Ganzinger-Bachmaier Model Existence Theorem for Propositional Logic

Ruzica Piskac

Digital Enterprise Research Institute, Innsbruck, Austria
ruzica.piskac@deri.org

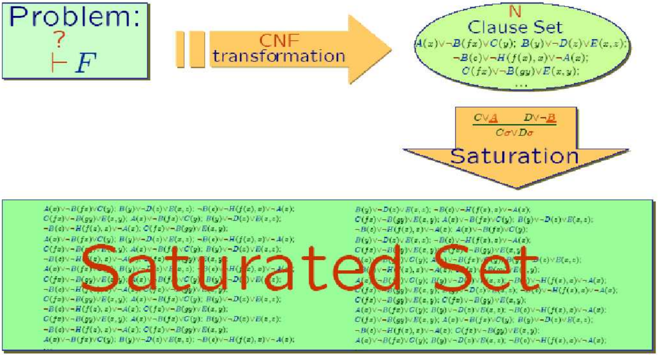
Lecture hold at EPFL, May 2007

Saturation Based First-Order Theorem Provers

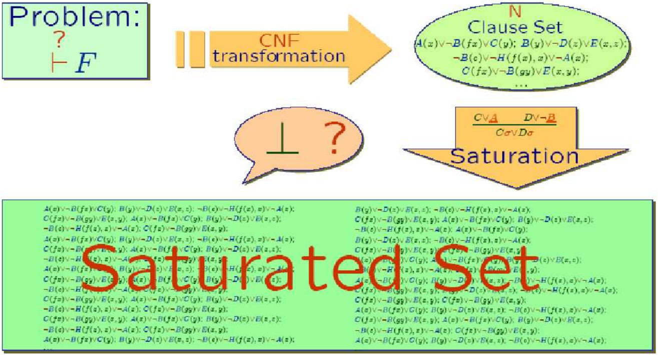
Problem:
$\overset{?}{\vdash} F$

Saturation Based First-Order Theorem Provers

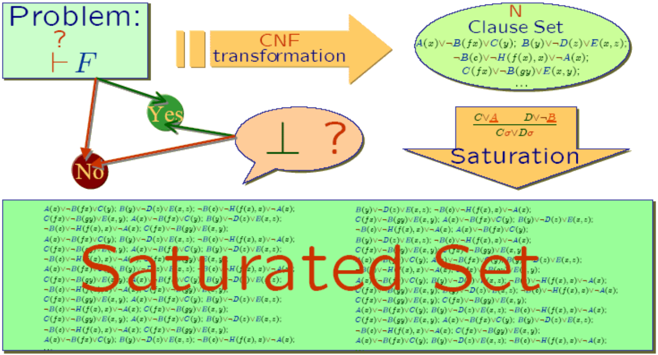
Saturation Based First-Order Theorem Provers



Saturation Based First-Order Theorem Provers



Saturation Based First-Order Theorem Provers



The Resolution Calculus *Res*

Definition

- Resolution inference rule

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

- (positive) factorisation

$$\frac{C \vee A \vee A}{C \vee A}$$

The Resolution Calculus *Res*

Example

1. $\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$ (given)
2. $P(f(a)) \vee Q(b)$ (given)
3. $\neg P(g(b, a)) \vee \neg Q(b)$ (given)
4. $P(g(b, a))$ (given)

The Resolution Calculus *Res*

Example

1. $\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$ (given)
2. $P(f(a)) \vee Q(b)$ (given)
3. $\neg P(g(b, a)) \vee \neg Q(b)$ (given)
4. $P(g(b, a))$ (given)
5. $\neg P(f(a)) \vee Q(b) \vee Q(b)$ (Res. 2. into 1.)

The Resolution Calculus Res

Example

1.	$\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$	(given)
2.	$P(f(a)) \vee Q(b)$	(given)
3.	$\neg P(g(b,a)) \vee \neg Q(b)$	(given)
4.	$P(g(b,a))$	(given)
5.	$\neg P(f(a)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(a)) \vee Q(b)$	(Fact. 5.)

