ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Implementing Combiners

Parallel Programming in Scala

Aleksandar Prokopec

Builders

Builders are used in sequential collection methods:

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```
trait Builder[T, Repr] {
  def +=(elem: T): this.type
  def result: Repr
}
```

Combiners

```scala
trait Combiner[T, Repr] extends Builder[T, Repr] {
  def combine(that: Combiner[T, Repr]): Combiner[T, Repr]
}
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Combiners

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How can we implement the `combine` method *efficiently*?

Combiners

- when `Repr` is a set or a map, `combine` represents union

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- when `Repr` is a set or a map, `combine` represents union
- when `Repr` is a sequence, `combine` represents concatenation

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The `combine` operation must be efficient, i.e. execute in $O(\log n + \log m)$ time, where n and m are the sizes of two input combiners.

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The `combine` operation must be efficient, i.e. execute in $O(\log n + \log m)$ time, where n and m are the sizes of two input combiners.

Question: Is the method `combine` *efficient*?

```
def combine(xs: Array[Int], ys: Array[Int]): Array[Int] = {
  val r = new Array[Int](xs.length + ys.length)
  Array.copy(xs, 0, r, 0, xs.length)
  Array.copy(ys, 0, r, xs.length, ys.length)
  r
}
```

- Yes.
- No.

Array Concatenation

Arrays cannot be efficiently concatenated.

Sets

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Most set implementations do not have efficient union operation.

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- array lists – amortized $O(1)$ append, $O(1)$ random accesss, otherwise $O(n)$

Mutable linked list can have $O(1)$ concatenation, but for most sequences, concatenation is $O(n)$.

Parallel Two-Phase Construction

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Two-Phase Construction

Most data structures can be constructed in parallel using *two-phase construction*.

The *intermediate data structure* is a data structure that:

- has an efficient `combine` method – $O(\log n + \log m)$ or better
- has an efficient `+=` method
- can be converted to the resulting data structure in $O(n/P)$ time

Example: Array Combiner

Let's implement a combiner for arrays.

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Two arrays cannot be efficiently concatenated, so we will do a *two-phase construction*.

```
class ArrayCombiner[T <: AnyRef: ClassTag](val parallelism: Int) {
  private var numElems = 0
  private val buffers = new ArrayBuffer[ArrayBuffer[T]]
  buffers += new ArrayBuffer[T]
```

Example: Array Combiner

First, we implement the += method:

```
def +=(x: T) = {
  buffers.last += x
  numElems += 1
  this
}
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```
def +=(x: T) = {
  buffers.last += x
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```

Amortized $O(1)$, low constant factors – as efficient as an array buffer.

Example: Array Combiner

Next, we implement the `combine` method:

```
def combine(that: ArrayCombiner[T]) = {
  buffers ++= that.buffers
  numElems += that.numElems
  this
}
```

Example: Array Combiner

Next, we implement the `combine` method:

```
def combine(that: ArrayCombiner[T]) = {
  buffers ++= that.buffers
  numElems += that.numElems
  this
}
```

$O(P)$, assuming that `buffers` contains no more than $O(P)$ nested array buffers.

Example: Array Combiner

Finally, we implement the `result` method:

```
def result: Array[T] = {
  val array = new Array[T](numElems)
  val step = math.max(1, numElems / parallelism)
  val starts = (0 until numElems by step) :+ numElems
  val chunks = starts.zip(starts.tail)
  val tasks = for ((from, end) <- chunks) yield task {
    copyTo(array, from, end)
  }
  tasks.foreach(_.join())
  array
}
```

Benchmark

Demo – we will test the performance of the `aggregate` method:

