Abstract Interpretation

(Cousot, Cousot 1977)

also known as

Data-Flow Analysis

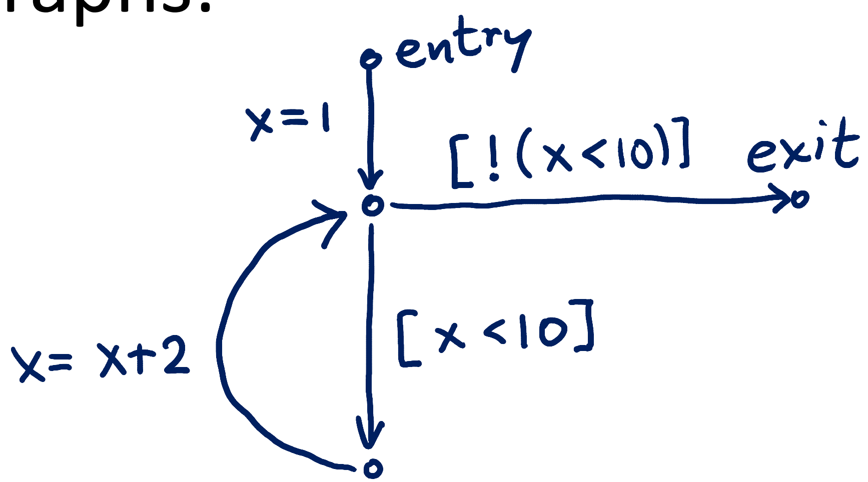
Goal of Data-Flow Analysis

Automatically compute information about the program

- Use it to report errors to user (like type errors)
- Use it to optimize the program

Works on control-flow graphs:
(like flow-charts)

```
x = 1
while (x < 10) {
  x = x + 2
}
```



Constant Propagation

```
int a, b, step, i;
 boolean c;
 a = 0;
 b = a + 10;
 step = -1;
 if (step > 0) {
   i = a;
 } else {
   i = b;
 }
 c = true;
 while (c) {
   print(i);
   i = i + step;    // can emit decrement
   if (step > 0) {
     c = (i < b);
   } else {
     c = (i > a); // can emit better instruction here
  } // insert here (a = a + step), redo analysis
}
```

Control-Flow **Graph**: (V,E)

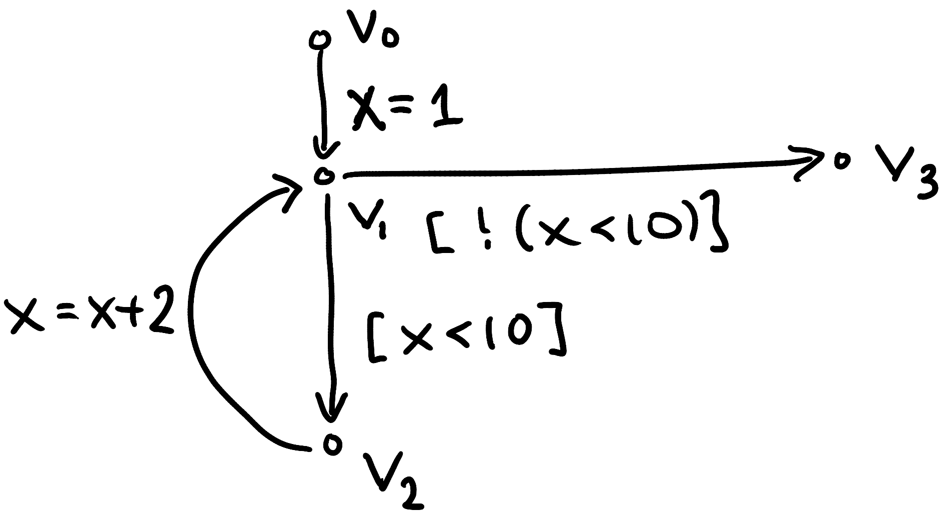
Set of nodes, V

Set of edges, which have statements on them

(v_1,st,v_2) in E

means there is edge from v_1 to v_2 labeled with statement st.

