ are also rays of $\mathcal{R}_1 \uplus \mathcal{R}_2$ and $-\boldsymbol{e}_\epsilon$ is a ray of $\mathcal{R}_1$, because $\mathcal{R}_1$ is a non-empty G-ϵ-polyhedron. □

Lemma A.13 Let $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$ and suppose that $\mathcal{R} \Rrightarrow_Y \mathcal{P}$. Let also $f \stackrel{\text{def}}{=} \lambda \boldsymbol{x} \in \mathbb{R}^n . A\boldsymbol{x} + \boldsymbol{b}$ be any affine transformation defined on $\mathbb{P}_n$ and define

$$g \stackrel{\text{def}}{=} \lambda \begin{pmatrix} \boldsymbol{x} \\ \epsilon \end{pmatrix} \in \mathbb{R}^{n+1} . \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \epsilon \end{pmatrix} + \begin{pmatrix} \boldsymbol{b} \\ 0 \end{pmatrix}$$

to be the corresponding affine transformation on $\mathbb{CP}_{n+1}$. Then $g(\mathcal{R}) \Rrightarrow_Y f(\mathcal{P})$.

Proof. Observe that, by definition of g, for any $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$ we have $g\bigl((\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}\bigr) = \bigl(f(\boldsymbol{v})^{\mathrm{T}}, e\bigr)^{\mathrm{T}}$. Thus the coefficient of the ϵ coordinate is not affected at all by the affine transformation, so that conditions (2) and (3) of Definition 5.2 and $f(\mathcal{P}) = [\![g(\mathcal{R})]\!]$ follow trivially from the hypothesis. Thus, $g(\mathcal{R}) \Rrightarrow_\epsilon f(\mathcal{P})$.

To complete the proof, we have to consider the cases when $\Rrightarrow_Y \in \{\Rrightarrow_C, \Rrightarrow_G\}$. First note that, if $\mathcal{R} = \varnothing$, then also $g(\mathcal{R}) = \varnothing$ and there is nothing to prove. Now assume $\mathcal{R} \neq \varnothing$. If $\mathcal{R}$ is a C-ϵ-polyhedron, then all the points in $\mathcal{R}$ satisfy the constraint $\epsilon \geqslant 0$. Since the ϵ coordinates are unaffected by the affine transformation, all the points in $g(\mathcal{R})$ satisfy the constraint $\epsilon \geqslant 0$, so that $g(\mathcal{R})$ is a C-ϵ-polyhedron too. Otherwise, $\mathcal{R}$ is a non-empty G-ϵ-polyhedron, so that $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R})$. We have $g(\mathcal{R}) \neq \varnothing$ and the ray is unaffected by the affine transformation, so that $-\boldsymbol{e}_\epsilon \in \operatorname{rays}\bigl(g(\mathcal{R})\bigr)$. □

Proof of Proposition 5.12. Items (1), (2) and (3) follow from Lemmas A.10, A.12 and A.13, respectively. □

A.5. Proofs of the results stated in Section 6

The proof of Proposition 6.3 requires a few preliminary lemmas.

Lemma A.14 Let $\mathcal{R} = \operatorname{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$ be a non-empty ϵ-polyhedron. Then $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R})$ if and only if $\mathcal{C} = \mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon$.

Proof. Suppose that $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R})$. Let $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. Then, for all $\rho \in \mathbb{R}_+$, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} + \rho(-\boldsymbol{e}_\epsilon) \in \mathcal{R}$. If $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}$, we have $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + s \cdot (e - \rho) \geqslant b$ for all $\rho \in \mathbb{R}_+$. Thus $s \leqslant 0$. Since our choice of $\beta \in \mathcal{C}$ was arbitrary, we obtain $\mathcal{C} = \mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon$.

Now suppose $\mathcal{C} = \mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon$. This means that, if $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}$, then $s \leqslant 0$. As $\mathcal{R}$ is non-empty, there exists a point $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. Also, since $s \leqslant 0$, for all $\rho \in \mathbb{R}_+$ we have $\langle \boldsymbol{a}, \boldsymbol{v} \rangle + s \cdot (e - \rho) \geqslant b$. As our choice of $\beta \in \mathcal{C}$ is arbitrary, $\boldsymbol{p} + \rho(-\boldsymbol{e}_\epsilon)$ satisfies all the constraints in $\mathcal{C}$ and is therefore in $\mathcal{R}$. Thus $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R})$, because also the choice of $\boldsymbol{p} \in \mathcal{R}$ was arbitrary. □

Lemma A.15 Let $\mathcal{R} = \operatorname{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Let also $\mathcal{C}' = \mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon \cup \{\epsilon \geqslant 0\}$ and $\mathcal{C}'' = \mathcal{C} \cup \{\epsilon \geqslant 0\}$. Then $\operatorname{con}(\mathcal{C}') \Rrightarrow_\epsilon \mathcal{P}$ and $\operatorname{con}(\mathcal{C}') = \operatorname{con}(\mathcal{C}'')$.

Proof. Let $\mathcal{R}' = \text{con}(\mathcal{C}')$, $\mathcal{R}'' = \text{con}(\mathcal{C}'')$, and $\mathcal{C}^* = \mathcal{C} \setminus (\mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon)$. Note that, by Definition 5.1 we have $\mathcal{P} = [\![\mathcal{R}]\!] = [\![\mathcal{R}'']\!]$. Moreover, by Proposition 5.3, since $\mathcal{R}$ is an ϵ-polyhedron, $\mathcal{R}''$ is also an ϵ-polyhedron. It remains for us to prove that $\mathcal{R}' = \mathcal{R}''$. Observe that, since $\mathcal{C}' \subseteq \mathcal{C}''$, $\mathcal{R}'' \subseteq \mathcal{R}'$.

We now show that $\mathcal{R}' \subseteq \mathcal{R}''$. Let $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$ so that $e \geqslant 0$. By hypothesis, $\mathcal{P} \neq \varnothing$ so that, as $\mathcal{P} = [\![\mathcal{R}'']\!]$, by Definition 5.1, there exists a point $\boldsymbol{q} = (\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}''$ such that $e_w > 0$. By hypothesis, both $\boldsymbol{p}$ and $\boldsymbol{q}$ must satisfy every constraint in $\mathcal{C}' = \mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon \cup \{\epsilon \geqslant 0\}$. We show that $\boldsymbol{p}$ also satisfies all the constraints in $\mathcal{C}^*$, so that $\boldsymbol{p} \in \mathcal{R}''$. Suppose, by contraposition, that $\boldsymbol{p}$ does not satisfy a constraint in $\mathcal{C}^*$. Let $\{\, \sigma \boldsymbol{p} + (1-\sigma)\boldsymbol{q} \mid 0 \leqslant \sigma \leqslant 1 \,\}$ be the set of points lying on the segment between $\boldsymbol{p}$ and $\boldsymbol{q}$. As $\boldsymbol{p} \notin \mathcal{R}''$ and $\boldsymbol{q} \in \mathcal{R}''$, there must exists a minimum value $0 \leqslant \tau < 1$ such that $\boldsymbol{p}' = (\boldsymbol{v}_\tau, e_\tau) = \tau \boldsymbol{p} + (1-\tau)\boldsymbol{q} \in \mathcal{R}''$, so that $\boldsymbol{p}'$ saturates some constraint $\beta^* = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}^*$. Note that, as $e \geqslant 0$, $e_w > 0$ and $\tau < 1$, we have $e_\tau > 0$. Also, by definition of $\mathcal{C}^*$, we have $s > 0$. As a consequence, $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ does not satisfy β^*, which implies $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}} \notin \mathcal{R}''$. However, since $\mathcal{R}''$ is an ϵ-polyhedron, this contradicts condition (3) of Definition 5.2. Thus $\boldsymbol{p} \in \mathcal{R}''$. As the choice of $\boldsymbol{p} \in \mathcal{R}'$ was arbitrary, $\mathcal{R}' \subseteq \mathcal{R}''$. □

Lemma A.16 Let $\mathcal{R} = \text{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. Let $\boldsymbol{p} \in \mathcal{R}$ be such that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$, where $e > 0$, and consider $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$. Then $\text{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geqslant) = \text{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geqslant)$ and $\text{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_> \cup \mathcal{C}_\epsilon) = \varnothing$.