```
xs.par.aggregate(newCombiner)(_ += _, _ combine _).result
```

Two-Phase Construction for Arrays

Two-phase construction works for in a similar way for other data structures. First, partition the elements, then construct parts of the final data structure in parallel:

1. partition the indices into subintervals
2. initialize the array in parallel

Two-Phase Construction for Hash Tables

1. partition the hash codes into buckets
2. allocate the table, and map hash codes from different buckets into different regions

Two-Phase Construction for Search Trees

1. partition the elements into non-overlapping intervals according to their ordering
2. construct search trees in parallel, and link non-overlapping trees

Two-Phase Construction for Spatial Data Structures

1. spatially partition the elements
2. construct non-overlapping subsets and link them

Implementing combiners

How can we implement combiners?

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1. Two-phase construction – the combiner uses an intermediate data structure with an efficient `combine` method to partition the elements. When `result` is called, the final data structure is constructed in parallel from the intermediate data structure.

Implementing combiners

How can we implement combiners?

1. Two-phase construction – the combiner uses an intermediate data structure with an efficient `combine` method to partition the elements. When `result` is called, the final data structure is constructed in parallel from the intermediate data structure.
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Implementing combiners

How can we implement combiners?

1. Two-phase construction – the combiner uses an intermediate data structure with an efficient `combine` method to partition the elements. When `result` is called, the final data structure is constructed in parallel from the intermediate data structure.
2. An efficient concatenation or union operation – a preferred way when the resulting data structure allows this.
3. Concurrent data structure – different combiners share the same underlying data structure, and rely on *synchronization* to correctly update the data structure when `+=` is called.

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Conc-Trees

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List Data Type

Let's recall the list data type in functional programming.

```
sealed trait List[+T] {
  def head: T
  def tail: List[T]
}

case class ::[T](head: T, tail: List[T])
extends List[T]

case object Nil extends List[Nothing] {
  def head = sys.error(”empty list”)
  def tail = sys.error(”empty list”)
}
```

List Data Type

How do we implement a `filter` method on lists?

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How do we implement a `filter` method on lists?

```scala
def filter[T](lst: List[T])(p: T => Boolean): List[T] = lst match {
  case x :: xs if p(x) => x :: filter(xs)(p)
  case x :: xs => filter(xs)(p)
  case Nil => Nil
}
```

Trees

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Trees allow parallel computations – their subtrees can be traversed in parallel.

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Trees allow parallel computations – their subtrees can be traversed in parallel.

```
sealed trait Tree[+T]

case class Node[T](left: Tree[T], right: Tree[T])
extends Tree[T]

case class Leaf[T](elem: T) extends Tree[T]

case object Empty extends Tree[Nothing]
```

Filter On Trees

How do we implement a filter method on trees?

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How do we implement a filter method on trees?

```
def filter[T](t: Tree[T])(p: T => Boolean): Tree[T] = t match {
  case Node(left, right) => Node(parallel(filter(left)(p), filter(right)(p)))
  case Leaf(elem) => if (p(elem)) t else Empty
  case Empty => Empty
}
```

Filter On Trees

How do we implement a filter method on trees?

```
def filter[T](t: Tree[T])(p: T => Boolean): Tree[T] = t match {
  case Node(left, right) => Node(parallel(filter(left)(p), filter(right)(p)))
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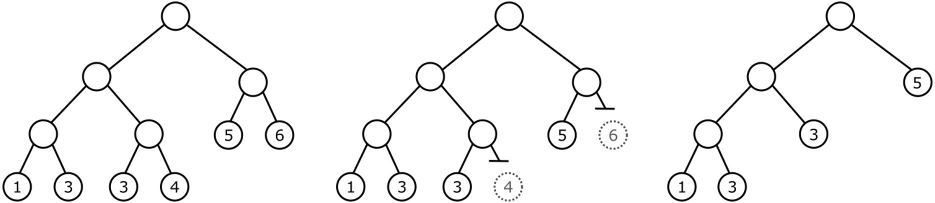


Conc Data Type

Trees are not good for parallelism unless they are balanced.

Conc Data Type

Trees are not good for parallelism unless they are balanced.