x = 1
while (x < 10) {
x = x + 2
}



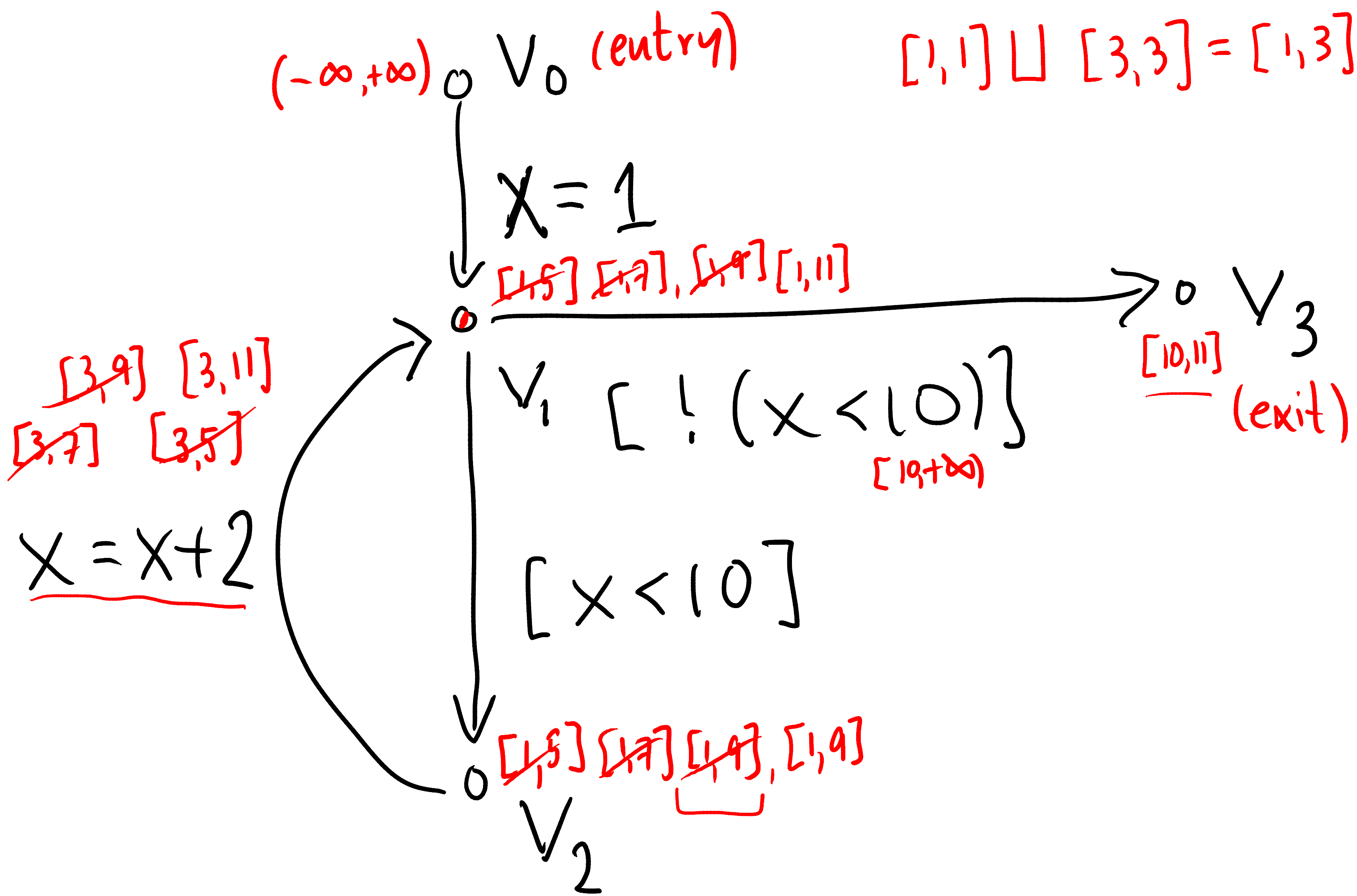
$V = \{v_0,v_1,v_2,v_3\}$

$E = \{(v_0,x=1,v_1), (v_1,[x<10],v_2),$
$(v_2,x=x+2,v_1), (v_1,[!(x<10)],v_3)\}$

Interpretation and Abstract Interpratation

- Control-Flow graph is similar to AST
- We can
  - interpret control flow graph
  - generate machine code from it (e.g. LLVM, gcc)
  - abstractly interpret it: do not push values, but **approximately compute supersets of possible values** (e.g. intervals, types, etc.)

Compute Range of x at Each Point



What we see today

1. How to compile abstract syntax trees into control-flow graphs
2. Lattices, as structures that describe abstractly sets of program states (facts)
3. Transfer functions that describe how to update facts (started)

   Next time:

4. Iterative analysis algorithm
5. Convergence

Generating Control-Flow Graphs

- Start with graph that has one entry and one exit node and label is entire program
- Recursively decompose the program to have more edges with simpler labels
- When labels cannot be decomposed further, we are done

