Ex 1

①

$$\dfrac{\dfrac{}{\{\} \vdash 3 : \text{Int}} \quad \dfrac{}{\{\} \vdash 5 : \text{Int}}}{\{\} \vdash 3 + 5 : \text{Int}}$$

②

$$\dfrac{\dfrac{}{\{\} \vdash 4 : \text{Int}} \quad \dfrac{\dfrac{\dfrac{(x,\text{Int}) \in \Gamma_1}{\Gamma_1 \vdash x : \text{Int}} \quad \dfrac{(x,\text{Int}) \in \Gamma_1}{\Gamma_1 \vdash x : \text{Int}}}{\Gamma_1 \vdash x + x : \text{Int}} \quad \dfrac{\dfrac{(x,\text{Int}) \in \Gamma_2}{\Gamma_2 \vdash x : \text{Int}} \quad \dfrac{(y,\text{Int}) \in \Gamma_2}{\Gamma_2 \vdash y : \text{Int}}}{\Gamma_2 \vdash x * y : \text{Int}}}{\Gamma_1 \vdash \text{val } y : \text{Int} = x + x;\ x * y : \text{Int}}}{\{\} \vdash \text{val } x : \text{Int} = 4;\ \text{val } y : \text{Int} = x + x;\ x * y : \text{Int}}$$

$$\Gamma_1 = \{ (x, \text{Int}) \} \qquad \Gamma_2 = \{ (x, \text{Int}), (y, \text{Int}) \}$$

③

$$\dfrac{\dfrac{}{\Gamma_0 \vdash 7 : \text{Int}} \quad \dfrac{\dfrac{\dfrac{(x,\text{Int}) \in \Gamma_1}{\Gamma_1 \vdash x : \text{Int}} \quad \dfrac{}{\Gamma_1 \vdash 100 : \text{Int}}}{\Gamma_1 \vdash x < 100 : \text{Boolean}} \quad \dfrac{\dfrac{(\text{power}, (\text{Int},\text{Int}) \Rightarrow \text{Int}) \in \Gamma_1}{\Gamma_1 \vdash \text{power} : (\text{Int},\text{Int}) \Rightarrow \text{Int}} \quad \dfrac{(x,\text{Int}) \in \Gamma_1}{\Gamma_1 \vdash x : \text{Int}} \quad \dfrac{}{\Gamma_1 \vdash 10 : \text{Int}}}{\Gamma_1 \vdash \text{power}(x, 10) : \text{Int}} \quad \dfrac{\dfrac{}{\Gamma_1 \vdash \text{"Too big!"} : \text{String}}}{\Gamma_1 \vdash \text{error}(\text{"Too big!"}) : \text{Int}}}{\Gamma_1 \vdash \text{if } (x < 100)\ \text{power}(x, 10) \text{ else error}(\text{"Too big!"}) : \text{Int}}}{\Gamma_0 \vdash \text{val } x : \text{Int} = 7;\ \text{if } (x < 100)\ \text{power}(x, 10) \text{ else error}(\text{"Too big!"}) : \text{Int}}$$

$$\Gamma_0 = \{ (x, \text{Boolean}), (\text{power}, (\text{Int}, \text{Int}) \Rightarrow \text{Int}) \} \qquad \Gamma_1 = \{ (x, \text{Int}), (\text{power}, (\text{Int}, \text{Int}) \Rightarrow \text{Int}) \}$$

$$
\dfrac{
\dfrac{
\dfrac{(x, \text{Boolean}) \in \Gamma_0}{\Gamma_0 \vdash x : \text{Boolean}}
\quad
\dfrac{}{\Gamma_0 \vdash 1 : \text{Int}}
\quad
\dfrac{}{\Gamma_0 \vdash 0 : \text{Int}}
}{\Gamma_0 \vdash \text{if } (x)\ 1 \text{ else } 0 : \text{Int}}
\qquad
\dfrac{
\dfrac{(x, \text{Int}) \in \Gamma_1}{\Gamma_1 \vdash x : \text{Int}}
\quad
\dfrac{}{\Gamma_1 \vdash 2 : \text{Int}}
}{\Gamma_1 \vdash x * 2 : \text{Int}}
}{\Gamma_0 \vdash \text{val } x : \text{Int} = \text{if } (x)\ 1 \text{ else } 0 \,;\ x * 2 : \text{Int}}
$$

$$\Gamma_0 = \{ (x, \text{Boolean}), (\text{power}, (\text{Int}, \text{Int}) \Rightarrow \text{Int}) \}$$

$$\Gamma_1 = \{ (x, \text{Int}), (\text{power}, (\text{Int}, \text{Int}) \Rightarrow \text{Int}) \}$$