Let's devise a data type called `Conc`, which represents balanced trees:

```
sealed trait Conc[+T] {
  def level: Int
  def size: Int
  def left: Conc[T]
  def right: Conc[T]
}
```

In parallel programming, this data type is known as the *conc-list* (introduced in the Fortress language).

Conc Data Type

Concrete implementations of the `Conc` data type:

```
case object Empty extends Conc[Nothing] {
  def level = 0
  def size = 0
}
class Single[T](val x: T) extends Conc[T] {
  def level = 0
  def size = 1
}
case class <>[T](left: Conc[T], right: Conc[T]) extends Conc[T] {
  val level = 1 + math.max(left.level, right.level)
  val size = left.size + right.size
}
```

Conc Data Type Invariants

In addition, we will define the following *invariants* for Conc-trees:

1. A <> node can never contain `Empty` as its subtree.
2. The `level` difference between the left and the right subtree of a <> node is always 1 or less.

Conc Data Type Invariants

In addition, we will define the following *invariants* for Conc-trees:

1. A <> node can never contain `Empty` as its subtree.
2. The `level` difference between the left and the right subtree of a <> node is always 1 or less.

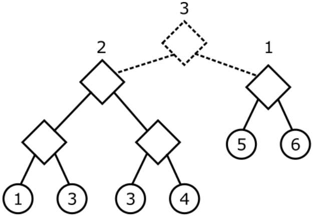
We will rely on these invariants to implement concatenation:

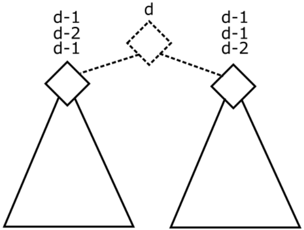
```
def <>(that: Conc[T]): Conc[T] = {
  if (this == Empty) that
  else if (that == Empty) this
  else concat(this, that)
}
```

Concatenation with the Conc Data Type

Concatenation needs to consider several cases.

First, the two trees could have height difference 1 or less:





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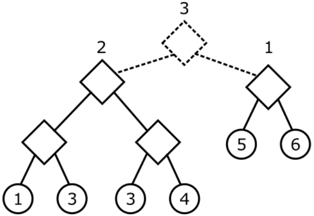

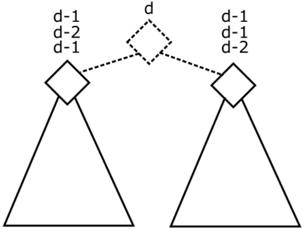


```
def concat[T](xs: Conc[T], ys: Conc[T]): Conc[T] = {
  val diff = ys.level - xs.level
  if (diff >= -1 && diff <= 1) new <>(xs, ys)
  else if (diff < -1) {
```

Concatenation with the Conc Data Type

Otherwise, let's assume that the left tree is higher than the right one.

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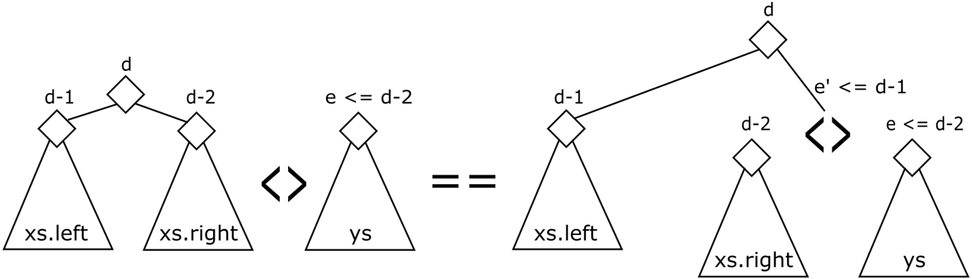
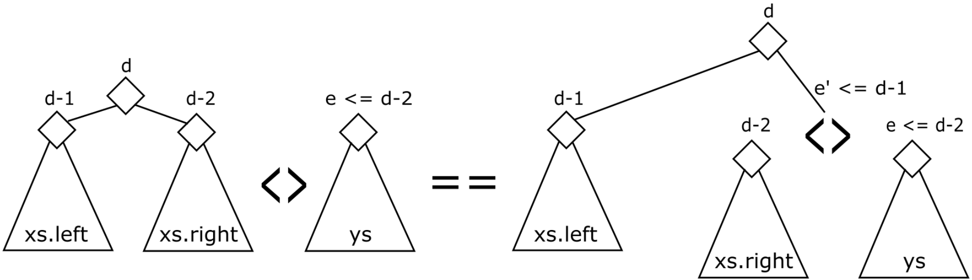