Proof. Let $\beta \in \mathcal{C}_\geqslant$, so that $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b\bigr)$; then $\beta \in \text{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geqslant)$ if and only if $\beta \in \text{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geqslant)$, so that $\text{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\geqslant) = \text{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geqslant)$. Consider now $\beta \in \mathcal{C}_>$, so that $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr)$ where $s < 0$; since $e > 0$, we obtain $\langle \boldsymbol{a}, \boldsymbol{v}\rangle > b$, so that $\boldsymbol{p}_0$ satisfies but does not saturate β; thus $\text{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_>) = \varnothing$. Consider now $\beta \in \mathcal{C}_\epsilon$, so that $\beta = (\epsilon \leqslant \delta)$ for some $\delta > 0$; then it follows that $\text{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_\epsilon) = \varnothing$. □

Lemma A.17 Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. Let $\beta \in \mathcal{C}_>$ be such that $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr)$ and consider $\beta_0 = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geqslant b\bigr)$. Then $\text{sat\_gen}\bigl(\beta_0, (\mathcal{G}_R, \mathcal{G}_C)\bigr) = \text{sat\_gen}\bigl(\beta, (\mathcal{G}_R, \mathcal{G}_C)\bigr)$.

Proof. If $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{G}_R \cup \mathcal{G}_C$, then $e = 0$. Thus $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + s \cdot e = 0$ if and only if $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + 0 \cdot e = 0$. Similarly, $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + s \cdot e = b$ if and only if $\langle \boldsymbol{a}, \boldsymbol{v}\rangle + 0 \cdot e = b$. Thus, if $\boldsymbol{p}$ is a ray encoding or $\boldsymbol{p}$ is a closure point encoding, $\boldsymbol{p}$ saturates β if and only if it saturates β_0. As $\boldsymbol{p}$ is an arbitrary ray or closure point encoding in $\mathcal{G}_R \cup \mathcal{G}_C$, we have the required result. □

Lemma A.18 Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. Let also $\beta \in \mathcal{C}_>$ be saturated by the point $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Then $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \text{gen}\bigl((\mathcal{G}_R, \mathcal{G}_C)\bigr)$.

Proof. Let $\mathcal{G} = (R, P)$ and $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr)$. Then, as $\beta \in \mathcal{C}_>$, $s < 0$. Since $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ saturates β, it holds $\langle \boldsymbol{a}, \boldsymbol{v}\rangle = b$, so that for all $e > 0$ we have $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \notin \mathcal{R}$. Therefore we can apply Lemma A.4, taking $e_{\max} = 0$, so that we obtain $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \text{gen}\bigl((\mathcal{G}_R, \mathcal{G}_P \cup \mathcal{G}_C)\bigr)$. By definition of 'gen', we conclude $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \text{gen}\bigl((\mathcal{G}_R, \mathcal{G}_C)\bigr)$. □

Lemma A.19 Let $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$. Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_Y \mathcal{P} \neq \varnothing$. If β is a strongly ϵ-redundant constraint in $\mathcal{C}$, then $\text{con}(\mathcal{C}') \Rrightarrow_Y \mathcal{P}$, where $\mathcal{C}' = \bigl(\mathcal{C} \setminus \{\beta\}\bigr) \cup \{\epsilon \leqslant 1\}$.

Proof. Suppose that $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr)$ is strongly ϵ-redundant in $\mathcal{C}$ so that, by Definition 6.2, we have $\beta \in \mathcal{C}_>$, $\boldsymbol{a} \neq \boldsymbol{0}$ and $s < 0$. Let $\mathcal{R}' = \text{con}(\mathcal{C}')$. As $\mathcal{P} \neq \varnothing$, by Definition 5.1, there exists $(\boldsymbol{w}^{\mathrm{T}}, e'_w)^{\mathrm{T}} \in \mathcal{R}$ for some $e'_w > 0$. As $\mathcal{R}$ is an ϵ-polyhedron, by condition (3) of Definition 5.2, $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Thus, as $\mathcal{R}$ is a convex set, for some $0 < e_w \leqslant 1$, $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}$. Since $e_w \leqslant 1$, we also have $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}'$. We show that, for all $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}' \setminus \mathcal{R}$,

$$e \geqslant 0, \tag{18}$$

$$(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}, \tag{19}$$

$$(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \text{ does not saturate } \beta. \tag{20}$$

We first prove (18). To do this we assume that $e < 0$ and derive a contradiction. Consider the line segment between $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}' \setminus \mathcal{R}$ and $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}' \cap \mathcal{R}$. Then there must be a point $(\boldsymbol{v}_1^{\mathrm{T}}, e_1)^{\mathrm{T}} \in \mathcal{R}$ on this segment that saturates β. As $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}$ satisfies β, $e_w > 0$ and $s < 0$, $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}}$ does not saturate β so that $\boldsymbol{v}_1 \neq \boldsymbol{w}$ and $e_1 < 0$. Thus $(\boldsymbol{v}_1^{\mathrm{T}}, 0)^{\mathrm{T}}$ does not satisfy β and hence $(\boldsymbol{v}_1^{\mathrm{T}}, 0)^{\mathrm{T}} \notin \mathcal{R}$; contradicting condition (3) of Definition 5.2. Therefore (18) holds.

We next show that (19) and (20) hold. Consider the closed segment between $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}' \setminus \mathcal{R}$ and $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}' \cap \mathcal{R}$. As $\mathcal{R}'$ is a convex set, for each $0 \leqslant \sigma \leqslant 1$, we have $(\boldsymbol{v}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} \in \mathcal{R}'$, where

$$(\boldsymbol{v}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} = \sigma(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} + (1-\sigma)(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}. \tag{21}$$

Let τ be the maximum value between 0 and 1 such that $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. We now show that $e_\tau > 0$. Since $e_w > 0$ and, by (18), $e \geqslant 0$, we obtain $e_\tau \geqslant 0$. Moreover, it cannot be $e_\tau = 0$, since otherwise we would have $e = 0$ and $\tau = 1$, contradicting the assumption that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \notin \mathcal{R}$. It follows that $e_\tau > 0$. Suppose that $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ saturates β. Then, by Lemma A.18, $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}} \in \operatorname{gen}\big((\mathcal{G}_R, \mathcal{G}_C)\big)$ so that $\operatorname{sat\_gen}\big(\beta, (\mathcal{G}_R, \mathcal{G}_C)\big) \not\sqsubseteq (\mathcal{G}_R, \varnothing)$. As a consequence, since by hypothesis β is strongly ϵ-redundant in $\mathcal{C}$, by Definition 6.2 there exists $\beta' = \big(\langle \boldsymbol{a}', \boldsymbol{x}\rangle + s' \cdot \epsilon \geqslant b'\big) \in \mathcal{C}'_{>}$ such that

$$\operatorname{sat\_gen}\big(\beta, (\mathcal{G}_R, \mathcal{G}_C)\big) \sqsubseteq \operatorname{sat\_gen}(\beta', \mathcal{G}).$$

Thus the point $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ also saturates β'. Since $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ and $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}$ both satisfy β', we can see that, using (21), $(\boldsymbol{v}_\tau^{\mathrm{T}}, e_\tau)^{\mathrm{T}}$, also satisfies β' so that, as $s' < 0$ and $\langle \boldsymbol{a}', \boldsymbol{v}_\tau\rangle = 0$, we have $e_\tau \leqslant 0$ which is a contradiction. Thus $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ does not saturate β. For all $\tau < \sigma \leqslant 1$, $(\boldsymbol{v}_\sigma^{\mathrm{T}}, 0)^{\mathrm{T}} \notin \mathcal{R}$ so that, as $\mathcal{R}$ is an ϵ-polyhedron, by condition (3) of Definition 5.2, $(\boldsymbol{v}_\sigma^{\mathrm{T}}, e_\sigma)^{\mathrm{T}} \notin \mathcal{R}$. Thus, as $(\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ does not saturate β, we must have $\tau = 1$. Hence, by (21), $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} = (\boldsymbol{v}_\tau^{\mathrm{T}}, 0)^{\mathrm{T}}$ and therefore (19) and (20) hold.

To prove $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$, we show that $\mathcal{R}'$ is an ϵ-polyhedron and $[\![\mathcal{R}]\!] = [\![\mathcal{R}']\!]$.