Ex 2

Q1

SEQ LIT

$$\frac{\Gamma \vdash x_1 : T \quad \ldots \quad \Gamma \vdash x_n : T}{\Gamma \vdash [x_1 \ldots x_n] : Seq[T]} \quad \text{for } n \geq 0$$

SEQ CONCAT

$$\frac{\Gamma \vdash e_1 : Seq[T] \quad \Gamma \vdash e_2 : Seq[T]}{\Gamma \vdash e_1 ++ e_2 : Seq[T]}$$

AT INDEX

$$\frac{\Gamma \vdash e_1 : Seq[T] \quad \Gamma \vdash e_2 : Int}{\Gamma \vdash atIndex(e_1, e_2) : T}$$

INDEX OF

$$\frac{\Gamma \vdash e_1 : Seq[T] \quad \Gamma \vdash e_2 : T}{\Gamma \vdash indexOf(e_1, e_2) : Int}$$

Q2

(1)

$$
\dfrac{\dfrac{\{\} \vdash 1 : Int \quad \{\} \vdash 2 : Int \quad \{\} \vdash 3 : Int}{\{\} \vdash [1,2,3] : Int} \qquad \dfrac{}{\{\} \vdash 1 : Int}}{\dfrac{\{\} \vdash atIndex([1,2,3], 1) : Int \qquad \dfrac{}{\{\} \vdash 2 : Int}}{\{\} \vdash atIndex([1,2,3], 1) == 2 : Boolean}}
$$

(2) Impossible...

(3)

$$
\dfrac{\dfrac{}{\{\} \vdash true : Boolean} \qquad \dfrac{\dfrac{(x, Boolean) \in \Gamma_1}{\Gamma_1 \vdash x : Boolean} \quad \dfrac{}{\Gamma_1 \vdash false : Boolean} \quad \dfrac{(x, Boolean) \in \Gamma_1}{\Gamma_1 \vdash x : Boolean}}{\Gamma_1 \vdash [x, false, x] : Seq[Boolean]}}{\{\} \vdash val\ x : Boolean = true;\ [x, false, x] : Seq[Boolean]}
$$

$$\Gamma_1 = \{(x, Boolean)\}$$

Let T be any type...

(4)

$$
\dfrac{\dfrac{\dfrac{}{\{\} \vdash [] : Seq[T]} \quad \dfrac{}{\{\} \vdash [] : Seq[T]}}{\{\} \vdash [] ++ [] : Seq[T]} \qquad \dfrac{}{\{\} \vdash 0 : Int}}{\{\} \vdash atIndex([] ++ [], 0) : T}
$$

This derivation works for any T.

Q3

Using forall:

$$\frac{\Gamma_{n+1} \vdash e : T \quad \forall i \in [1,n].\ \Gamma_i \vdash e_i : Seq[T_i] \quad \Gamma_1 = \Gamma \quad \forall i \in [1,n].\ \Gamma_{(i+1)} = \Gamma_i \oplus \{(x_i, T_i)\}}{\Gamma \vdash for\ \{\ x_1 \leftarrow e_1; \ldots; x_n \leftarrow e_n\ \}\ yield\ e : Seq[T]}\ (n \geq 1)$$

Also possible: (with 2 rules)

$$\frac{\Gamma \vdash e_1 : Seq[T_1] \quad \Gamma \oplus \{(x, T_1)\} \vdash e_2 : T_2}{\Gamma \vdash for\ \{\ x \leftarrow e_1\}\ yield\ e_2 : Seq[T_2]}$$

$$\frac{\Gamma \vdash e_1 : T_1 \quad \Gamma \oplus \{(x, T_1)\} \vdash for\ \{x_2 \leftarrow e_2; \ldots; x_n \leftarrow e_n\}\ yield\ e : Seq[T]}{\Gamma \vdash for\ \{\ x_1 \leftarrow e_1; \ldots; x_n \leftarrow e_n\ \}\ yield\ e : Seq[T]}\ (n \geq 2)$$