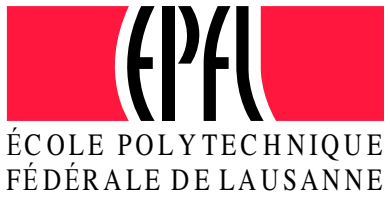
A Taste of CVC4

Part 2: Quantified Formulas

Cesare Tinelli Andrew Reynolds Clark Barrett



Quantified Formulas in SMT

$\forall x.P(x)$

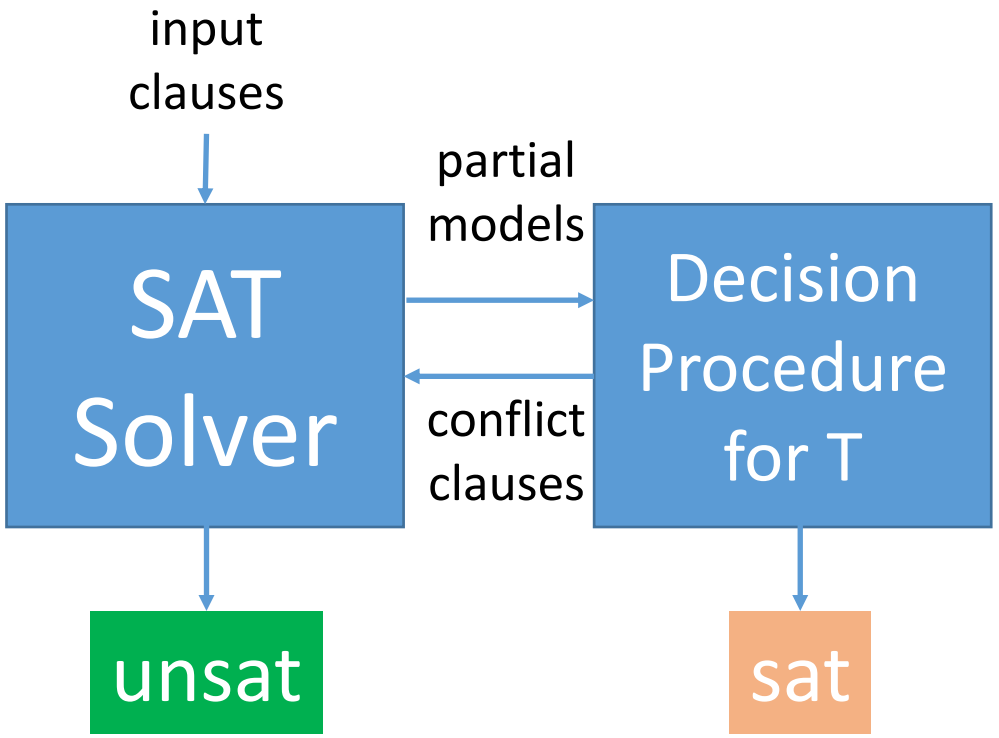
P is true for all x, where P is a formula involving some background theory

- Satisfiability problem is undecidable in general
- $\forall$ are critical for applications:
  - Automated Theorem Proving
  - Software/Hardware verification
  - Synthesis, planning, …
- $\forall$ are handled in SMT solvers by a variety of techniques:
  - Complete techniques for certain fragments
  - Heuristic techniques for the general case