By taking $\delta = 1$, the inclusion $\mathcal{R}' \subseteq \operatorname{con}\big(\{\epsilon \leqslant \delta\}\big)$ holds trivially, because the constraint $\epsilon \leqslant 1$ has been explicitly added in $\mathcal{C}'$. Thus condition (2) of Definition 5.2 holds. By (19), if $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ is an arbitrary point in $\mathcal{R}'$, we have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Since $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ obviously satisfies the constraint $\epsilon \leqslant 1$, we have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'$, so that condition (3) of Definition 5.2 also holds and $\mathcal{R}'$ is an ϵ-polyhedron.

To prove the inclusion $[\![\mathcal{R}]\!] \subseteq [\![\mathcal{R}']\!]$, let $\boldsymbol{v} \in [\![\mathcal{R}]\!]$. Thus, there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$. By condition (3) of Definition 5.2, we also have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ so that, as $\mathcal{R}$ is a convex set, there exists $0 < e' \leqslant 1$ such that $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}$. Note that $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}}$ satisfies all the constraints in $\mathcal{C}$ and it also satisfies the constraint $\epsilon \leqslant 1$; as a consequence, $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}'$ and $\boldsymbol{v} \in [\![\mathcal{R}']\!]$, as required.

To show the other inclusion $[\![\mathcal{R}']\!] \subseteq [\![\mathcal{R}]\!]$, let $\boldsymbol{v} \in [\![\mathcal{R}']\!]$. Thus, there exists $e > 0$ such that $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}'$. By (19) and (20), we know that $\langle \boldsymbol{a}, \boldsymbol{v}\rangle > b$. Thus, by letting $e' = \min\Big(\big\{e, \frac{b - \langle \boldsymbol{a}, \boldsymbol{v}\rangle}{s}\big\}\Big)$, we obtain $e' > 0$ and $(\boldsymbol{v}^{\mathrm{T}}, e')^{\mathrm{T}} \in \mathcal{R}$. Thus $\boldsymbol{v} \in [\![\mathcal{R}]\!]$.

Suppose next that $\mathcal{R} \Rrightarrow_C \mathcal{P}$. Thus $\mathcal{R} \subseteq \operatorname{con}\big(\{\epsilon \geqslant 0\}\big)$. By the first part of the proof, $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$ so that it remains to show that $\mathcal{R}' \subseteq \operatorname{con}\big(\{\epsilon \geqslant 0\}\big)$. Suppose $(\boldsymbol{u}^{\mathrm{T}}, e_u)^{\mathrm{T}} \in \mathcal{R}'$. If $(\boldsymbol{u}^{\mathrm{T}}, e_u)^{\mathrm{T}} \in \mathcal{R}$, then, as $\mathcal{R}$ is a C-ϵ-polyhedron, $e_u \geqslant 0$. On the other hand, if $(\boldsymbol{u}^{\mathrm{T}}, e_u)^{\mathrm{T}} \in \mathcal{R}' \setminus \mathcal{R}$, then, by (18), we again have $e_u \geqslant 0$. Thus $\operatorname{con}(\mathcal{C}') \subseteq \operatorname{con}\big(\{\epsilon \geqslant 0\}\big)$ and $\mathcal{R}' \Rrightarrow_C \mathcal{P}$.

Finally, suppose that $\mathcal{R} \Rrightarrow_G \mathcal{P}$. By the first part of the proof $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$. Note that, since $\mathcal{P} \neq \varnothing$, we also have $\mathcal{R} \neq \varnothing$ and $\mathcal{R}' \neq \varnothing$. Thus, by Definition 5.9, $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R})$ and, to complete the proof, we need to show that $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R}')$. By Lemma A.14, $\mathcal{C} = \mathcal{C}_{>} \cup \mathcal{C}_{\geqslant} \cup \mathcal{C}_\epsilon$. Since $(\epsilon \leqslant 1) \in \mathcal{C}'_\epsilon$, we also have $\mathcal{C}' = \mathcal{C}'_{>} \cup \mathcal{C}'_{\geqslant} \cup \mathcal{C}'_\epsilon$ so that, again by Lemma A.14, $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R}')$. Thus, $\mathcal{R}' \Rrightarrow_G \mathcal{P}$. □

Lemma A.20 Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ and $\{\boldsymbol{p}, \boldsymbol{p}'\} \subseteq \mathcal{G}_P$, where $\boldsymbol{p} \neq \boldsymbol{p}'$ and $\operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant}) \subseteq \operatorname{sat\_con}(\boldsymbol{p}', \mathcal{C}_{\geqslant})$. Let also $\mathcal{G} = (R, P)$, $\mathcal{G}' = \big(R, P \setminus \{\boldsymbol{p}\}\big)$ and $\mathcal{R}' = \operatorname{gen}(\mathcal{G}')$. Then

$$\mathcal{R} \cap \operatorname{con}\big(\{\epsilon = 0\}\big) = \mathcal{R}' \cap \operatorname{con}\big(\{\epsilon = 0\}\big).$$

Proof. Since $\mathcal{G}' \sqsubset \mathcal{G}$, we have $\mathcal{R}' \subseteq \mathcal{R}$, which implies $\mathcal{R} \cap \operatorname{con}\big(\{\epsilon = 0\}\big) \supseteq \mathcal{R}' \cap \operatorname{con}\big(\{\epsilon = 0\}\big)$. To prove the other inclusion $\mathcal{R} \cap \operatorname{con}\big(\{\epsilon = 0\}\big) \subseteq \mathcal{R}' \cap \operatorname{con}\big(\{\epsilon = 0\}\big)$, we assume that $\boldsymbol{q} = (\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ and show that $\boldsymbol{q}$ is also in $\mathcal{R}'$.

Let $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}}$ and $\boldsymbol{p}' = (\boldsymbol{y}^{\mathrm{T}}, e_y)^{\mathrm{T}}$ so that, since they are both in $\mathcal{G}_P$, we obtain $e_v > 0$ and $e_y > 0$. Consider $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ and $\boldsymbol{p}'_0 = (\boldsymbol{y}^{\mathrm{T}}, 0)^{\mathrm{T}}$. As $\mathcal{R}$ is an ϵ-polyhedron, by condition (3) of Definition 5.2, we have $\{\boldsymbol{p}_0, \boldsymbol{p}'_0\} \subseteq \mathcal{R}$. By Theorem 3.2, $\boldsymbol{p}_0 \in \mathcal{R}$ can be obtained by suitably combining the generators in $\mathcal{G}$; in particular, $\boldsymbol{p}_0$ can be rewritten as $\boldsymbol{p}_0 = \sigma \boldsymbol{p} + (1 - \sigma)\boldsymbol{p}_-$, where $0 \leqslant \sigma \leqslant 1$ and the point $\boldsymbol{p}_- = (\boldsymbol{v}^{\mathrm{T}}, e_-)^{\mathrm{T}}$ is such that $\boldsymbol{p}_- \in \operatorname{gen}(\mathcal{G}') = \mathcal{R}'$. Since $e_v > 0$, we obtain $e_- \leqslant 0$. Since $\boldsymbol{p}' \in \mathcal{R}'$, which is a convex set, then $\mathcal{R}'$ contains the whole segment $[\boldsymbol{p}_-, \boldsymbol{p}']$ and, in particular, by taking $\boldsymbol{q}_1 = (\boldsymbol{w}_1^{\mathrm{T}}, 0)^{\mathrm{T}}$ to be the point on this segment having a zero ϵ coordinate, we obtain $\boldsymbol{q}_1 \in \mathcal{R}'$ (note that there exists exactly one such a $\boldsymbol{q}_1$, because $e_y > 0$). By applying Lemma A.16 to $\boldsymbol{p}$ and $\boldsymbol{p}'$ we obtain $\operatorname{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_{\geqslant}) = \operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant})$ and $\operatorname{sat\_con}(\boldsymbol{p}'_0, \mathcal{C}_{\geqslant}) = \operatorname{sat\_con}(\boldsymbol{p}', \mathcal{C}_{\geqslant})$ so that, by hypothesis, $\operatorname{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_{\geqslant}) \subseteq \operatorname{sat\_con}(\boldsymbol{p}'_0, \mathcal{C}_{\geqslant})$. Thus, as $\boldsymbol{q}_1$ lies on the segment $[\boldsymbol{p}_0, \boldsymbol{p}'_0]$, we obtain $\operatorname{sat\_con}(\boldsymbol{p}_0, \mathcal{C}_{\geqslant}) \subseteq \operatorname{sat\_con}(\boldsymbol{q}_1, \mathcal{C}_{\geqslant})$ and hence,

$$\operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant}) \subseteq \operatorname{sat\_con}(\boldsymbol{q}_1, \mathcal{C}_{\geqslant}). \tag{22}$$

Let $\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{q}_1$. If $\boldsymbol{r} = \boldsymbol{0}$, then $\boldsymbol{q} = \boldsymbol{q}_1 \in \mathcal{R}'$. Otherwise, let $\boldsymbol{r} \neq \boldsymbol{0}$. By hypothesis, $\operatorname{rays}(\mathcal{R}) = \operatorname{rays}(\mathcal{R}')$. Thus, if $\boldsymbol{r} \in \operatorname{rays}(\mathcal{R})$, we also have $\boldsymbol{r} \in \operatorname{rays}(\mathcal{R}')$ so that $\boldsymbol{q} = \boldsymbol{q}_1 + \boldsymbol{r} \in \mathcal{R}'$.

