Exercises 03: First-Order Logic

1 Definitions

Consider countably infinite sets of variables V, constants C, function symbols F, and predicate symbols P. Each function symbol f has an *arity* denoted by $\mathsf{ar}(f)$, and so do predicate symbols.

In *first-order logic*, a *term* t can be: a variable $x \in V$, a constant $c \in C$, or (inductively) a function f applied to terms: $f(t_1, \dots, t_k)$, where $\mathsf{ar}(f) = k$.

A *first-order formula* F is defined inductively to be:

$$\begin{aligned} F ::= \quad & p(t_1, \dots, t_k) \mid \top \mid \bot \mid \\ & \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \\ & \exists x.\, F \mid \forall x.\, F \end{aligned}$$

where $p \in P$ and $t_1, \dots, t_k$ are terms with $k = \mathsf{ar}(p)$.

An *interpretation* I, is a pair (D, e) where D is a non-empty set of values, and e maps constants and variables to values in D, function symbols $f \in F$ to functions $D^{\mathsf{ar}(f)} \to D$, and predicate symbols $p \in P$ to relations in $D^{\mathsf{ar}(f)}$. We assume that the predicate symbols always contain an equality sign '=' whose arity is 2, and which is interpreted as the equality over D, i.e. $e(=)$ is Δ_D. We extend e to function applications (recursively) as follows:

$$e(f(t_1, \dots, t_k)) \triangleq e(f)(e(t_1), \dots, e(t_k))$$

Then, we use the notation $I \models F$ to say that F is *true* under interpretation I. Otherwise, we write $\neg(I \models F)$, and say that F is *false* under interpretation I.

$$\begin{array}{lcl} I \models \top & & \\ \neg(I \models \bot) & & \\ & & \\ I \models p(t_1, \dots, t_k) & \triangleq & (e(t_1), \dots, e(t_k)) \in e(p) \\ & & \\ I \models \neg F & \triangleq & \neg(I \models F) \\ I \models F_1 \wedge F_2 & \triangleq & I \models F_1 \text{ and } I \models F_2 \\ I \models F_1 \vee F_2 & \triangleq & I \models F_1 \text{ or } I \models F_2 \\ & & \\ I \models \exists x.\, F & \triangleq & \text{there exists } d \in D \text{ such that } I[x \mapsto d] \models F \\ I \models \forall x.\, F & \triangleq & \text{for all } d \in D, I[x \mapsto d] \models F \end{array}$$

2 Density

Consider the formula:

$$\mathsf{Dense} \triangleq \forall x, y.\ x < y \to \exists z.\ x < z \land z < y$$

Find two interpretations, one under which Dense is false, and one under which it is true.

Solution: Under the interpretation $I_{\mathbb{N}} = (\mathbb{N}, e_{\mathbb{N}})$, where values are natural numbers, and the predicate '<' is interpreted as the less-than relation, we have $\neg(I_{\mathbb{N}} \models \mathsf{Dense})$. However, under the interpretation $I_{\mathbb{R}}$ where values are real numbers, we have $I_{\mathbb{R}} \models \mathsf{Dense}$.

3 Boundedness

Same question for the formula:

$$\mathsf{AtLeastThreeElements} \triangleq \exists x, y, z.\ \neg x = y \land \neg y = z \land \neg x = z$$

Solution: This formula is true under the interpretation $I_{\mathbb{N}}$, but not on I_2 where the set of values is $\{0, 1\}$.

4 FOL validity

Determine whether each formula is valid or not. If it is valid prove it, otherwise give a counterexample (i.e. give an interpretation under which the formula is false).

1. $(\exists x.\ p(x) \lor q(x)) \quad \leftrightarrow \quad ((\exists x.\ p(x)) \lor (\exists x.\ q(x)))$

   Solution: Valid. Let $I = (D, e)$ be an interpretation. We have:

   $$\begin{array}{ll} & I \models \exists x.\ p(x) \lor q(x) \\ \iff & \text{there exists } d \in D \text{ such that } I[x \mapsto d] \models p(x) \lor q(x) \\ \iff & \text{there exists } d \in D \text{ such that } I[x \mapsto d] \models p(x) \text{ or } I[x \mapsto d] \models q(x) \\ \iff & \text{there exists } d \in D \text{ such that } I[x \mapsto d] \models p(x) \text{ or there exists } d \in D \text{ such that } I[x \mapsto d] \models q(x) \\ \iff & I \models \exists x.\ p(x) \text{ or } I \models \exists x.\ q(x) \\ \iff & I \models (\exists x.\ p(x)) \lor (\exists x.\ q(x)) \end{array}$$

   Therefore:

   $$I \models (\exists x.\ p(x) \lor q(x)) \quad \leftrightarrow \quad ((\exists x.\ p(x)) \lor (\exists x.\ q(x)))$$

2. $(\exists x.\ p(x) \land q(x)) \quad \leftrightarrow \quad ((\exists x.\ p(x)) \land (\exists x.\ q(x)))$

   Solution: This formula is false in the interpretation $(\mathbb{N}, e)$ where $e(p) = \{0\}$ and $e(q) = \{1\}$. There exists a natural number which is equal to 0, and a natural number which is equal to 1 (right-hand-side) but there does not exist a natural number which is equal to 0 and 1 (left-hand-side).

