Exercises 03: First-Order Logic

1 Definitions

Consider countably infinite sets of variables V, constants C, function symbols F, and predicate symbols P. Each function symbol f has an *arity* denoted by $\mathsf{ar}(f)$, and so do predicate symbols.

In *first-order logic*, a *term* t can be: a variable $x \in V$, a constant $c \in C$, or (inductively) a function f applied to terms: $f(t_1, \ldots, t_k)$, where $\mathsf{ar}(f) = k$.

A *first-order formula* F is defined inductively to be:

$$\begin{aligned} F ::= \quad & p(t_1, \ldots, t_k) \mid \top \mid \bot \mid \\ & \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \\ & \exists x.\, F \mid \forall x.\, F \end{aligned}$$

where $p \in P$ and $t_1, \ldots, t_k$ are terms with $k = \mathsf{ar}(p)$.

An *interpretation* I, is a pair (D, e) where D is a non-empty set of values, and e maps constants and variables to values in D, function symbols $f \in F$ to functions $D^{\mathsf{ar}(f)} \to D$, and predicate symbols $p \in P$ to relations in $D^{\mathsf{ar}(f)}$. We assume that the predicate symbols always contain an equality sign '=' whose arity is 2, and which is interpreted as the equality over D, i.e. $e(=)$ is Δ_D. We extend e to function applications (recursively) as follows:

$$e(f(t_1, \ldots, t_k)) \triangleq e(f)(e(t_1), \ldots, e(t_k))$$

Then, we use the notation $I \models F$ to say that F is *true* under interpretation I. Otherwise, we write $\neg(I \models F)$, and say that F is *false* under interpretation I.

$$\begin{aligned} & I \models \top \\ & \neg(I \models \bot) \\ \\ & I \models p(t_1, \ldots, t_k) && \triangleq && (e(t_1), \ldots, e(t_k)) \in e(p) \\ \\ & I \models \neg F && \triangleq && \neg(I \models F) \\ & I \models F_1 \wedge F_2 && \triangleq && I \models F_1 \text{ and } I \models F_2 \\ & I \models F_1 \vee F_2 && \triangleq && I \models F_1 \text{ or } I \models F_2 \\ \\ & I \models \exists x.\, F && \triangleq && \text{there exists } d \in D \text{ such that } I[x \mapsto d] \models F \\ & I \models \forall x.\, F && \triangleq && \text{for all } d \in D,\, I[x \mapsto d] \models F \end{aligned}$$

2 Density

Consider the formula:

$$\mathsf{Dense} \triangleq \forall x, y.\ x < y \to \exists z.\ x < z \land z < y$$

Find two interpretations, one under which Dense is false, and one under which is true.

3 Boundedness

Same question for the formula:

$$\mathsf{AtLeastThreeElements} \triangleq \exists x, y, z.\ \neg x = y \land \neg y = z \land \neg x = z$$

4 FOL validity

Determine whether each formula is valid or not. If it is valid prove it, otherwise give a counterexample (i.e. give an interpretation under which the formula is false).

1. $(\exists x.\ p(x) \lor q(x)) \quad \leftrightarrow \quad ((\exists x.\ p(x)) \lor (\exists x.\ q(x)))$
2. $(\exists x.\ p(x) \land q(x)) \quad \leftrightarrow \quad ((\exists x.\ p(x)) \land (\exists x.\ q(x)))$
3. $(\forall x.\ p(x) \lor q(x)) \quad \leftrightarrow \quad ((\forall x.\ p(x)) \lor (\forall x.\ q(x)))$
4. $(\forall x.\ p(x) \land q(x)) \quad \leftrightarrow \quad ((\exists x.\ p(x)) \land (\forall).\ xq(x)))$
5. $(p \to (\exists x.\ q(x))) \quad \leftrightarrow \quad (\exists x.\ p \to q(x))$
6. $((\exists x.\ p(x)) \to q) \quad \leftrightarrow \quad (\forall x.\ p(x) \to q)$
7. $(p \to (\forall x.\ q(x))) \quad \leftrightarrow \quad (\forall x.\ p \to q(x))$
8. $\forall x.\ ((p(x) \to q(x)) \to p(x)) \to p(x))$
9. $(\forall x, y.\ p(x, y) \to p(y, x)) \to \forall z. p(z, z)$
10. $\forall x, y.\ p(x, y) \to p(y, x) \to \forall z. p(z, z)$
11. $(\exists x.\ p(x)) \to \forall y. p(y)$
12. $(\forall x.\ p(x)) \to \exists y. p(y)$
13. $\exists x, y.\ (p(x, y) \to (p(y, x) \to \forall z. p(z, z)))$

5 Unboundedness

Find a formula F which is true for infinite interpretations, but false for finite interpretations.