Flattening Expressions

E_1, E_2 – complex expressions

t_1, t_2 – fresh variables

$x = E_1 * E_2 \quad \Rightarrow \quad t_1 = E_1;\ t_2 = E_2;\ x = t_1 * t_2$

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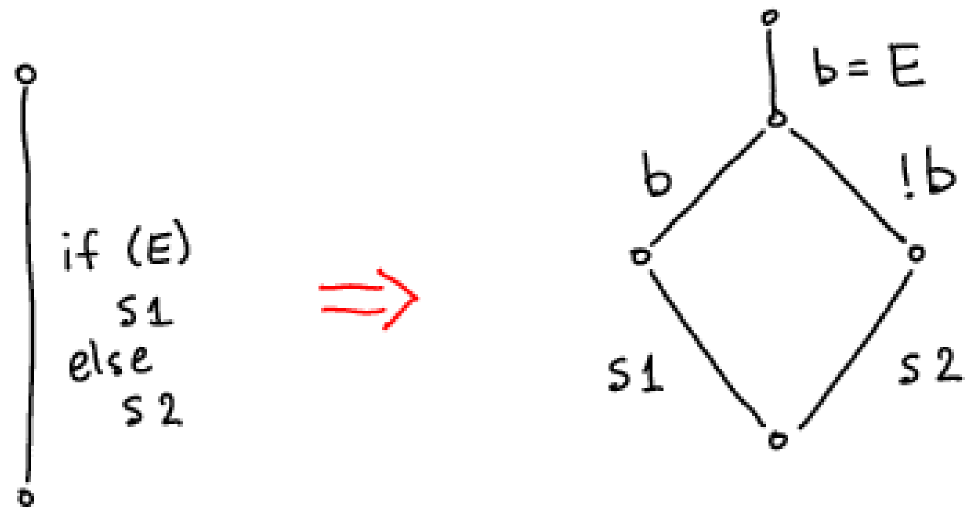
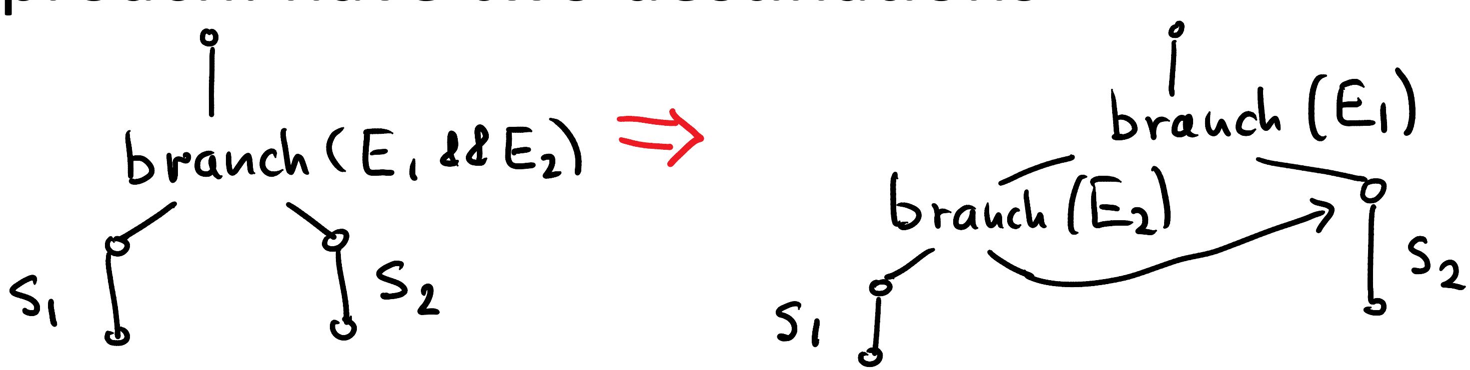
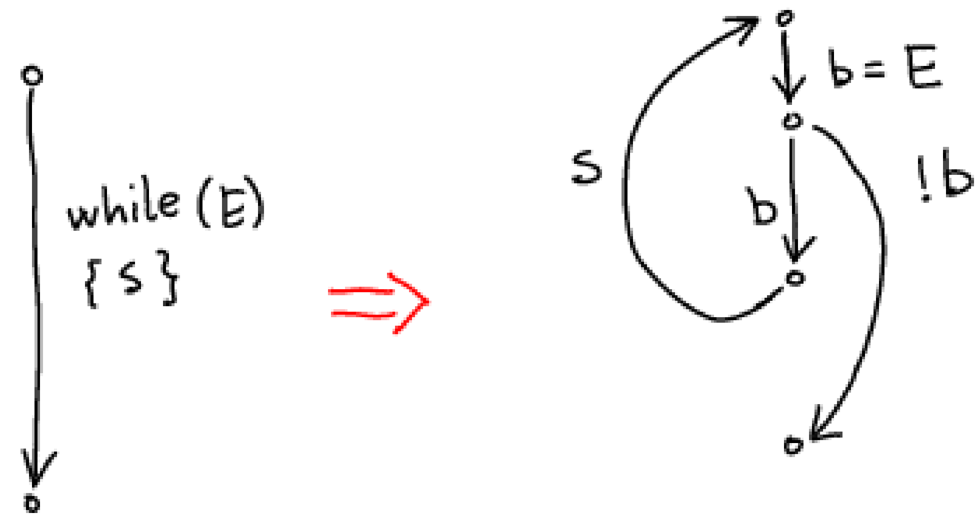
If-Then-Else



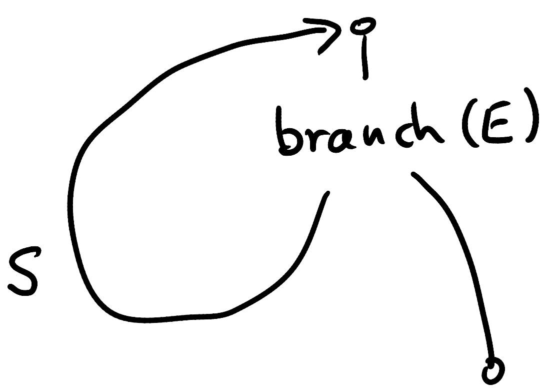
Better translation uses the "branch" instruction approach: have two destinations