Case 1: The left tree is left-leaning.

Recursively concatenate the right subtree.

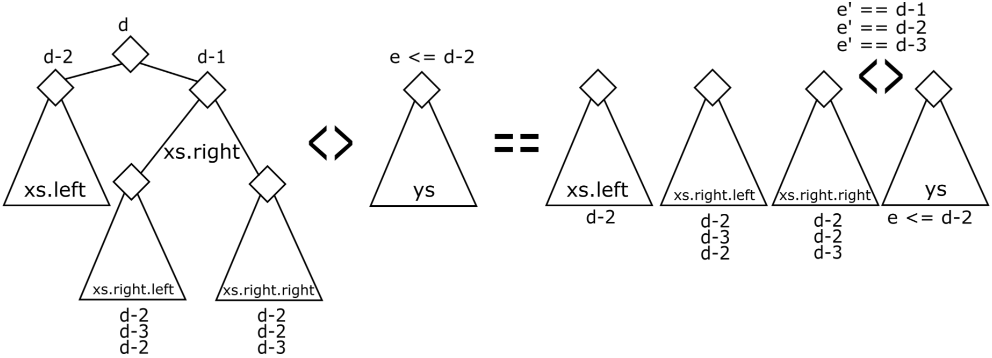
Concatenation with the Conc Data Type

```
if (xs.left.level >= xs.right.level) {
  val nr = concat(xs.right, ys)
  new <>(xs.left, nr)
} else {
```



Concatenation with the Conc Data Type

Case 2: The left tree is right-leaning.



Concatenation with the Conc Data Type

```
} else {
  val nrr = concat(xs.right.right, ys)
  if (nrr.level == xs.level - 3) {
    val nl = xs.left
    val nr = new <>(xs.right.left, nrr)
    new <>(nl, nr)
  } else {
    val nl = new <>(xs.left, xs.right.left)
    val nr = nrr
    new <>(nl, nr)
  }
}
```

Summary

Question: What is the complexity of <> method?

- $O(\log n)$
- $O(h_1 - h_2)$
- $O(n)$
- $O(1)$

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- $O(\log n)$
- $O(h_1 - h_2)$
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Concatenation takes $O(h_1 - h_2)$ time, where h_1 and h_2 are the heights of the two trees.

Amortized Conc-Tree Appends

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Constant Time Appends in Conc-Trees

Let's use Conc-Trees to implement a `Combiner`.

How could we implement `+=` method?

```
var xs: Conc[T] = Empty
def +=(elem: T) {
  xs = xs <> Single(elem)
}
```

Constant Time Appends in Conc-Trees

Let's use Conc-Trees to implement a `Combiner`.

How could we implement `+=` method?

```
var xs: Conc[T] = Empty
def +=(elem: T) {
  xs = xs <> Single(elem)
}
```

This takes $O(\log n)$ time – can we do better than that?

Constant Time Appends in Conc-Trees

To achieve $O(1)$ appends with low constant factors, we need to extend the Conc-Tree data structure.