Overview

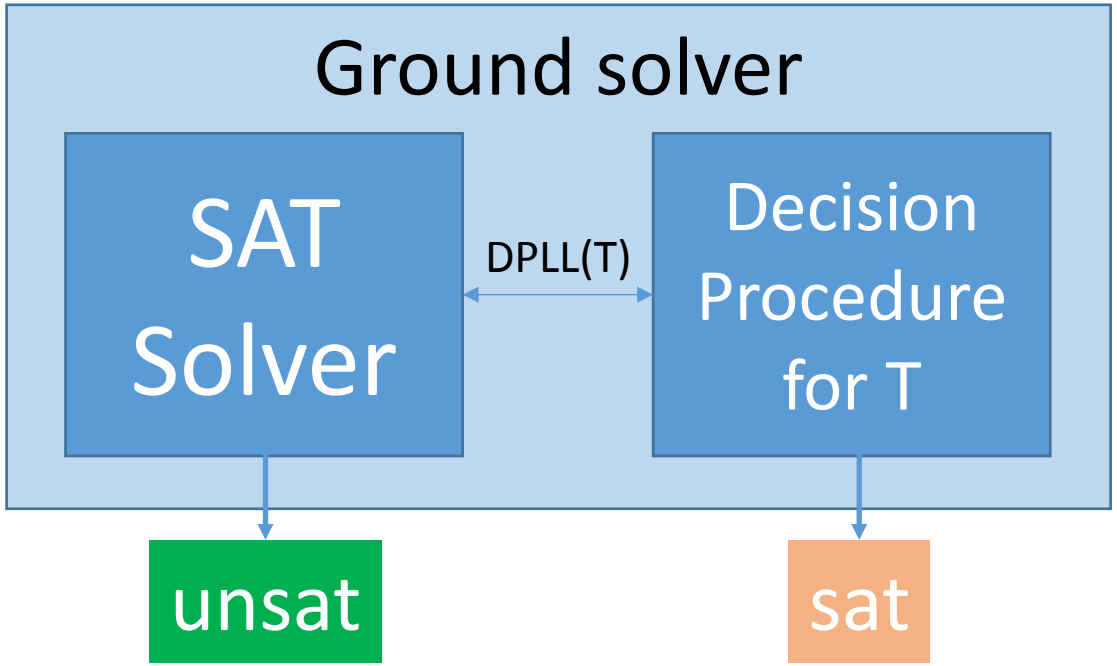
- How do we extend SMT solvers for quantified formulas?
- Quantifier Instantiation in CVC4:
  - Heuristic (E-matching)
  - Model-based
  - Conflict-based
- More advanced techniques in CVC4:
  - Finite Model Finding
  - Function synthesis

DPLL(T)-based SMT Solver



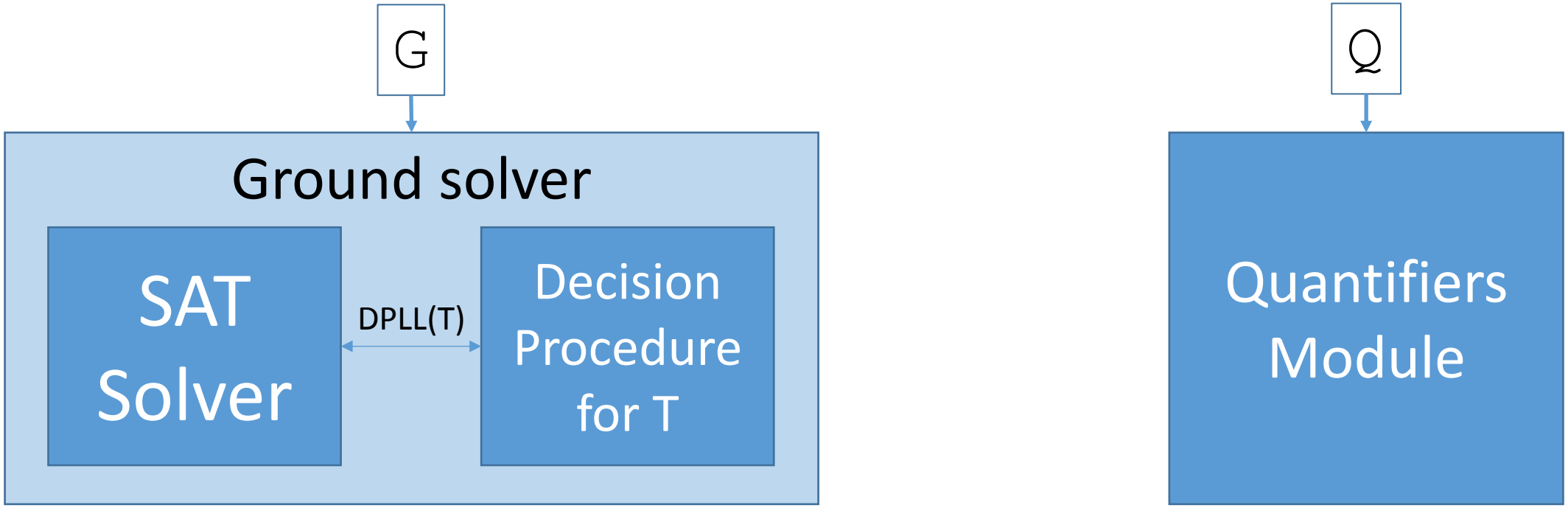
- DPLL(T)-based SMT solver
  - SAT solver maintains a set of propositional clauses
  - Decision Procedure for T determines satisfiability of conjunctions of T-literals