Suppose now that $\boldsymbol{r} \neq \mathbf{0}$ and that $\boldsymbol{r} \notin \operatorname{rays}(\mathcal{R})$. Then there must exist a maximum value $\rho_2 \geqslant 0$ such that $\boldsymbol{q}_1 + \rho_2 \boldsymbol{r} \in \mathcal{R}$. Note that, since $\boldsymbol{q} = \boldsymbol{q}_1 + \boldsymbol{r} \in \mathcal{R}$, it must be $\rho_2 \geqslant 1$. Thus let $\boldsymbol{q}_2 = \boldsymbol{q}_1 + \rho_2 \boldsymbol{r} = (\boldsymbol{w}_2^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Note that, as $\rho_2 \geqslant 1$, $\boldsymbol{q}_2 \neq \boldsymbol{q}_1$. Thus, by choice of ρ_2, there must exist a constraint $\beta \in \mathcal{C}$ that saturates $\boldsymbol{q}_2$ but not $\boldsymbol{q}_1$. Since no constraint in $\mathcal{C}_\epsilon$ can be saturated by $\boldsymbol{q}_2$, we have $\beta \notin \mathcal{C}_\epsilon$. Suppose that $\beta \in \mathcal{C}_>$. Then, by Lemma A.18, $\boldsymbol{q}_2 \in \operatorname{gen}\bigl((\mathcal{G}_R, \mathcal{G}_C)\bigr)$; since $(\mathcal{G}_R, \mathcal{G}_C) \sqsubseteq \mathcal{G}'$, we obtain $\boldsymbol{q}_2 \in \mathcal{R}'$. Suppose now that $\beta \in \mathcal{C}_\geqslant$; then, as $\beta \notin \operatorname{sat\_con}(\boldsymbol{q}_1, \mathcal{C}_\geqslant)$, by (22), we obtain $\beta \notin \operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geqslant)$. Similarly, supposing now $\beta \in \mathcal{C} \setminus (\mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon)$, then $s > 0$ so that, as $\boldsymbol{p}_0 \in \mathcal{R}$, $\beta \notin \operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C})$. In both cases, as $\beta \in \operatorname{sat\_con}(\boldsymbol{q}_2, \mathcal{C})$, $\boldsymbol{q}_2$ is generated by $\operatorname{sat\_gen}\bigl(\beta, (\mathcal{G}_R, \mathcal{G}_C)\bigr)$, so that $\boldsymbol{q}_2 \in \operatorname{gen}(\mathcal{G}') = \mathcal{R}'$. Thus in all cases $\boldsymbol{q}_2 \in \mathcal{R}'$. As $\boldsymbol{q}$ lies on the segment $[\boldsymbol{q}_1, \boldsymbol{q}_2]$ and $\mathcal{R}'$ is a convex set, we have $\boldsymbol{q} \in \mathcal{R}'$ as required. $\square$

Lemma A.21 Let $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$. Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_Y \mathcal{P} \neq \varnothing$. If $\boldsymbol{p}$ is a strongly ϵ-redundant generator in $\mathcal{G} = (R, P)$, then $\operatorname{gen}(\mathcal{G}') \Rrightarrow_Y \mathcal{P}$, where $\mathcal{G}' = \bigl(R, P \setminus \{\boldsymbol{p}\}\bigr)$.

Proof. Let $\mathcal{R}' = \operatorname{gen}(\mathcal{G}')$ and $P' = P \setminus \{\boldsymbol{p}\}$, so that $\mathcal{G}' = (R, P')$. Note that $\mathcal{G}' \sqsubseteq \mathcal{G}$ and hence, as the function 'gen' is monotonic, $\mathcal{R}' \subseteq \mathcal{R}$.

Suppose that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ is strongly ϵ-redundant in $\mathcal{G}$ so that, by Definition 6.2, $\boldsymbol{p} \in P$, $e > 0$ and there exists a point $\boldsymbol{p}' = (\boldsymbol{y}^{\mathrm{T}}, e')^{\mathrm{T}}$ such that $\boldsymbol{p}' \in P'$, $e' > 0$ and

$$\operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geqslant) \subseteq \operatorname{sat\_con}(\boldsymbol{p}', \mathcal{C}). \tag{23}$$

Note that $\boldsymbol{p}' \in \mathcal{R}'$. Letting $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ and $\boldsymbol{p}'_0 = (\boldsymbol{y}^{\mathrm{T}}, 0)^{\mathrm{T}}$, by condition (3) of Definition 5.2, we have $\{\boldsymbol{p}_0, \boldsymbol{p}'_0\} \subseteq \mathcal{R}$. By applying Lemma A.16 twice and using (23), we have

$$\begin{aligned} \operatorname{sat\_con}\bigl(\boldsymbol{p}_0, \mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon\bigr) &= \operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geqslant) \subseteq \operatorname{sat\_con}(\boldsymbol{p}', \mathcal{C}_\geqslant) = \operatorname{sat\_con}(\boldsymbol{p}'_0, \mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon) \\ &\subseteq \operatorname{sat\_con}(\boldsymbol{p}'_0, \mathcal{C}). \end{aligned} \tag{24}$$

Also, by applying Lemma A.20 and using (23) we obtain that, for all $\boldsymbol{w} \in \mathbb{R}^n$,

$$(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R} \iff (\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'. \tag{25}$$

In order to show that $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$, we first prove that $\mathcal{R}'$ is an ϵ-polyhedron. Consider condition (2) of Definition 5.2. As $\mathcal{R}' \subseteq \mathcal{R}$, $\mathcal{R}'$ satisfies condition (2) by taking the same value δ used for $\mathcal{R}$. Consider now condition (3) of Definition 5.2. Let $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}'$. Since $\mathcal{R}' \subseteq \mathcal{R}$, we have $(\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}} \in \mathcal{R}$; since $\mathcal{R}$ is an ϵ-polyhedron, $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Then, by (25), we obtain $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'$. Thus $\mathcal{R}'$ is an ϵ-polyhedron.

To show that $\mathcal{R}'$ is an ϵ-polyhedron for $\mathcal{P}$, it remains to prove that $[\![\mathcal{R}]\!] = [\![\mathcal{R}']\!]$. Since $\mathcal{R}' \subseteq \mathcal{R}$, the inclusion $[\![\mathcal{R}']\!] \subseteq [\![\mathcal{R}]\!]$ holds by monotonicity of function $[\![\cdot]\!]$. We now prove that $[\![\mathcal{R}]\!] \subseteq [\![\mathcal{R}']\!]$. By Proposition 5.5, $[\![\mathcal{R}]\!] = \operatorname{gen}\bigl(\operatorname{gen\_enc}(\mathcal{G})\bigr)$ and $[\![\mathcal{R}']\!] = \operatorname{gen}\bigl(\operatorname{gen\_enc}(\mathcal{G}')\bigr)$. Thus, to show that $[\![\mathcal{R}]\!] \subseteq [\![\mathcal{R}']\!]$, we just have to show that $\boldsymbol{v} \in [\![\mathcal{R}']\!]$. Now let $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr) \in (\mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon) \setminus \operatorname{sat\_con}\bigl(\boldsymbol{p}_0, \mathcal{C}\bigr)$. Then, as $\boldsymbol{p}$ satisfies β, $e > 0$ and $s \leqslant 0$, we have $\langle \boldsymbol{a}, \boldsymbol{v}\rangle \geqslant b$. However, if $s = 0$, then, as $\beta \notin \operatorname{sat\_con}(\boldsymbol{p}_0, \mathcal{C})$, we have $\langle \boldsymbol{a}, \boldsymbol{v}\rangle \neq b$ and, if $s < 0$ then $\langle \boldsymbol{a}, \boldsymbol{v}\rangle > b$. Thus, in both cases, we have $\langle \boldsymbol{a}, \boldsymbol{v}\rangle > b$. Consider the set

$$\left\{ \frac{\langle \boldsymbol{a}, \boldsymbol{y}\rangle - \langle \boldsymbol{a}, \boldsymbol{v}\rangle}{\langle \boldsymbol{a}, \boldsymbol{v}\rangle - b} \in \mathbb{R}_+ \;\middle|\; \begin{array}{l} \beta \in (\mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon) \setminus \operatorname{sat\_con}\bigl(\boldsymbol{p}_0, \mathcal{C}\bigr), \\ \beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geqslant b\bigr), \langle \boldsymbol{a}, \boldsymbol{y}\rangle > \langle \boldsymbol{a}, \boldsymbol{v}\rangle \end{array} \right\}.$$