We will introduce a new `Append` node with different semantics:

```
case class Append[T](left: Conc[T], right: Conc[T]) extends Conc[T] {
  val level = 1 + math.max(left.level, right.level)
  val size = left.size + right.size
}
```

Constant Time Appends in Conc-Trees

One possible `appendLeaf` implementation:

```
def appendLeaf[T](xs: Conc[T], y: T): Conc[T] = Append(xs, new Single(y))
```

Constant Time Appends in Conc-Trees

One possible `appendLeaf` implementation:

```
def appendLeaf[T](xs: Conc[T], y: T): Conc[T] = Append(xs, new Single(y))
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Can we still do $O(\log n)$ concatenation? I.e. can we eliminate `Append` nodes in $O(\log n)$ time?

Constant Time Appends in Conc-Trees

One possible `appendLeaf` implementation:

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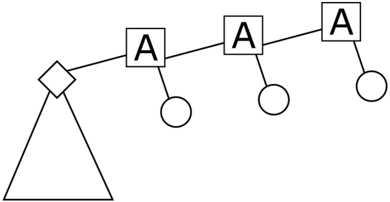
def appendLeaf[T](xs: Conc[T], y: T): Conc[T] = Append(xs, new Single(y))
```

Can we still do $O(\log n)$ concatenation? I.e. can we eliminate `Append` nodes in $O(\log n)$ time?

This implementation breaks the $O(\log n)$ bound on the concatenation.



Counting in a Binary Number System

0

W=2^0

Counting in a Binary Number System

1

W=2^0

Counting in a Binary Number System

$$\underset{W=2^0}{1} \;+\; 1$$

$$\underset{W=2^1}{1}\;\;\underset{W=2^0}{0}$$

Counting in a Binary Number System

$$\underset{W=2^0}{1} + 1$$

$$\underset{W=2^1}{1}\ \underset{W=2^0}{0} + 1$$

$$\underset{W=2^1}{1}\ \underset{W=2^0}{1}$$

Counting in a Binary Number System

		1	+	1
		$W=2^0$		
	1	0	+	1
	$W=2^1$	$W=2^0$		
	1	1	+	1
	$W=2^1$	$W=2^0$		
1	0	0		
$W=2^2$	$W=2^1$	$W=2^0$		

Counting in a Binary Number System

$$\underset{W=2^0}{1} + 1$$

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$$\underset{W=2^2}{1}\ \underset{W=2^1}{0}\ \underset{W=2^0}{0}$$

- To count up to n in the binary number system, we need $O(n)$ work.

Counting in a Binary Number System

		1	+	1
		W=2^0		
	1	0	+	1
	W=2^1	W=2^0		
	1	1	+	1
	W=2^1	W=2^0		
1	0	0		
W=2^2	W=2^1	W=2^0		

- To count up to n in the binary number system, we need $O(n)$ work.
- A number n requires $O(\log n)$ digits.

Counting in a Binary Number System

$1 + 1$

$W=2^0$

$1 \quad 0 + 1$

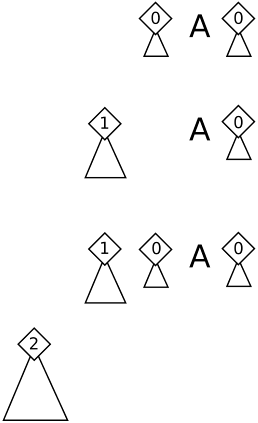
$W=2^1 \quad W=2^0$

$1 \quad 1 + 1$

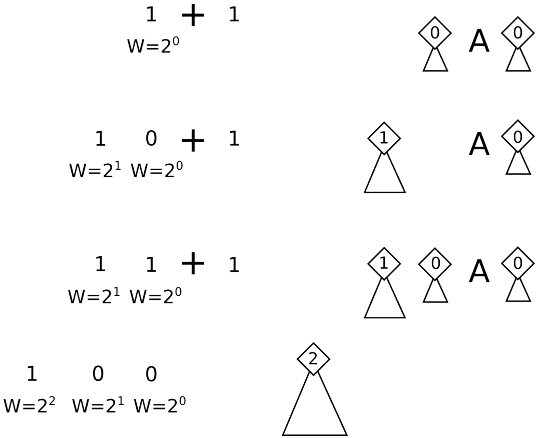
$W=2^1 \quad W=2^0$

$1 \quad 0 \quad 0$

$W=2^2 \quad W=2^1 \quad W=2^0$



Counting in a Binary Number System



- To add n leaves to an Append list, we need $O(n)$ work.
- Storing n leaves requires $O(\log n)$ Append nodes.