While



Better translation uses the "branch" instruction

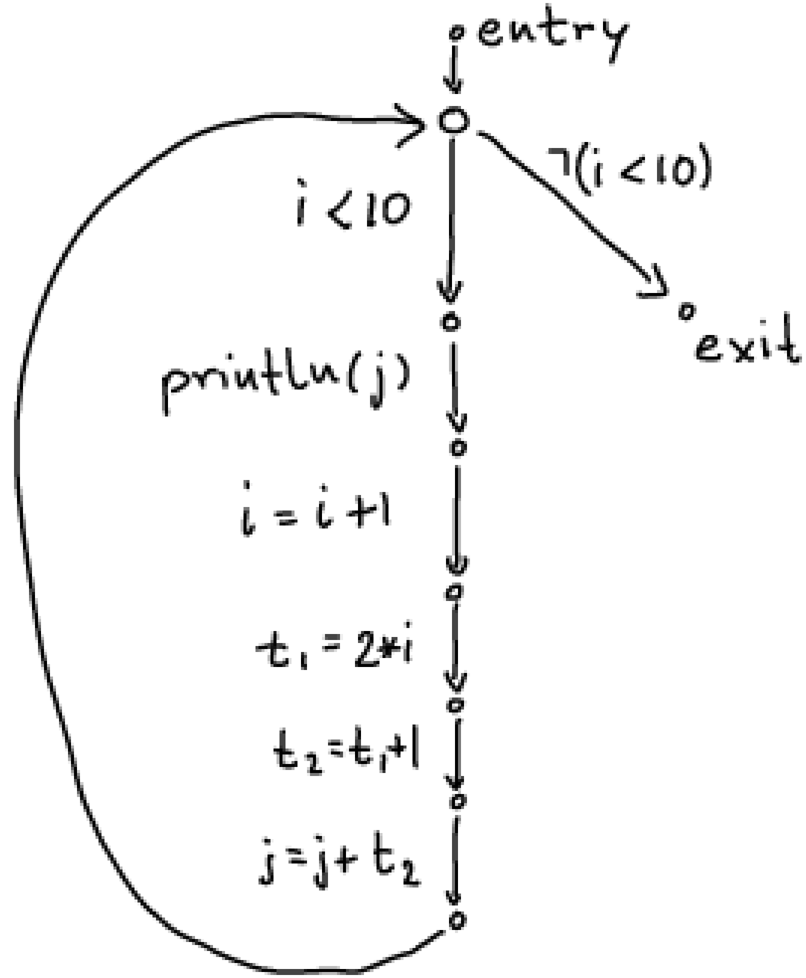


Example 1: Convert to CFG

```
while (i < 10) {
  println(j);
  i = i + 1;
  j = j +2*i + 1;
}
```

Example 1 Result

```
while (i < 10) {
  println(j);
  i = i + 1;
  j = j +2*i + 1;
}
```

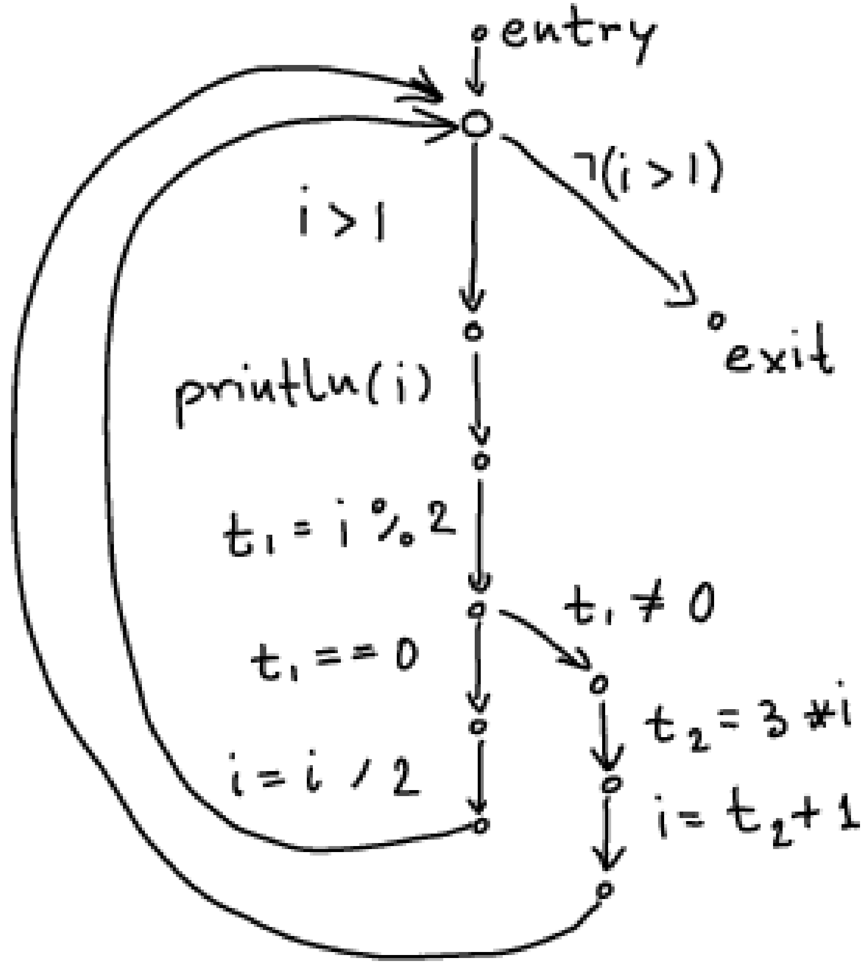


Example 2: Convert to CFG

```
int i = n;
while (i > 1) {
  println(i);
  if (i % 2 == 0) {
    i = i / 2;
  } else {
    i = 3*i + 1;
  }
}
```