DPLL(T)-based SMT Solver



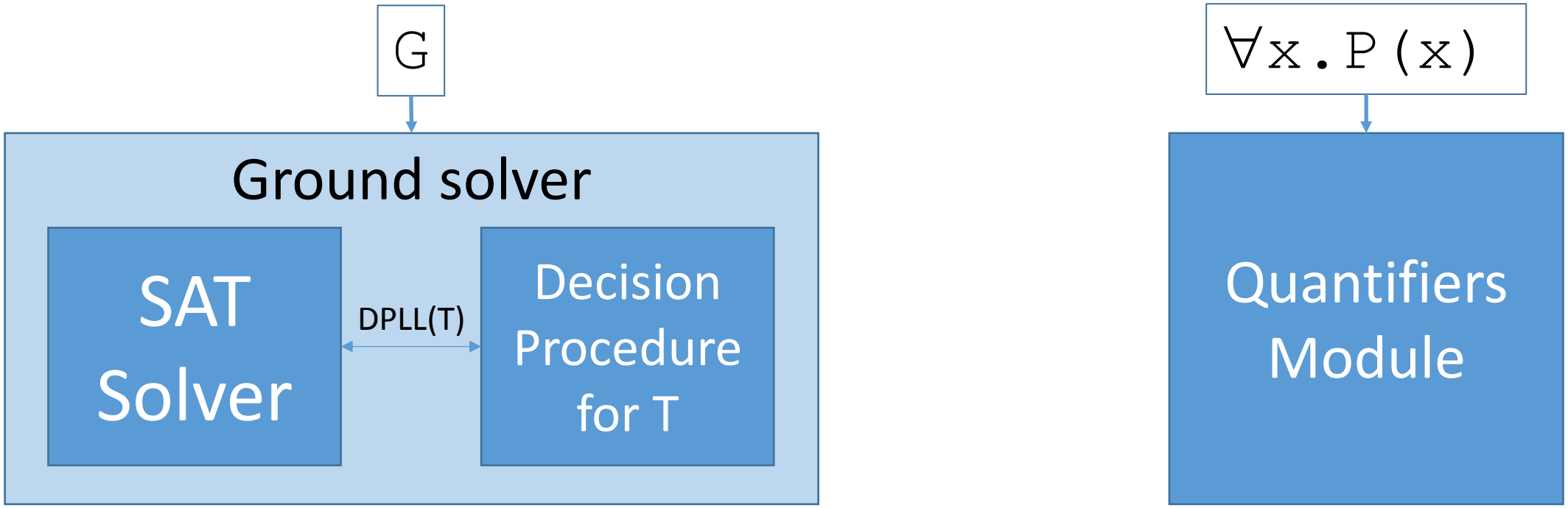
- Ground solver = SAT solver + Decision Procedure for T