If this set is non-empty, let ρ be the minimum of this set; otherwise, let $\rho = 1$. Then $\rho > 0$. Consider the affine combination

$$\boldsymbol{q}_\rho = (1 + \rho)\boldsymbol{p}_0 - \rho \boldsymbol{p}'_0.$$

By construction, $\boldsymbol{q}_\rho = (\boldsymbol{w}_\rho^{\mathrm{T}}, 0)^{\mathrm{T}}$ satisfies all the constraints in $(\mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon) \setminus \operatorname{sat\_con}(\boldsymbol{p}_0, \mathcal{C})$. Since $\rho > 0$, $\boldsymbol{p}_0$ lies on the segment $[\boldsymbol{q}_\rho, \boldsymbol{p}'_0]$ so that, by (24), $\boldsymbol{q}_\rho$ saturates every constraint in $\operatorname{sat\_con}(\boldsymbol{p}_0, \mathcal{C})$. By Lemma A.15, $\boldsymbol{q}_\rho$ also satisfies every constraint in $\mathcal{C} \setminus (\mathcal{C}_> \cup \mathcal{C}_\geqslant \cup \mathcal{C}_\epsilon)$. Therefore $\boldsymbol{q}_\rho$ satisfies every constraint in $\mathcal{C}$ so that $\boldsymbol{q}_\rho \in \mathcal{R}$ and hence, by (25), $\boldsymbol{q}_\rho \in \mathcal{R}'$. Letting $\sigma = \frac{1}{1+\rho}$ we obtain $0 < \sigma < 1$ and $\boldsymbol{v} = \sigma \boldsymbol{w}_\rho + (1 - \sigma)\boldsymbol{y}$. Thus $\bigl(\boldsymbol{v}^{\mathrm{T}}, (1 - \sigma)e'\bigr)^{\mathrm{T}} = \sigma \boldsymbol{q}_\rho + (1 - \sigma)\boldsymbol{p}' \in \mathcal{R}'$. As $e' > 0$ and $\sigma < 1$, we have $(1 - \sigma)e' > 0$ and hence, by Definition 5.1, $\boldsymbol{v} \in [\![\mathcal{R}']\!]$ as required.

Suppose next that $\mathcal{R} \Rrightarrow_C \mathcal{P}$. By the first part of the proof $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$. By Definition 5.6, $\mathcal{R} \subseteq \operatorname{con}\bigl(\{\epsilon \geqslant 0\}\bigr)$. Since $\mathcal{R}' \subseteq \mathcal{R}$, we also obtain $\mathcal{R}' \subseteq \operatorname{con}\bigl(\{\epsilon \geqslant 0\}\bigr)$, so that, by Definition 5.6, $\mathcal{R}' \Rrightarrow_C \mathcal{P}$.

Finally, suppose that $\mathcal{R} \Rrightarrow_G \mathcal{P}$. By the first part of the proof, $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$. By hypothesis, $\mathcal{G}' = \bigl(R, P \setminus \{\boldsymbol{p}\}\bigr)$ so that $\operatorname{rays}(\mathcal{R}) = \operatorname{rays}(\mathcal{R}')$. Since $\mathcal{R} \neq \varnothing$, by Definition 5.9, we have that $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R})$. Thus $-\boldsymbol{e}_\epsilon \in \operatorname{rays}(\mathcal{R}')$ and $\mathcal{R}' \Rrightarrow_G \mathcal{P}$, as required. $\square$

Proof of Proposition 6.3. Properties (6) and (7) follow from Lemmas A.19 and A.21, respectively. □

The proof of Proposition 6.4 requires the following lemma.

Lemma A.22 Let $\mathcal{P} \in \mathbb{CP}_n$ and $\boldsymbol{r} \in \mathbb{R}^n$, where $\boldsymbol{r} \neq \boldsymbol{0}$. Suppose that there exists $\boldsymbol{p} \in \mathcal{P}$ such that, for all $\rho \in \mathbb{R}_+$, $\boldsymbol{p} + \rho \boldsymbol{r} \in \mathcal{P}$. Then $\boldsymbol{r} \in \operatorname{rays}(\mathcal{P})$.

Proof. Let $\boldsymbol{q} \in \mathcal{P}$ and let $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr) \in \mathcal{C}$. Then $\langle \boldsymbol{a}, \boldsymbol{q} \rangle \geqslant b$. By hypothesis, for all $\rho \in \mathbb{R}_+$, $\boldsymbol{p} + \rho \boldsymbol{r} \in \mathcal{P}$ so that

$$\langle \boldsymbol{a}, \boldsymbol{p} \rangle + \rho \cdot \langle \boldsymbol{a}, \boldsymbol{r} \rangle \geqslant b.$$

Thus $\langle \boldsymbol{a}, \boldsymbol{r} \rangle \geqslant 0$ and hence

$$\langle \boldsymbol{a}, \boldsymbol{q} \rangle + \rho \cdot \langle \boldsymbol{a}, \boldsymbol{r} \rangle \geqslant b.$$

As this holds for all constraints in $\mathcal{C}$ and all points in $\mathcal{P}$, $\boldsymbol{r} \in \operatorname{rays}(\mathcal{P})$. □

Proof of Proposition 6.4. Since, by hypothesis $\mathcal{P} \neq \varnothing$, we can apply Lemma A.7 twice to obtain

$$(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R} \iff \boldsymbol{v} \in \mathbb{C}(\mathcal{P}) \iff (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'. \tag{26}$$

To prove property (8), let $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geqslant b\bigr)$ be a non-strict inequality encoding in $\mathcal{C}$ and $\boldsymbol{q} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}}$ be any point of $\mathcal{R}'$. By condition (3) of Definition 5.2, $\boldsymbol{q}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}'$ so that, by (26), we have $\boldsymbol{q}_0 \in \mathcal{R}$. In particular, $\boldsymbol{q}_0$ satisfies β and, since the coefficient of ϵ in β is 0, $\boldsymbol{q}$ also satisfies β. As the choice of $\boldsymbol{q} \in \mathcal{R}'$ is arbitrary, $\mathcal{R}' \subseteq \operatorname{con}\bigl(\{\beta\}\bigr)$.

To prove property (9), let $\boldsymbol{r} = (\boldsymbol{s}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{G}_R$. By hypothesis, $\mathcal{R}$ is not empty; thus, considering any $\boldsymbol{q} = (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$, by condition (3) of Definition 5.2, we obtain $\boldsymbol{q}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Since $\boldsymbol{r} \in \operatorname{rays}(\mathcal{R})$, for all $\rho \in \mathbb{R}_+$ we also have

$$\boldsymbol{q}_\rho = \boldsymbol{q}_0 + \rho \boldsymbol{r} = \bigl((\boldsymbol{v} + \rho \boldsymbol{s})^{\mathrm{T}}, 0\bigr)^{\mathrm{T}} \in \mathcal{R}.$$

By (26), we have $\boldsymbol{q}_0 \in \mathcal{R}'$ and $\boldsymbol{q}_\rho \in \mathcal{R}'$, for all $\rho \in \mathbb{R}_+$. Thus, by Lemma A.22, $\boldsymbol{r} \in \operatorname{rays}(\mathcal{R}')$.