The Resolution Calculus Res

Example

1.	$\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$	(given)
2.	$P(f(a)) \vee Q(b)$	(given)
3.	$\neg P(g(b,a)) \vee \neg Q(b)$	(given)
4.	$P(g(b,a))$	(given)
5.	$\neg P(f(a)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(a)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)

The Resolution Calculus *Res*

Example

1.	$\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$	(given)
2.	$P(f(a)) \vee Q(b)$	(given)
3.	$\neg P(g(b,a)) \vee \neg Q(b)$	(given)
4.	$P(g(b,a))$	(given)
5.	$\neg P(f(a)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(a)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)

The Resolution Calculus *Res*

Example

1.	$\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$	(given)
2.	$P(f(a)) \vee Q(b)$	(given)
3.	$\neg P(g(b,a)) \vee \neg Q(b)$	(given)
4.	$P(g(b,a))$	(given)
5.	$\neg P(f(a)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(a)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)
9.	$\neg P(g(b,a))$	(Res. 8. into 3.)

The Resolution Calculus Res

Example

1.	$\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$	(given)
2.	$P(f(a)) \vee Q(b)$	(given)
3.	$\neg P(g(b,a)) \vee \neg Q(b)$	(given)
4.	$P(g(b,a))$	(given)
5.	$\neg P(f(a)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(a)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)
9.	$\neg P(g(b,a))$	(Res. 8. into 3.)
10.	$\perp$	(Res. 4. into 9.)

Refutational Completeness of Resolution

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$,
  or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.

Refutational Completeness of Resolution

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$,
  or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.
- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.

Refutational Completeness of Resolution

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$,
  or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.
- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
- The limit interpretation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e.g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e.g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)
2. Extend $\succ$ to an ordering $\succ_L$ on ground literals:

$$\begin{array}{lcll} [\neg]A & \succ_L & [\neg]B & \text{, if } A \succ B \\ \neg A & \succ_L & A & \end{array}$$

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e.g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)
2. Extend $\succ$ to an ordering $\succ_L$ on ground literals:

$$\begin{array}{ccccl} [\neg]A & \succ_L & [\neg]B & , \text{if } A \succ B \\ \neg A & \succ_L & A & \end{array}$$

3. Extend $\succ_L$ to an ordering $\succ_C$ on ground clauses:
$\succ_C = (\succ_L)_{\mathrm{mul}}$, the multi-set extension of $\succ_L$.
Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Clause Orderings

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$.
Order the following clauses:

$\neg A_1 \vee \neg A_4 \vee A_3$

$\neg A_1 \vee A_2$

$\neg A_1 \vee A_4 \vee A_3$

$A_0 \vee A_1$

$\neg A_5 \vee A_5$

$A_1 \vee A_2$

Clause Orderings

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$.
Then:

$$\begin{array}{rc} & A_0 \lor A_1 \\ \prec & A_1 \lor A_2 \\ \prec & \neg A_1 \lor A_2 \\ \prec & \neg A_1 \lor A_4 \lor A_3 \\ \prec & \neg A_1 \lor \neg A_4 \lor A_3 \\ \prec & \neg A_5 \lor A_5 \end{array}$$

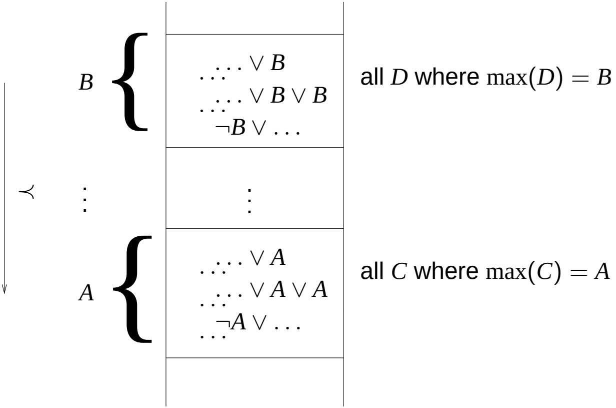
Properties of the Clause Ordering

Theorem

1. *The orderings on literals and clauses are total and well-founded.*
2. *Let C and D be clauses with $A = \max(C)$, $B = \max(D)$, where $\max(C)$ denotes the maximal atom in C.*
   (i) *If $A \succ B$ then $C \succ D$.*
   (ii) *If $A = B$, A occurs negatively in C but only positively in D, then $C \succ D$.*

Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:



Closure of Clause Sets under *Res*

Definition

$$Res(N) = \{C \mid C \text{ is concl. of a rule in } Res \text{ w/ premises in } N\}$$
$$Res^0(N) = N$$
$$Res^{n+1}(N) = Res(Res^n(N)) \cup Res^n(N), \text{ for } n \geq 0$$
$$Res^*(N) = \bigcup_{n\geq 0} Res^n(N)$$

N is called saturated (wrt. resolution), if $Res(N) \subseteq N$.