DPLL(T) + Quantifiers



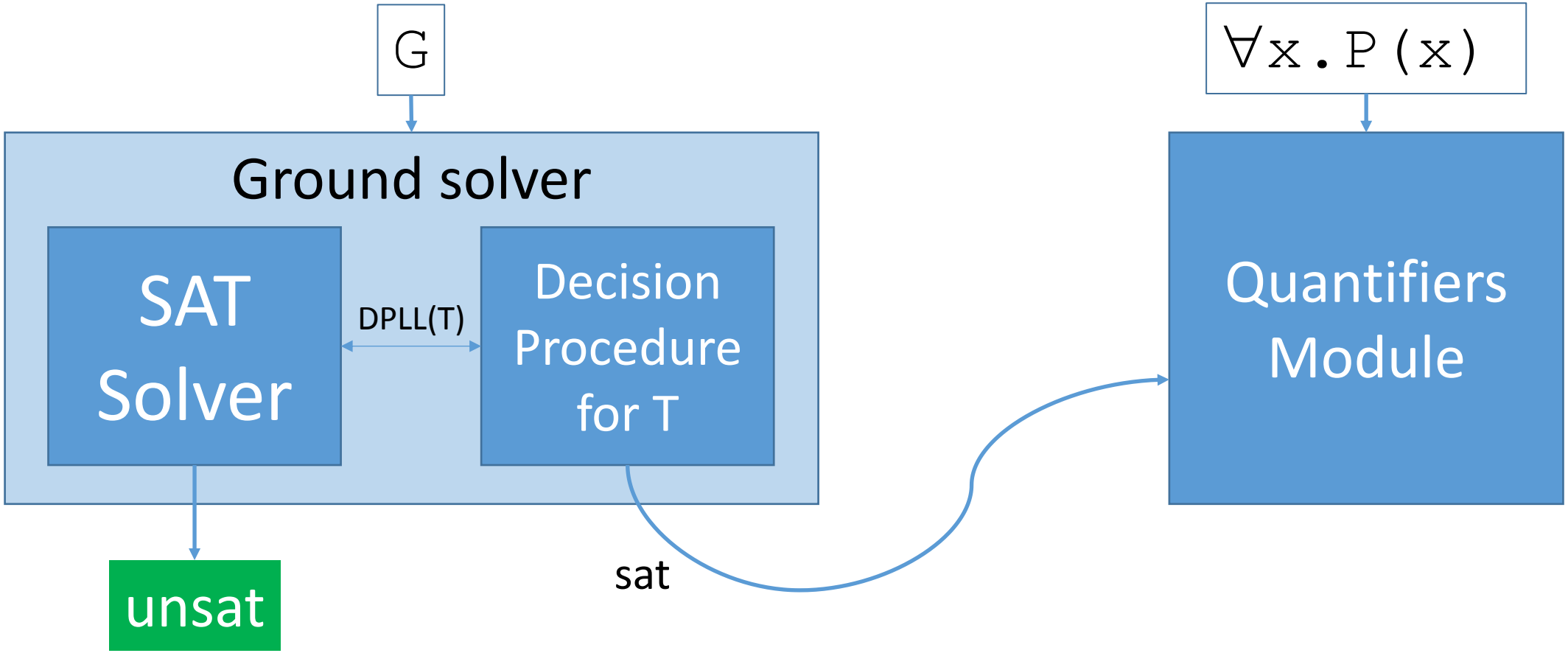
- SMT solver consists of:
  - Ground solver maintains a set of ground (quantifier-free) constraints $\mathbb{G}$
  - Quantifiers Module maintains a set of quantified formulas $\mathbb{Q}$

DPLL(T) + Quantifier Instantiation



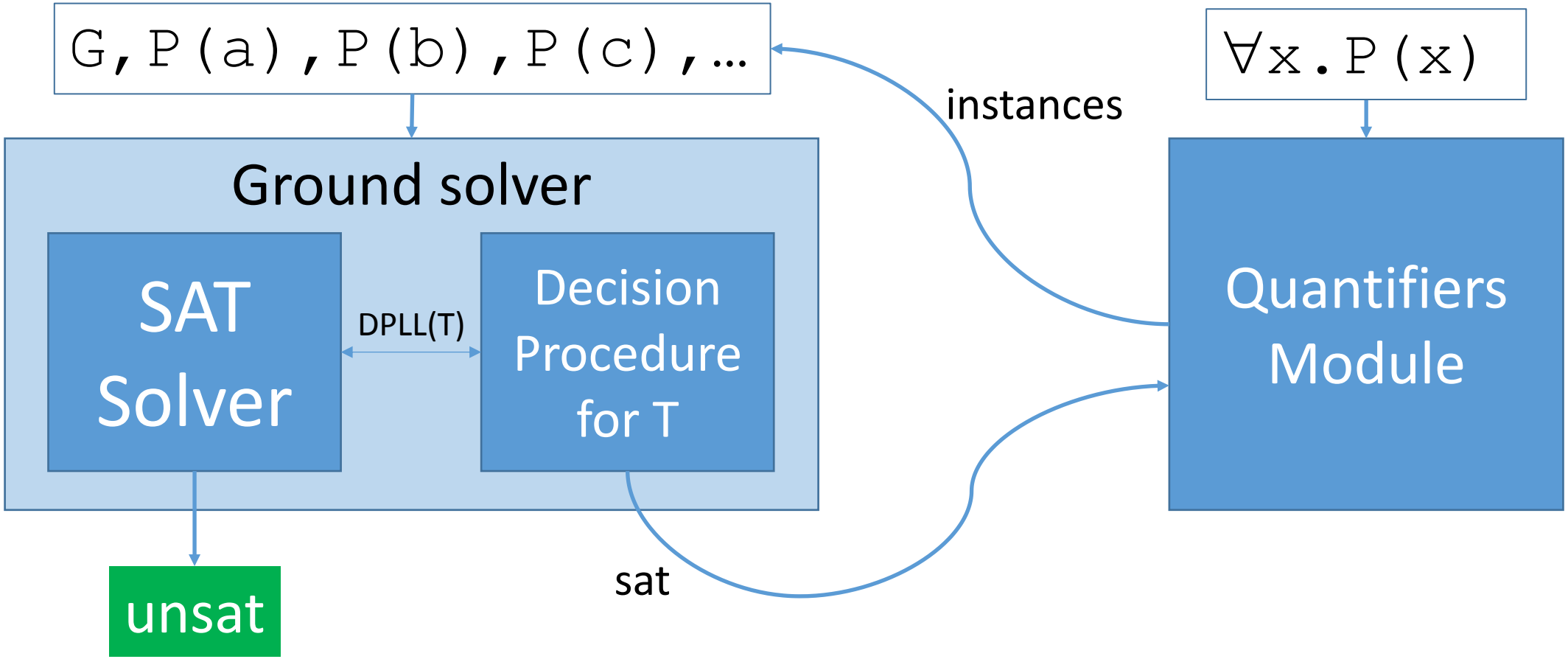
- Primary technique for quantifiers in this talk: **Quantifier Instantiation**