Example 2 Result

```
int i = n;
while (i > 1) {
  println(i);
  if (i % 2 == 0) {
    i = i / 2;
  } else {
    i = 3*i + 1;
  }
}
```



Analysis Domains

Abstract Intepretation Generalizes Type Inference

Type Inference

- computes types
- type rules
  - can be used to compute types of expression from subtypes
- types fixed for a variable

Abstract Interpretation

- computes **facts** from a domain
  - types
  - intervals
  - formulas
  - set of initialized variables
  - set of live variables
- transfer functions
  - compute facts for one program point from facts at previous program points
- facts change as the values of vars change (*flow-sensitivity*)

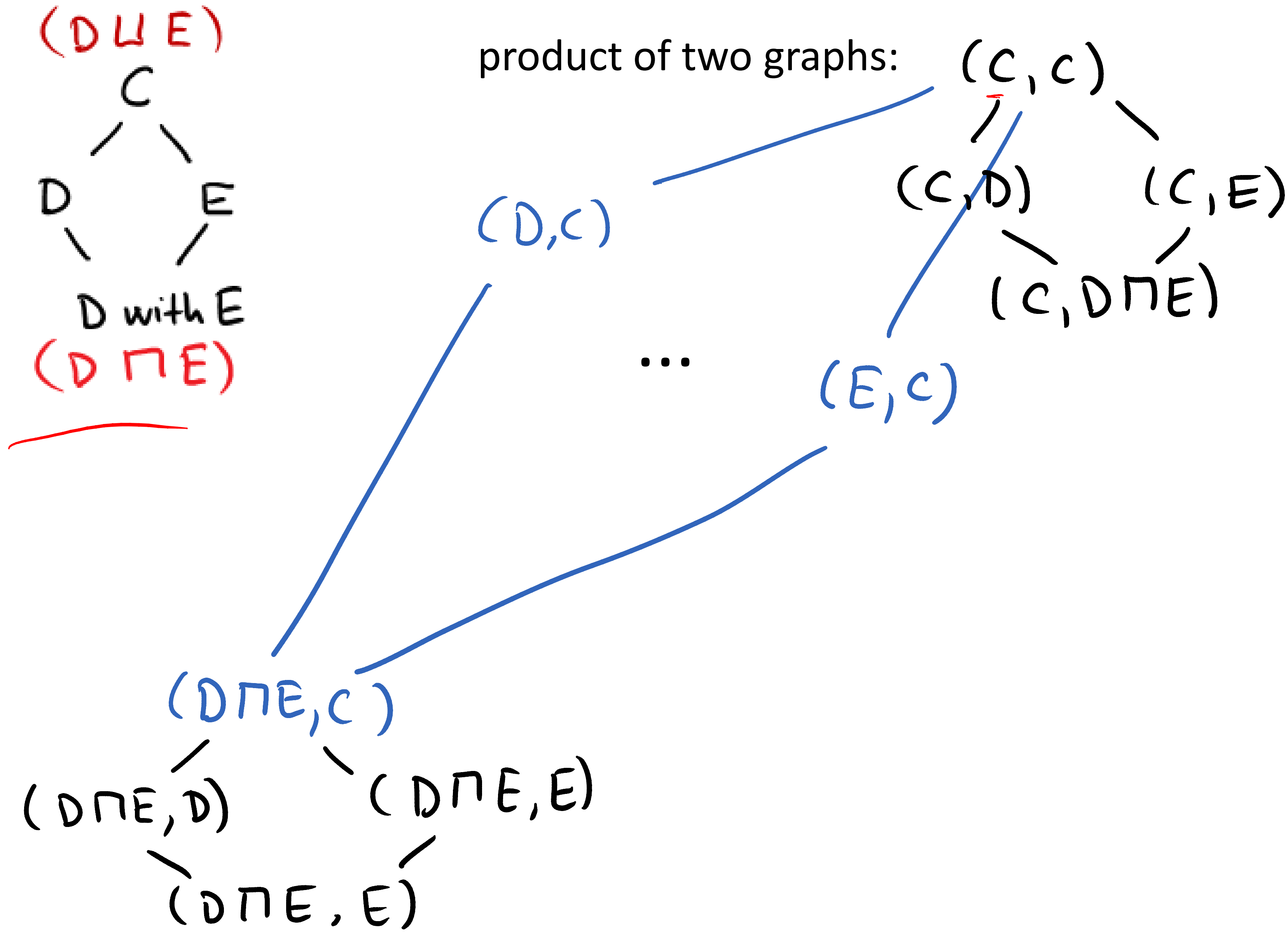
scalac computes types. Try in REPL:

```
class C
class D extends C
class E extends C
val p = false
val d = new D()
val e = new E()
val z = if (p) d else e

val u = if (p) (d,e) else (d,d)
val v = if (p) (d,e) else (e,d)

val f1 = if (p) ((d1:D) => 5) else ((e1:E) => 5)
val f2 = if (p) ((d1:D) => d) else ((e1:E) => e)
```