Binary Number Representation

- 0 digit corresponds to a missing tree
- 1 digit corresponds to an existing tree

Constant Time Appends in Conc-Trees

```
def appendLeaf[T](xs: Conc[T], ys: Single[T]): Conc[T] = xs match {
  case Empty => ys
  case xs: Single[T] => new <>(xs, ys)
  case _ <> _ => new Append(xs, ys)
  case xs: Append[T] => append(xs, ys)
}
```

Constant Time Appends in Conc-Trees

```
@tailrec private def append[T](xs: Append[T], ys: Conc[T]): Conc[T] = {
  if (xs.right.level > ys.level) new Append(xs, ys)
  else {
    val zs = new <>(xs.right, ys)
    xs.left match {
      case ws @ Append(_, _) => append(ws, zs)
      case ws if ws.level <= zs.level => ws <> zs
      case ws => new Append(ws, zs)
    }
  }
}
```

Constant Time Appends in Conc-Trees

We have implemented an *immutable* data structure with:

- $O(1)$ appends
- $O(\log n)$ concatenation

Next, we will see if we can implement a more efficient, *mutable* data Conc-tree variant, which can implement a `Combiner`.

Conc-Tree Combiners

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Conc Buffers

The `ConcBuffer` appends elements into an array of size k.

When the array gets full, it is stored into a `Chunk` node and added into the Conc-tree.

```
class ConcBuffer[T: ClassTag](val k: Int, private var conc: Conc[T]) {
  private var chunk: Array[T] = new Array(k)
  private var chunkSize: Int = 0
```

Conc Buffers

The += operation in most cases just adds an element to the chunk array:

```
final def +=(elem: T): Unit = {
  if (chunkSize >= k) expand()
  chunk(chunkSize) = elem
  chunkSize += 1
}
```

Occasionally, the chunk array becomes full, and needs to be expanded.

Chunk Nodes

`Chunk` nodes are similar to `Single` nodes, but instead of a single element, they hold an array of elements.

```
class Chunk[T](val array: Array[T], val size: Int) extends Conc[T] {
  def level = 0
}
```

Expanding the Conc Buffer

The `expand` method inserts the chunk into the Conc-tree, and allocates a new chunk:

```
private def expand() {
  conc = appendLeaf(conc, new Chunk(chunk, chunkSize))
  chunk = new Array(k)
  chunkSize = 0
}
```

Combine Method

The `combine` method is straightforward:

```
final def combine(that: ConcBuffer[T]): ConcBuffer[T] = {
  val combinedConc = this.result <> that.result
  new ConcBuffer(k, combinedConc)
}
```

Above, the `combine` method relies on the `result` method to obtain the Conc-trees from both buffers.

Result Method

The `result` method packs `chunk` array into the tree and returns the resulting tree:

```
def result: Conc[T] = {
  conc = appendLeaf(conc, new Chunk(chunk, chunkSize))
  conc
}
```

Result Method

The `result` method packs `chunk` array into the tree and returns the resulting tree:

```
def result: Conc[T] = {
  conc = appendLeaf(conc, new Chunk(chunk, chunkSize))
  conc
}
```

Summary:

- $O(\log n)$ `combine` concatenation
- fast $O(1)$ `+=` operation
- $O(1)$ `result` operation

Conc Buffer Demo

Demo – run the same benchmark as we did for the `ArrayCombiner`:

```
xs.par.aggregate(new ConcBuffer[String])(_ += _, _ combine _).result
```