To prove property (10), let $\boldsymbol{p} \in \mathcal{G}_C$, so that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$. Then, by (26), $\boldsymbol{p} \in \mathcal{R}'$ as required. □

The proof of Proposition 6.5 is based on some preliminary lemmas. In the following proofs, .for each (strict or non-strict) linear constraint $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr)$, where $\boldsymbol{a} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and $\bowtie \in \{\geqslant, >\}$, the corresponding non-strict constraint is denoted by $\operatorname{geq}(\beta) \stackrel{\text{def}}{=} \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr)$. Similarly, for each constraint system $\mathcal{C}$, we define $\operatorname{geq}(\mathcal{C}) \stackrel{\text{def}}{=} \bigl\{ \operatorname{geq}(\beta) \bigm| \beta \in \mathcal{C} \bigr\}$.

Lemma A.23 Let $\operatorname{con}(\mathcal{C}) = \mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$. Then the constraint $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}$ is singular for $\mathcal{R}$ if and only if $s = 0$ and $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr) \in \operatorname{con\_enc}(\mathcal{C})$ is singular for $\mathcal{P}$.

Proof. Suppose that $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}$ is singular for $\mathcal{R}$. Then every point in $\mathcal{R}$ must saturate β. As $\mathcal{P} \neq \varnothing$, by Definition 5.1, there exists a point $(\boldsymbol{w}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$ with $e \neq 0$; by condition (3) of Definition 5.2, we also have $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$; since both these points saturate β, it must be $s = 0$. By Definition 5.4, we have $\beta_1 = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr) \in \operatorname{con\_enc}(\mathcal{C})$. Let $\boldsymbol{v} \in \mathcal{P}$ so that, by Definition 5.1 and 5.2, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$; as β is singular, $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}$ saturates β so that $\boldsymbol{v}$ saturates β_1. As $\boldsymbol{v} \in \mathcal{P}$ was arbitrary, every point in $\mathcal{P}$ saturates β_1; and hence β_1 is singular for $\mathcal{P}$.

Conversely, suppose there is a constraint $\beta_1 \in \operatorname{con\_enc}(\mathcal{C})$ that is singular for $\mathcal{P} \neq \varnothing$; thus, β_1 must be non-strict so that, for some $\boldsymbol{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, $\beta_1 = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr)$. By Definition 5.4, $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geqslant b\bigr) \in \mathcal{C}$. As β_1 is singular for $\mathcal{P}$, every point $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$ saturates β_1; thus every point $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ saturates β. By condition (3) of Definition 5.2, $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}$ only if $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$; thus every point in $\mathcal{R}$ saturates β; and hence β is singular for $\mathcal{R}$. □

Lemma A.24 Let $\mathcal{R} = \operatorname{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ and suppose that $\mathcal{C}$ is in minimal and orthogonal form, but not in ϵ-minimal form. Then $\mathcal{C}$ contains a strongly ϵ-redundant constraint.

Proof. Suppose $\mathcal{R} = \operatorname{gen}(\mathcal{G})$, where $\mathcal{G} = (R, P)$. Let $\mathcal{C}_1 = \operatorname{con\_enc}(\mathcal{C})$ and $\mathcal{G}_1 = (R_1, P_1, C_1) = \operatorname{gen\_enc}(\mathcal{G})$. It follows from Proposition 5.5 that $\mathcal{P} = \operatorname{con}(\mathcal{C}_1) = \operatorname{gen}(\mathcal{G}_1)$; also, $\mathbb{C}(\mathcal{P}) = \operatorname{con}\bigl(\operatorname{geq}(\mathcal{C}_1)\bigr) = \operatorname{gen}\bigl((R_1, P_1 \cup C_1)\bigr)$.

Let $\operatorname{eq}(\mathcal{C}) = \operatorname{eq}(\mathcal{R}) \cap \mathcal{C}$ and $\operatorname{eq}(\mathcal{C}_1) = \operatorname{eq}(\mathcal{P}) \cap \mathcal{C}_1$ so that, by Lemma A.23, $\operatorname{eq}(\mathcal{C}_1) = \operatorname{con\_enc}\bigl(\operatorname{eq}(\mathcal{C})\bigr)$. Let also $\operatorname{ineq}(\mathcal{C}) = \mathcal{C} \setminus \operatorname{eq}(\mathcal{C})$ and $\operatorname{ineq}(\mathcal{C}_1) = \mathcal{C}_1 \setminus \operatorname{eq}(\mathcal{C}_1)$. As $\mathcal{P} \neq \varnothing$, $\operatorname{eq}(\mathcal{C}_1)$ contains no strict constraints so that, by Definition 5.4, we also have $\operatorname{ineq}(\mathcal{C}_1) = \operatorname{con\_enc}\bigl(\operatorname{ineq}(\mathcal{C})\bigr)$. Since $\mathcal{C}$ is in orthogonal form, for all $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant$

$b\big) \in \mathrm{eq}(\mathcal{C})$ and $\beta' = \big(\langle \boldsymbol{a}', \boldsymbol{x} \rangle + s' \cdot \epsilon \geqslant b'\big) \in \mathrm{ineq}(\mathcal{C})$ we have $\big\langle \mathrm{slope}(\beta), \mathrm{slope}(\beta') \big\rangle = 0$; by Lemma A.23, $s = 0$ and hence $\langle \boldsymbol{a}, \boldsymbol{a}' \rangle = 0$, so that $\mathcal{C}_1$ is in orthogonal form too.

As $\mathcal{C}$ is not in ϵ-minimal form, by Definition 6.1, $\mathcal{C}_1$ is not in minimal form. Thus there exists a constraint $\eta = \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\big) \in \mathcal{C}_1$, where $\bowtie \in \{\geqslant, >\}$, which is redundant for $\mathcal{C}_1$. By Definition 5.4, there exists $s \leqslant 0$ such that $\beta = \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\big) \in \mathcal{C}$. We distinguish two main cases:

Case 1. $\mathrm{geq}(\eta)$ is saturated by none of the closure points of $\mathcal{P}$;
Case 2. $\mathrm{geq}(\eta)$ is saturated by at least one closure point of $\mathcal{P}$.

Consider first **Case 1**. By the assumption for this case, we obtain $\mathrm{sat\_gen}\big(\mathrm{geq}(\eta), (R_1, P_1 \cup C_1)\big) \sqsubseteq (R_1, \varnothing)$. It is easy to observe that η is non-singular for $\mathcal{P}$, because any singular constraint has to be saturated by all the points of $\mathbb{C}(\mathcal{P}) \neq \varnothing$, including those in $P_1 \cup C_1$. We now show that η is a strict constraint. To do this we assume by contraposition η is non-strict. Thus $\eta = \mathrm{geq}(\eta)$ and, by Definition 5.4, $s = 0$. Since no point in $P_1 \cup C_1$ saturates η, then no point in P saturates β. Thus β is redundant in $\mathcal{C}$, contradicting the hypothesis that $\mathcal{C}$ is in minimal form. Therefore, η is a strict constraint, so that, by Definition 5.4, $s < 0$ and $\beta \in \mathcal{C}_>$. The above saturation assumption implies $\mathrm{sat\_gen}\big(\beta, (\mathcal{G}_R, \mathcal{G}_C)\big) \sqsubseteq (\mathcal{G}_R, \varnothing)$ and hence, by Definition 6.2, β is strongly ϵ-redundant in $\mathcal{C}$.

Consider now **Case 2**. As η is redundant in $\mathcal{C}_1$ and $\mathrm{geq}(\eta)$ is saturated by at least one closure point of $\mathcal{P}$, there exists $k > 0$ such that $\boldsymbol{a} = \sum_{i=1}^{k} \rho_i \boldsymbol{a}_i$ and $b = \sum_{i=1}^{k} \rho_i b_i$ where, for $1 \leqslant i \leqslant k$, we have $\rho_i > 0$, $\eta_i = \big(\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \bowtie_i b_i\big) \in \mathcal{C}_1 \setminus \{\eta\}$ and $\bowtie_i \in \{\geqslant, >\}$. Therefore, for $1 \leqslant i \leqslant k$, we have

$$\mathrm{sat\_gen}\big(\mathrm{geq}(\eta), (R_1, P_1 \cup C_1)\big) \sqsubseteq \mathrm{sat\_gen}\big(\mathrm{geq}(\eta_i), (R_1, P_1 \cup C_1)\big). \tag{27}$$

By Definition 5.4, for $1 \leqslant i \leqslant k$, there exists $s_i \leqslant 0$ such that $\beta_i = \big(\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + s_i \cdot \epsilon \geqslant b_i\big) \in \mathcal{C} \setminus \{\beta\}$.