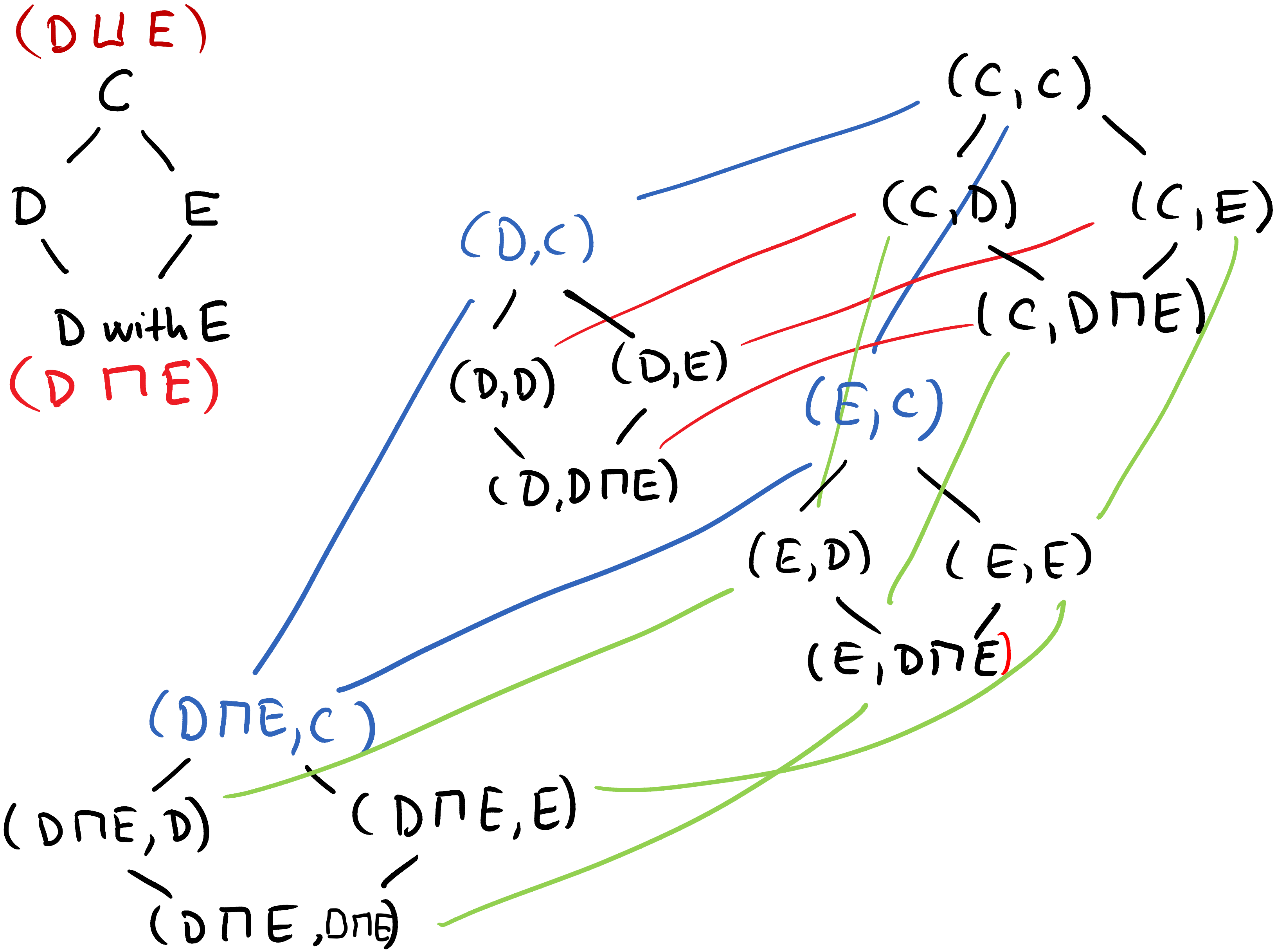
Finds "Best Type" for Expression

```
class C
class D extends C
class E extends C
val p = false
val d = new D()                                          //   d:D
val e = new E()                                          //   e:E
val z = if (p) d else e                                  //   z:C

val u = if (p) (d,e) else (d,d)                          //   u:(D,C)
val v = if (p) (d,e) else (e,d)                          //   v:(C,C)

val f1 = if (p) ((d1:D) => 5) else ((e1:E) => 5)         //   f1: ((D with E) => Int)
val f2 = if (p) ((d1:D) => d) else ((e1:E) => e)         //   f2: ((D with E) => C)
```