3. $(\forall x.\ p(x) \lor q(x)) \quad \leftrightarrow \quad ((\forall x.\ p(x)) \lor (\forall x.\ q(x)))$

   Solution: This formula is false in the interpretation $(\mathbb{N}, e)$ where $e(p) = \{n \in \mathbb{N} \mid \text{n is even}\}$ and $e(q) = \{n \in \mathbb{N} \mid \text{n is odd}\}$. Every natural number is either even or odd (left-hand-side), but it is not the case that they are all even, or all odd (right-hand-side).

4. $(\forall x.\ p(x) \land q(x)) \quad \leftrightarrow \quad ((\forall x.\ p(x)) \land (\forall x.\ q(x)))$

   Solution: Valid (similar to (1)).

5. $(p \to (\exists x.\ q(x))) \quad \leftrightarrow \quad (\exists x.\ p \to q(x))$

   Solution: Valid when x does not appear free in p. In the proof we use the fact that interpretation domains are not empty.

6. $((\exists x.\ p(x)) \to q) \quad \leftrightarrow \quad (\forall x.\ p(x) \to q)$

   Solution: Valid.

7. $(p \to (\forall x.\ q(x))) \quad \leftrightarrow \quad (\forall x.\ p \to q(x))$

   Solution: Valid when x does not appear free in p.

8. $\forall x.\ ((p(x) \to q(x)) \to p(x)) \to p(x))$

   Solution: Valid. Let $I = (D, e)$ be an interpretation. For every $d \in D$, consider two cases. Either $d \in e(p)$, and the formula is true because the right-hand-side of the implication is true, or $d \notin e(p)$, and the formula is true because the left-hand-side of the implication is false.

9. $(\forall x, y.\ p(x, y) \to p(y, x)) \to \forall z.p(z, z)$

   Solution: False under the interpretation $(\mathbb{N}, e)$ where $e(p) = \{(a, b) \mid a \neq b\}$.

10. $\forall x, y.\ p(x, y) \to p(y, x) \to \forall z.p(z, z)$

    Solution: False under the interpretation $(\mathbb{N}, e)$ where $e(p) = \{(a, b) \mid a \neq b\}$ (take $x = 0$ and $y = 1$).

11. $(\exists x.\ p(x)) \to \forall y.p(y)$

    Solution: False under the interpretation $(\mathbb{N}, e)$ where $e(p) = \{0\}$.

12. $(\forall x.\ p(x)) \to \exists y.p(y)$

    Solution: Valid because domains are not empty.

13. $\exists x, y.\ (p(x, y) \to (p(y, x) \to \forall z.p(z, z)))$

    Solution: Valid. Let $I = (D, e)$ be an interpretation. We consider two cases. Either $I \models \forall z.p(z, z)$, and the whole formula is valid by picking any two x and y (D is not empty). Otherwise, there exists an element $d \in D$ such that $(d, d) \notin e(p)$. In that case, we can pick x and y to be this d.

5 Unboundedness

Find a formula F which is true for some infinite interpretations, but false for all finite interpretations.

Solution: Define F to be

$$
\begin{aligned}
&(\forall x, y, z.\ p(x, y) \land p(y, z) \to p(x, z)) \land && (\mathit{transitivity}) \\
&(\forall x.\ \neg p(x, x) \land && (\mathit{irreflexive}) \\
&(\forall x.\exists y.p(x, y) && (\mathit{totality})
\end{aligned}
$$

We can verify that, for any finite interpretation, F is false. On the other hand, F is true under the interpretation $(\mathbb{N}, e)$ where $e(p) = \{(a, b) \mid a < b\}$.

To answer the additional question that we had in class, it is also the case that for any infinite set D, there exists e, such that $(D, e) \models F$ holds. For instance with $D = [0, 1]$, we can choose

$$e(p) = \{(a, b) \mid a < b \wedge b \neq 1\} \cup \{(1, 0)\}$$

(i.e. we move 1 to be before 0).

In general, for an infinite set D, we can pick an infinite sequence of elements $d_0, d_1, d_2, d_3, \dots \in D$ (this requires the axiom of choice) to put them in relation in $e(p)$, and put all other elements from D before d_0. Formally:

$$e(p) = \{(d_i, d_j) \mid i < j\} \cup \{(d, d_0) \mid d \in D \wedge \forall i. d \neq d_i\}$$