Suppose first that $\{\eta_1, \dots, \eta_k\} \subseteq \mathrm{eq}(\mathcal{C}_1)$ so that $\eta \in \mathrm{eq}(\mathcal{C}_1)$. By Lemma A.23, $\beta \in \mathrm{eq}(\mathcal{C})$, $s = 0$ and, for each $1 \leqslant i \leqslant k$, $\beta_i \in \mathrm{eq}(\mathcal{C}) \setminus \{\beta\}$ and $s_i = 0$. As a consequence, $\mathrm{slope}(\beta) = \sum_{i=1}^{k} \big(\rho_i \, \mathrm{slope}(\beta_i)\big)$ so that β is redundant in $\mathrm{eq}(\mathcal{C})$; contradicting the hypothesis that $\mathcal{C}$ is in minimal form. Thus $\{\eta_1, \dots, \eta_k\} \setminus \mathrm{eq}(\mathcal{C}_1) \neq \varnothing$.

It follows that $\{\eta_1, \dots, \eta_k\} \cap \mathrm{ineq}(\mathcal{C}_1) \neq \varnothing$, so that $\eta \in \mathrm{ineq}(\mathcal{C}_1)$ and hence, by Lemma A.23, $\beta \in \mathrm{ineq}(\mathcal{C})$. Since, by the hypothesis for this case, $\mathrm{geq}(\eta)$ is saturated by at least one closure point of $\mathcal{P}$, if η is a strict constraint, then at least one constraint in $\{\eta_1, \dots, \eta_k\} \cap \mathrm{ineq}(\mathcal{C}_1)$ must be strict. Thus we will assume, without losing generality, that $\eta_k \in \mathrm{ineq}(\mathcal{C}_1)$ and η_k is strict if η is strict. By Lemma A.23, we have $\beta_k \in \mathrm{ineq}(\mathcal{C}) \setminus \{\beta\}$.

We show that η is a strict constraint. To do this we assume by contraposition η is non-strict. Thus $\eta = \mathrm{geq}(\eta)$ and, by Definition 5.4, $s = 0$. Let $\beta'_k = \big(\langle \boldsymbol{a}_k, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geqslant b_k\big)$; by Proposition 5.3, $\mathcal{R} \subseteq \mathrm{con}\big(\{\beta'_k\}\big)$, so that β'_k is a valid constraint for $\mathcal{R}$. Consider the generator systems $\mathcal{H} = (R_\beta, P_\beta) = \mathrm{sat\_gen}(\beta, \mathcal{G})$ and $\mathcal{H}'_k = (R'_k, P'_k) = \mathrm{sat\_gen}(\beta'_k, \mathcal{G})$ defined by the saturators of β and β'_k; we now prove that $\mathcal{H} \sqsubseteq \mathcal{H}'_k$. Let $\boldsymbol{r} = (\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}} \in R_\beta$. As $s = 0$, $\boldsymbol{v}$ saturates η. If $\boldsymbol{v} = \boldsymbol{0}$, then we trivially obtain $\boldsymbol{r} \in R'_k$. Suppose now that $\boldsymbol{v} \neq \boldsymbol{0}$; thus, by Lemma A.2, we have $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathrm{rays}(\mathcal{R})$ and, by Lemma A.8, $\boldsymbol{v} \in \mathrm{rays}(\mathcal{P})$, which also implies $\boldsymbol{v} \in \mathrm{rays}\big(\mathbb{C}(\mathcal{P})\big)$. Since $\boldsymbol{v}$ saturates η, it can also be obtained as a non-negative combination of the rays in R_1 that saturate η so that, by (27), $\boldsymbol{v}$ also saturates $\mathrm{geq}(\eta_k)$. As a consequence, $\boldsymbol{r}$ saturates β'_k and hence $\boldsymbol{r} \in R'_k$. Now let $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}} \in P_\beta$. By Definition 5.2, $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}$ so that, by Lemma A.7, $\boldsymbol{v} \in \mathbb{C}(\mathcal{P})$. As $s = 0$, $\boldsymbol{v}$ saturates η. Thus, by (27), $\boldsymbol{v}$ also saturates $\mathrm{geq}(\eta_k)$, so that $\boldsymbol{p}$ saturates β'_k and $\boldsymbol{p} \in P'_k$. Hence $\mathcal{H} \sqsubseteq \mathcal{H}'_k$. Note that, as β'_k is non-singular, $\mathcal{C}$ is in minimal form and $\beta \in \mathcal{C}$, we cannot have $\mathcal{H} \sqsubset \mathcal{H}'_k$, so that $\mathcal{H} = \mathcal{H}'_k$. Since $\mathcal{C}$ is also in orthogonal form, $\beta = \beta'_k$. Therefore, by Definition 5.4, as $\{\eta, \eta_k\} \subseteq \mathcal{C}_1 = \mathrm{con\_enc}(\mathcal{C})$, $\eta = \eta_k$; a contradiction. It follows that η must be a strict constraint.

By construction, since η is strict, η_k is also strict so that, by Definition 5.4, we have $s < 0$ and $s_k < 0$; hence $\{\beta, \beta_k\} \subseteq \mathcal{C}_>$. Suppose that the constraint β is saturated by a ray encoding $\boldsymbol{r} = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{G}_R$. Then, $\boldsymbol{v} \in R_1$ saturates $\mathrm{geq}(\eta)$; by (27), $\boldsymbol{v}$ also saturates $\mathrm{geq}(\eta_k)$ and hence, $\boldsymbol{r}$ saturates β_k. Similarly, suppose that β is saturated by a closure point encoding $\boldsymbol{p}_0 = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{G}_C$. By Definition 5.4, either $\boldsymbol{v} \in C_1$ or $\boldsymbol{v} \in P_1$; in both cases, $\boldsymbol{v}$ saturates $\mathrm{geq}(\eta)$ and, by (27), $\boldsymbol{v}$ also saturates $\mathrm{geq}(\eta_k)$; hence, the closure point encoding $\boldsymbol{p}_0$ saturates β_k. Thus we obtain $\mathrm{sat\_gen}\big(\beta, (\mathcal{G}_R, \mathcal{G}_C)\big) \sqsubseteq \mathrm{sat\_gen}(\beta_k, \mathcal{G})$ so that, by Definition 6.2, β is strongly ϵ-redundant in $\mathcal{C}$. □

Lemma A.25 Let $\mathcal{R} = \mathrm{gen}(\mathcal{G}) \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ and suppose that $\mathcal{G}$ is in minimal and orthogonal form, but not in ϵ-minimal form. Then $\mathcal{G}$ contains a strongly ϵ-redundant generator.

Proof. Let $\mathcal{C}_1 = \mathrm{con\_enc}(\mathcal{C})$. Observe that, by Definition 5.4, for each constraint $\big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\big) \in \mathcal{C}$ where $\boldsymbol{a} \neq \boldsymbol{0} \in \mathbb{R}^n$ and $s, b \in \mathbb{R}$, we have $\big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\big) \in \mathrm{geq}(\mathcal{C}_1)$. Let $\mathcal{G} = (R, P)$ and suppose that $\mathcal{G}_1 = (R_1, P_1, C_1) = \mathrm{gen\_enc}(\mathcal{G})$. It follows from Proposition 5.5 that $\mathcal{P} = \mathrm{con}(\mathcal{C}_1) = \mathrm{gen}(\mathcal{G}_1)$; moreover, $\mathbb{C}(\mathcal{P}) = \mathrm{con}\big(\mathrm{geq}(\mathcal{C}_1)\big) = \mathrm{gen}\big((R_1, P_1 \cup C_1)\big)$. By hypothesis, $\mathcal{P} \neq \varnothing$ so that, by Definition 5.1, there exists a point in $\mathcal{R}$ having an ϵ coordinate strictly greater than 0. Since $\mathcal{G}$ is not in ϵ-minimal form, by Definition 6.1, the generator system $\mathcal{G}_1$ is not in minimal form. Thus there exists a redundant generator $\boldsymbol{v}$ in $\mathcal{G}_1$. To prove the thesis, we will show that $\boldsymbol{v} \in P_1$ and, for some $e_v > 0$, $(\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}} \in \mathcal{G}_P$ is strong ϵ-redundant in $\mathcal{G}$.