DPLL(T) + Quantifier Instantiation



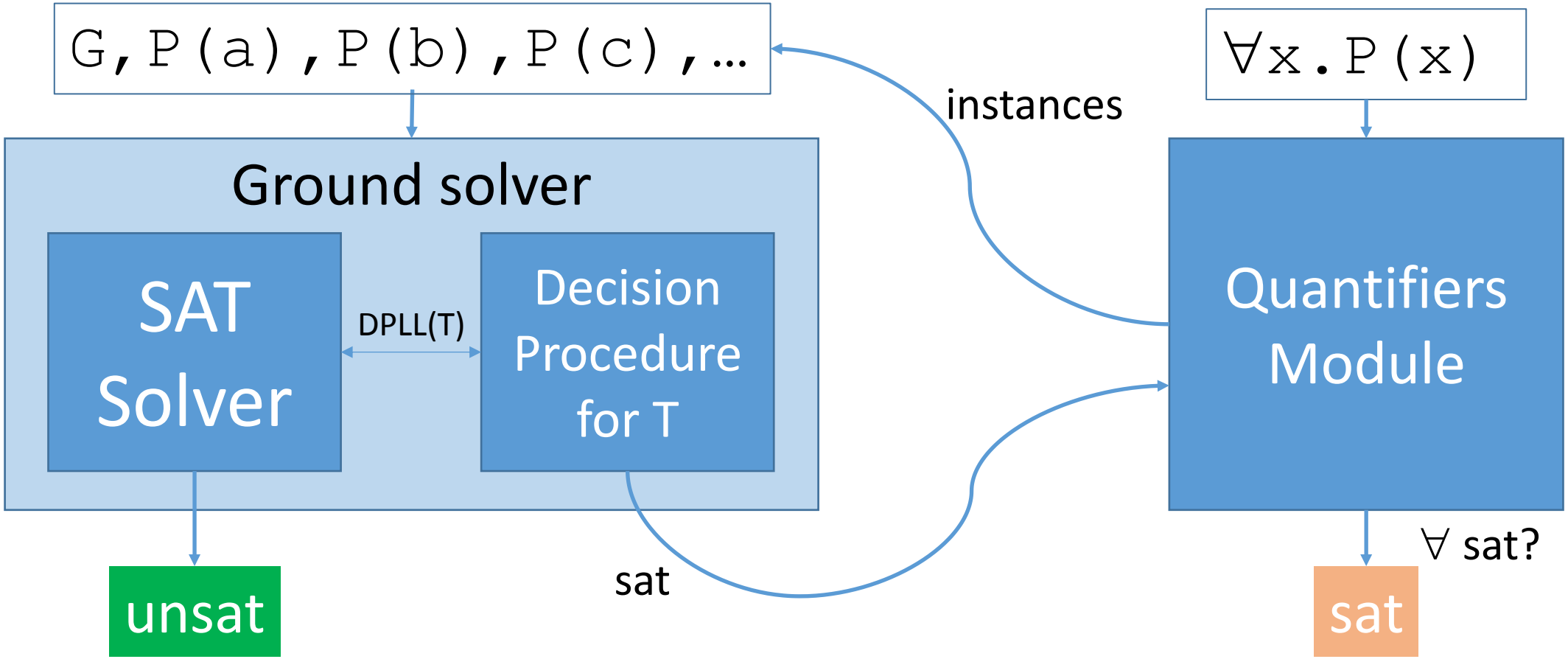
- If G is T-satisfiable, invoke quantifiers module