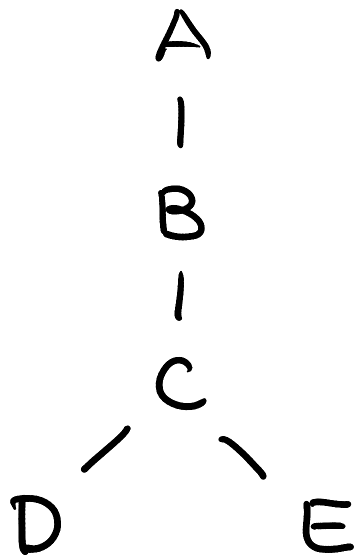
Subtyping Relation in this Example



Subtyping Relation in this Example



Least Upper Bound (lub, join)



A,B,C are all upper bounds on both D and E (they are above each of then in the picture, they are supertypes of D and supertypes of E). Among these upper bounds, C is the least one (the most specific one).
We therefore say C is the **least upper bound**,

$$C = D \sqcup E$$

In any partial order $\leq$, if S is a set of elements (e.g. S={D,E}) then:

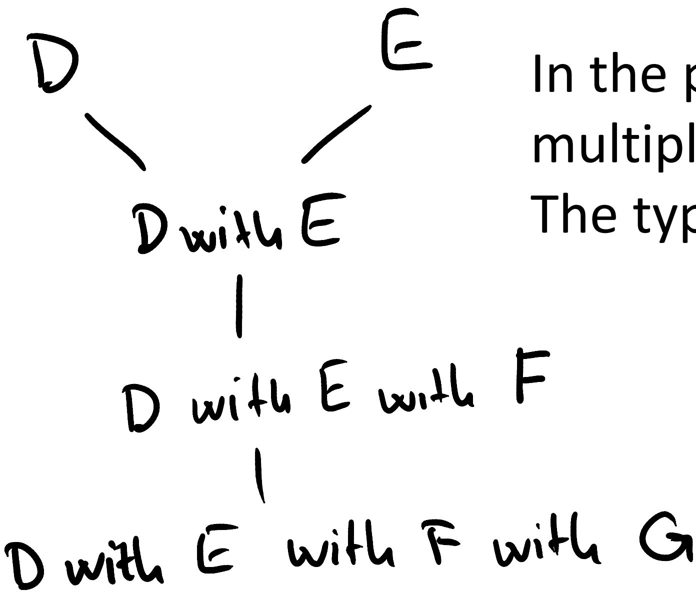
U is **upper bound** on S iff $x \leq U$ for every x in S.

U_0 is the **least upper bound (lub)** of S, written $U_0 = \bigsqcup S$, or U_0=lub(S) iff:

U_0 is upper bound and

if U is any upper bound on S, then $U_0 \leq U$

Greatest Lower Bound (glb, meet)



In the presence of traits or interfaces, there are multiple types that are subtypes of both D and E. The type (D with E) is the largest of them.

$D \sqcap E$

In any partial order $\leq$, if S is a set of elements (e.g. S={D,E}) then:

L is **lower bound** on S iff $L \leq x$ for every x in S.

L_0 is the **greatest upper bound (glb)** of S, written $L_0 = \sqcup S$, or L_0=glb(S), iff:

m_0 is upper bound and

if m is any upper bound on S, then $m_0 \leq m$

Computing lub and glb for tuple and function types

$$(x_1, y_1) \sqcup (x_2, y_2) = (x_1 \sqcup x_2,\ y_1 \sqcup y_2)$$

$$(x_1, y_1) \sqcap (x_2, y_2) = (x_1 \sqcap x_2,\ y_1 \sqcap y_2)$$

$$(x_1 \to y_1) \sqcup (x_2 \to y_2) = (x_1 \sqcap y_1) \to (y_1 \sqcup y_2)$$

$$(x_1 \to y_1) \sqcap (x_2 \to y_2) = (x_1 \sqcup y_1) \to (y_1 \sqcap y_2)$$

Lattice

Partial order: binary relation $\leq$ (subset of some D^2) which is

- reflexive: $x \leq x$
- anti-symmetric: $x \leq y \wedge y \leq x \rightarrow x = y$
- transitive: $x \leq y \wedge y \leq z \rightarrow x \leq z$

Lattice is a partial order in which every **two-element** set has **lub** and **glb**

- Lemma: if $(D, \leq)$ is lattice and D is finite, then lub and glb exist for every finite set

Idea of Why Lemma Holds

- $lub(x_1,lub(x_2,...,lub(x_{n-1},x_n)))$ is $lub(\{x_1,...x_n\})$
- $glb(x_1,glb(x_2,...,glb(x_{n-1},x_n)))$ is $glb(\{x_1,...x_n\})$
- lub of all elements in D is maximum of D
  - by definition, glb({}) is the maximum of D
- glb of all elements in D is minimum of D
  - by definition, lub({}) is the minimum of D

Graphs and Partial Orders

- If the domain is finite, then partial order can be represented by directed graphs
  - if $x \leq y$ then draw edge from x to y
- For partial order, no need to draw $x \leq z$ if $x \leq y$ and $y \leq z$. So we only draw non-transitive edges
- Also, because always $x \leq x$, we do not draw those self loops
- Note that the resulting graph is acyclic: if we had a cycle, the elements must to be equal

Defining Abstract Interpretation

Abstract Domain D (elements are data-flow **facts**), describing which information to compute, e.g.