We first show that $\boldsymbol{v} \notin R_1$. To do this we assume by contraposition $\boldsymbol{v} \in R_1$. Since it is redundant in $\mathcal{G}_1$, then $\boldsymbol{v}$ is a non-negative combination of the rays in $R_1 \setminus \{\boldsymbol{v}\}$. By Definition 5.4, for each $\boldsymbol{w} \in R_1$ there exists $(\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R$. As a consequence $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R$ is a non-negative combination of the rays in $R \setminus \{(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}}\}$, so that $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R$ is redundant in $\mathcal{G}$; contradicting the hypothesis that $\mathcal{G}$ is in minimal form.

It follows that $\boldsymbol{v} \in P_1 \cup C_1$. By Definition 5.4, $P_1 \cap C_1 = \varnothing$. Thus, since $\boldsymbol{v}$ is redundant in $\mathcal{G}_1$, there must exist another vector $\boldsymbol{w} \in (P_1 \cup C_1) \setminus \{\boldsymbol{v}\}$ such that

$$\operatorname{sat\_con}\bigl(\boldsymbol{v}, \operatorname{geq}(\mathcal{C}_1)\bigr) \subseteq \operatorname{sat\_con}\bigl(\boldsymbol{w}, \operatorname{geq}(\mathcal{C}_1)\bigr). \tag{28}$$

By Definition 5.4, there exist $e_v, e_w \in \mathbb{R}_+$ such that $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, e_v)^{\mathrm{T}}$, $\boldsymbol{q} = (\boldsymbol{w}^{\mathrm{T}}, e_w)^{\mathrm{T}}$ and $\{\boldsymbol{p}, \boldsymbol{q}\} \subseteq P$.

We next show that $\boldsymbol{v} \notin C_1$. To do this we assume by contraposition $\boldsymbol{v} \in C_1$ so that, by Definition 5.4, $e_v = 0$. Let $\boldsymbol{q}_0 = (\boldsymbol{w}^{\mathrm{T}}, 0)^{\mathrm{T}}$ so that, by condition (3) of Definition 5.2, $\boldsymbol{q}_0 \in \mathcal{R}$. Consider $\boldsymbol{\ell} \in \operatorname{lines}(\mathcal{R})$. Since $\mathcal{G}$ is in orthogonal form and $\boldsymbol{q} \in P$, we have $\langle \boldsymbol{q}, \boldsymbol{\ell} \rangle = 0$. Since $\{\boldsymbol{\ell}, -\boldsymbol{\ell}\} \subseteq \operatorname{rays}(\mathcal{R})$, by applying Lemma A.3 twice we obtain $\boldsymbol{\ell} = (\boldsymbol{\ell}_1^{\mathrm{T}}, 0)^{\mathrm{T}}$. Thus $\langle \boldsymbol{w}, \boldsymbol{\ell}_1 \rangle = 0$, so that $\langle \boldsymbol{q}_0, \boldsymbol{\ell} \rangle = \langle \boldsymbol{w}, \boldsymbol{\ell}_1 \rangle + 0 \cdot 0 = 0$. As this holds for all vectors in $\operatorname{lines}(\mathcal{R})$, we obtain $\boldsymbol{q}_0 \in \operatorname{lines}(\mathcal{R})^{\perp}$. Consider now $\beta = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\bigr) \in \operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C})$. Suppose first that $\boldsymbol{a} \neq \boldsymbol{0}$. Then, by Definition 5.4, there exists $\eta \in \mathcal{C}_1$ such that $\operatorname{geq}(\eta) = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b\bigr) \in \operatorname{geq}(\mathcal{C}_1)$. Since $\boldsymbol{p}$ saturates β and $e_v = 0$, then $\boldsymbol{v}$ saturates $\operatorname{geq}(\eta)$. Thus, by (28), $\boldsymbol{w}$ saturates $\operatorname{geq}(\eta)$. Then $\beta \in \operatorname{sat\_con}(\boldsymbol{q}_0, \mathcal{C})$. On the other hand, if $\boldsymbol{a} = \boldsymbol{0}$, since $\boldsymbol{p}$ saturates β and $e_v = 0$, we obtain $\langle \boldsymbol{0}, \boldsymbol{v} \rangle + s \cdot 0 = b$. Thus $b = 0$ and again $\beta \in \operatorname{sat\_con}(\boldsymbol{q}_0, \mathcal{C})$. Therefore we obtain $\operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}) \subseteq \operatorname{sat\_con}(\boldsymbol{q}_0, \mathcal{C})$. From this property, since $\boldsymbol{q}_0 \in \operatorname{lines}(\mathcal{R})^{\perp}$ and $\mathcal{G}$ is a generator system in minimal form, we obtain $\boldsymbol{p} = \boldsymbol{q}_0$ and hence $\boldsymbol{v} = \boldsymbol{w}$, which is a contradiction.

It follows that $\boldsymbol{v} \in P_1$ so that, by Definition 5.4, $e_v > 0$ and $\boldsymbol{p} \in \mathcal{G}_P$. Consider the constraint system

$$\mathcal{C}_1' = \operatorname{con\_enc}\bigl(\operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant})\bigr) = \Bigl\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geqslant b \Bigm| \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geqslant b\bigr) \in \operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant}) \Bigr\}.$$

Then $\boldsymbol{v}$ saturates all the constraints in $\mathcal{C}_1'$. As $\boldsymbol{p} \in \mathcal{R}$ and $e_v > 0$, for all $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geqslant b\bigr) \in \operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant})$ and all $s < 0$, we have $\bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geqslant b\bigr) \notin \mathcal{C}$. Thus, by Definition 5.4, $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and hence $\mathcal{P} \subseteq \operatorname{con}(\mathcal{C}_1')$. Let $\mathcal{G}' = \bigl(R, P \setminus \{\boldsymbol{p}\}\bigr)$ and $\mathcal{G}_1' = \operatorname{gen\_enc}(\mathcal{G}') = \bigl(R_1, P_1 \setminus \{\boldsymbol{v}\}, C_1\bigr)$. As $\boldsymbol{v} \in \mathcal{P}$ is redundant in $\mathcal{G}_1$, we have $\operatorname{gen}(\mathcal{G}_1) = \operatorname{gen}(\mathcal{G}_1')$ and, in particular, $\boldsymbol{v} \in \operatorname{gen}(\mathcal{G}_1')$. Thus $\boldsymbol{v} = \sigma \boldsymbol{y} + (1 - \sigma)\boldsymbol{z}$ where $0 < \sigma < 1$, $\boldsymbol{y} \in P_1 \setminus \{\boldsymbol{v}\}$ and $\boldsymbol{z} \in \mathbb{C}(\mathcal{P})$. This implies that $\operatorname{sat\_con}\bigl(\boldsymbol{v}, \mathcal{C}_1\bigr) \subseteq \operatorname{sat\_con}\bigl(\boldsymbol{y}, \mathcal{C}_1\bigr)$; in particular, since $\mathcal{C}_1' \subseteq \mathcal{C}_1$, we obtain $\operatorname{sat\_con}(\boldsymbol{v}, \mathcal{C}_1') \subseteq \operatorname{sat\_con}(\boldsymbol{y}, \mathcal{C}_1')$. By Definition 5.4, $\boldsymbol{p}' = (\boldsymbol{y}^{\mathrm{T}}, e_y)^{\mathrm{T}} \in P \setminus \{\boldsymbol{p}\}$, where $e_y > 0$, so that $\boldsymbol{p}' \in \mathcal{G}_P'$. Observe that, as $\boldsymbol{y}$ saturates every constraint in $\mathcal{C}_1'$, $\boldsymbol{p}'$ saturates every constraint in $\operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant})$. It follows that $\operatorname{sat\_con}(\boldsymbol{p}, \mathcal{C}_{\geqslant}) \subseteq \operatorname{sat\_con}(\boldsymbol{p}', \mathcal{C}_{\geqslant})$ and, by Definition 6.2, $\boldsymbol{p}$ is strongly ϵ-redundant in $\mathcal{G}$. □

Proof of Proposition 6.5. Items (1) and (2) follow from Lemmas A.24 and A.25, respectively. □

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