DPLL(T) + Quantifier Instantiation



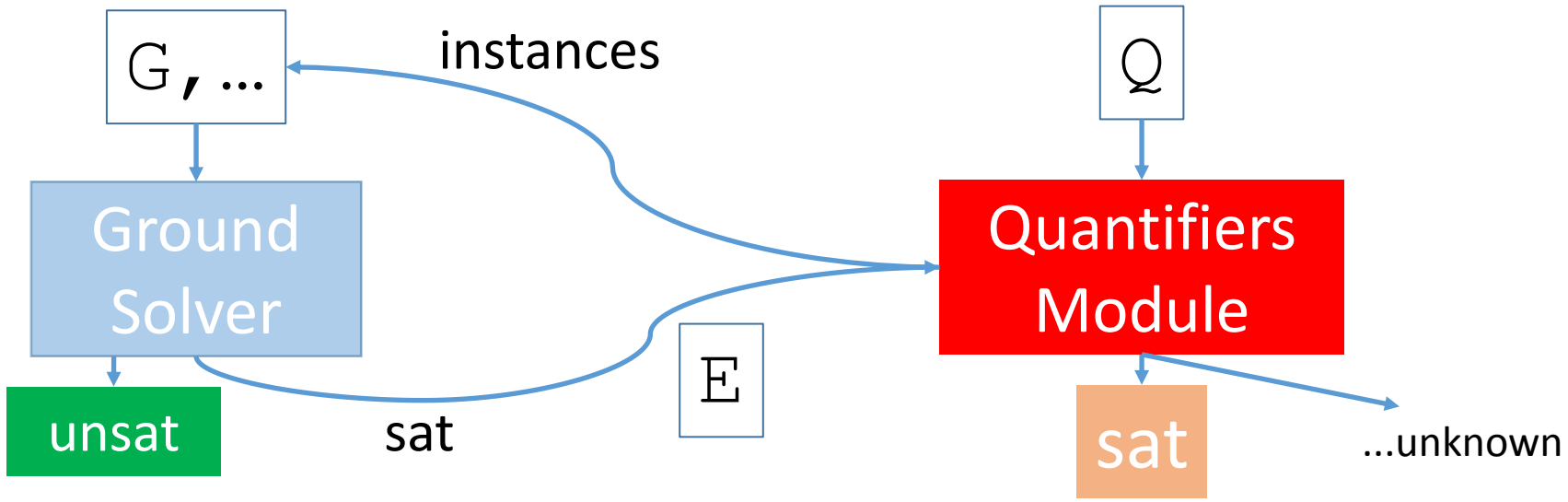
- Add instances of axioms to G

DPLL(T) + Quantifier Instantiation



- …and repeat, generally a sound but incomplete procedure
  - Difficult to answer sat (when have we added enough instances of $\forall x.P(x)$?)

Quantifiers Module: Overview



- Inputs:
  - Set of ground formulas $\mathtt{G}$
- Outputs:
  - "G is T-unsat", or
  - "G is T-sat", set of literals $\mathtt{E} \models_p \mathtt{G}$

- Inputs:
  - Set of ground T-literals $\mathtt{E}$
  - Set of quantified T-formulas $\mathtt{Q}$
- Outputs:
  - "$\mathtt{E} \wedge \mathtt{Q}$ is T-sat"
  - Set of instances of $\mathtt{Q}$ to add to $\mathtt{G}$
  - …"unknown" (give up)

Quantifier Instantiation : Design Decisions

- When do we invoke it?
  - Eagerly, during the DPLL(T) search **[deMoura/Bjorner CAV07]**, or
  - Lazily, only after ground solver answers “sat”

Quantifier Instantiation : Design Decisions

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  - …

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  - …
- Can we terminate?

  i.e. can we ever answer “sat”?