- inferred types for each variable: x:C, y:D
- interval for each variable x:[a,b], y:[a',b']

Transfer Functions, [[**st**]] for each statement **st**, how this statement affects the facts

- Example:

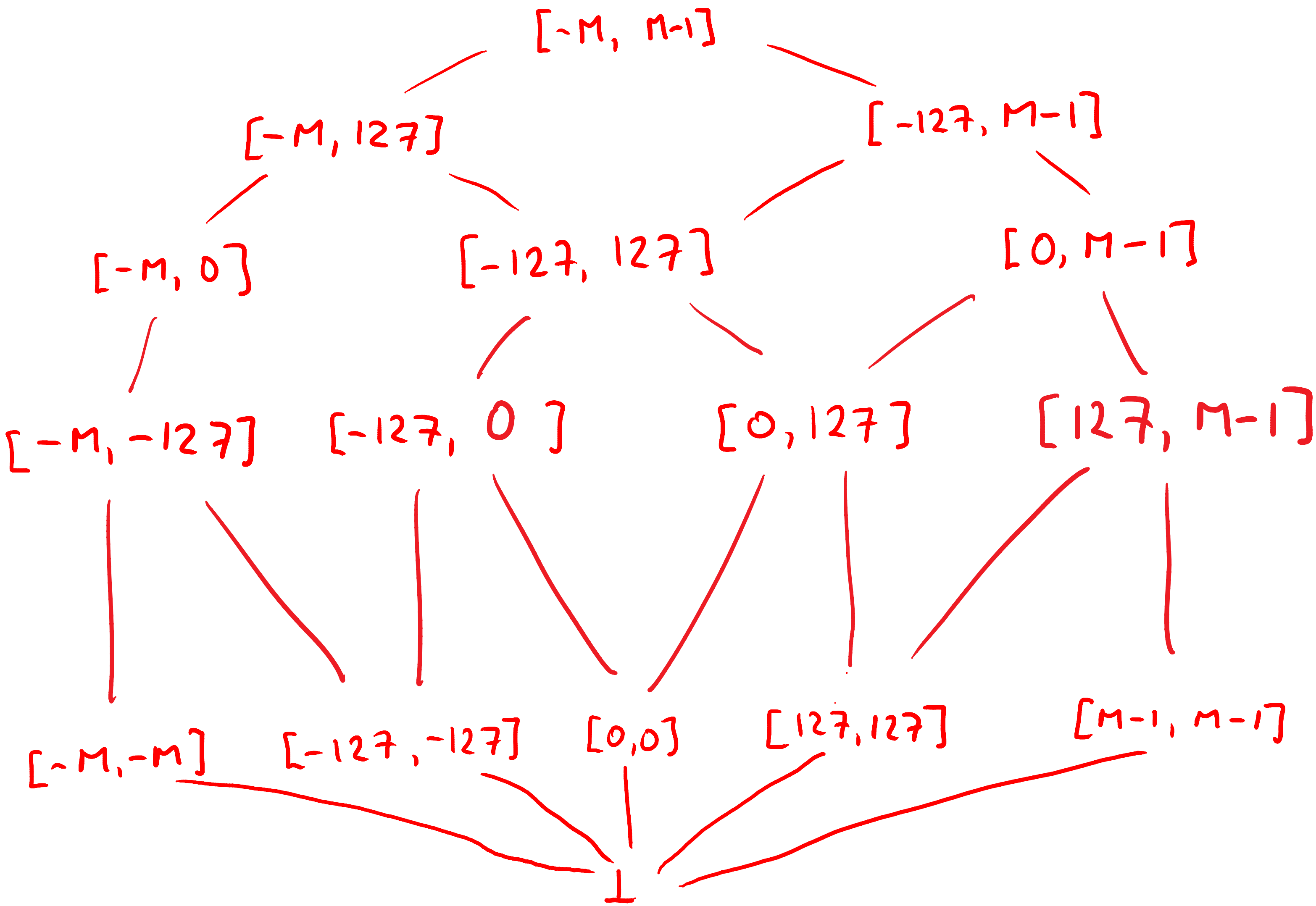
$$[\![x = x+2]\!](x:[a,b], \ldots) = (x:[a+2, b+2], \ldots)$$

x:[a,b] y:[c,d]

$\downarrow$ x = x+2

x:[a+2, b+2], y:[c,d]

Domain of Intervals [a,b] where a,b∈{-M,-127,0,127,M-1}



Find Transfer Function: Plus

Suppose we have only two integer variables: x,y

$x: [a,b]$ $y: [c,d]$

$x = x+y$

$x: [a',b']$ $y: [c',d']$

If $a \le x \le b$ $c \le y \le d$

and we execute **x= x+y**

then $x' = x+y$
$y' = y$

so

$\le x' \le$

$\le y' \le$

So we can let

$a' = a+c$ $b' = b+d$

$c' = c$ $d' = d$

Find Transfer Function: Minus

Suppose we have only two integer variables: x,y

$x : [a,b] \quad y : [c,d]$

$y = x - y$

$x : [a',b'] \quad y : [c',d']$

If

and we execute **y= x-y**

then

So we can let

$a' = a \qquad b' = b$

$c' = a - d \qquad d' = b - c$