Construction of Interpretations

Given:

set N of ground clauses, atom ordering $\succ$.

Construction of Interpretations

Given:

set N of ground clauses, atom ordering $\succ$.

Wanted:

Herbrand interpretation I such that

- "many" clauses from N are valid in I;
- $I \models N$, *if* N *is saturated and* $\bot \notin N$.

Construction of Interpretations

Given:

set N of ground clauses, atom ordering $\succ$.

Wanted:

Herbrand interpretation I such that

- "many" clauses from N are valid in I;
- *$I \models N$, if N is saturated and $\bot \notin N$.*

Construction according to $\succ$, starting with the minimal clause.

Construction of Interpretations

Example

Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

	clauses C	I_C	Δ_C	Remarks
1	$\neg A_0$	$\emptyset$	$\emptyset$	true in I_C
2	$A_0 \lor A_1$	$\emptyset$	$\{A_1\}$	A_1 maximal
3	$A_1 \lor A_2$	$\{A_1\}$	$\emptyset$	true in I_C
4	$\neg A_1 \lor A_2$	$\{A_1\}$	$\{A_2\}$	A_2 maximal
5	$\neg A_1 \lor A_4 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\{A_4\}$	A_4 maximal
6	$\neg A_1 \lor \neg A_4 \lor A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	A_3 not maximal; *min. counter-ex.*
7	$\neg A_1 \lor A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set
$\Rightarrow$ there exists a counterexample.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.

Main Ideas of the Construction

- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should be maintained the way it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding A will make C become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of A has no effect on smaller clauses*).

Resolution Reduces Counterexamples

Example

$$\frac{\neg A_1 \lor A_4 \lor A_3 \lor A_0 \qquad \neg A_1 \lor \neg A_4 \lor A_3}{\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_0$	$\emptyset$	$\emptyset$	
$A_0 \lor A_1$	$\emptyset$	$\{A_1\}$	
$A_1 \lor A_2$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \lor A_2$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\emptyset$	A_3 occurs twice *minimal counter-ex.*
$\neg A_1 \lor A_4 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\{A_4\}$	
$\neg A_1 \lor \neg A_4 \lor A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	counterexample
$\neg A_1 \lor A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

The same I, but smaller counterexample, hence some progress was made.

Factorization Reduces Counterexamples

Example

$$\frac{\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0}{\neg A_1 \lor \neg A_1 \lor A_3 \lor A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_0$	$\emptyset$	$\emptyset$	
$A_0 \lor A_1$	$\emptyset$	$\{A_1\}$	
$A_1 \lor A_2$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \lor A_2$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \lor \neg A_1 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\{A_3\}$	
$\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_1 \lor A_4 \lor A_3 \lor A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	
$\neg A_1 \lor \neg A_4 \lor A_3$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_3 \lor A_5$	$\{A_1, A_2, A_3\}$	$\{A_5\}$	

The resulting $I = \{A_1, A_2, A_3, A_5\}$ is a model of the clause set.

Construction of Candidate Models Formally

Definition

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N, C = C' \vee A, A \succ C', I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

Construction of Candidate Models Formally

Definition

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that C produces A, if $\Delta_C = \{A\}$.

Construction of Candidate Models Formally

Definition

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

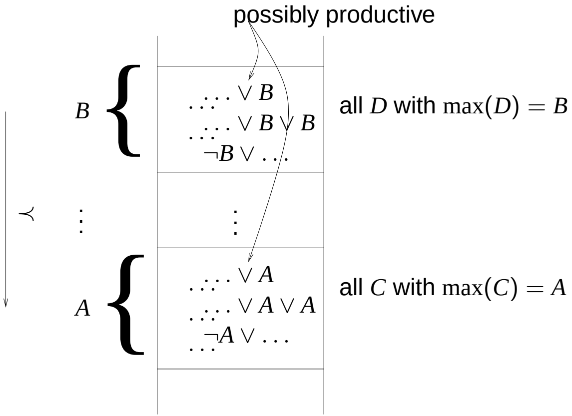
$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that C produces A, if $\Delta_C = \{A\}$.

The candidate model for N (wrt. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$.
We also simply write I_N, or I, for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.

Structure of $N, \succ$

Let $A \succ B$; producing a new atom does not affect smaller clauses.



Model Existence Theorem

Theorem

(Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let N be saturated wrt. Res, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.

Model Existence Theorem

Theorem

(Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let N be saturated wrt. Res, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.

Proof

Easy exercise! :-)