Quantifier Instantiation : in CVC4

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  - Eagerly, during the DPLL(T) search [deMoura/Bjorner CAV09], or
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Quantifier Instantiation : in CVC4

- When do we invoke it?
  - Eagerly, during the DPLL(T) search [deMoura/Bjorner CAV09], or
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- Which instances do we add?
  - E-matching **[Detlefs et al 03]**
  - Model-based quantifier instantiation **[Ge/de Moura CAV09]**
  - Conflict-based quantifier instantiation **[Reynolds et al FMCAD14]**
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Quantifier Instantiation : in CVC4

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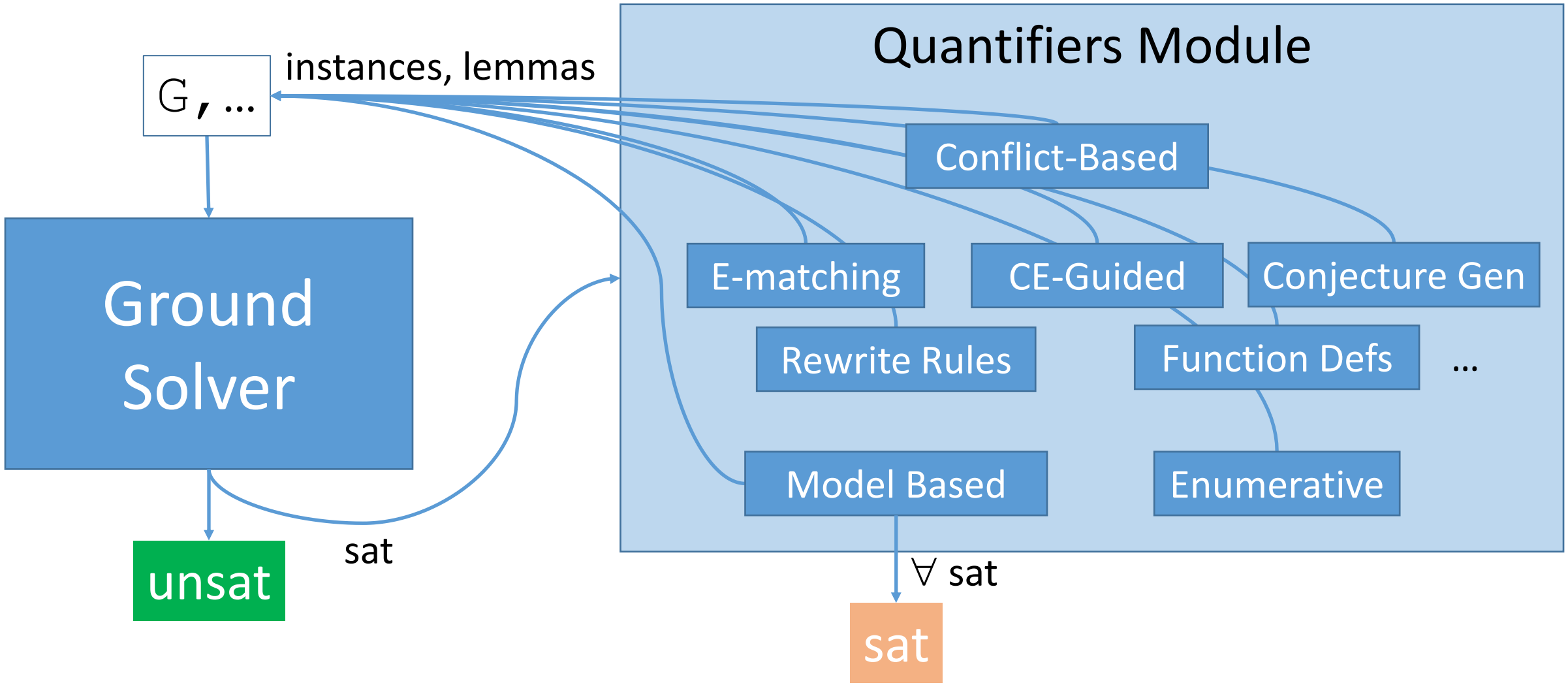
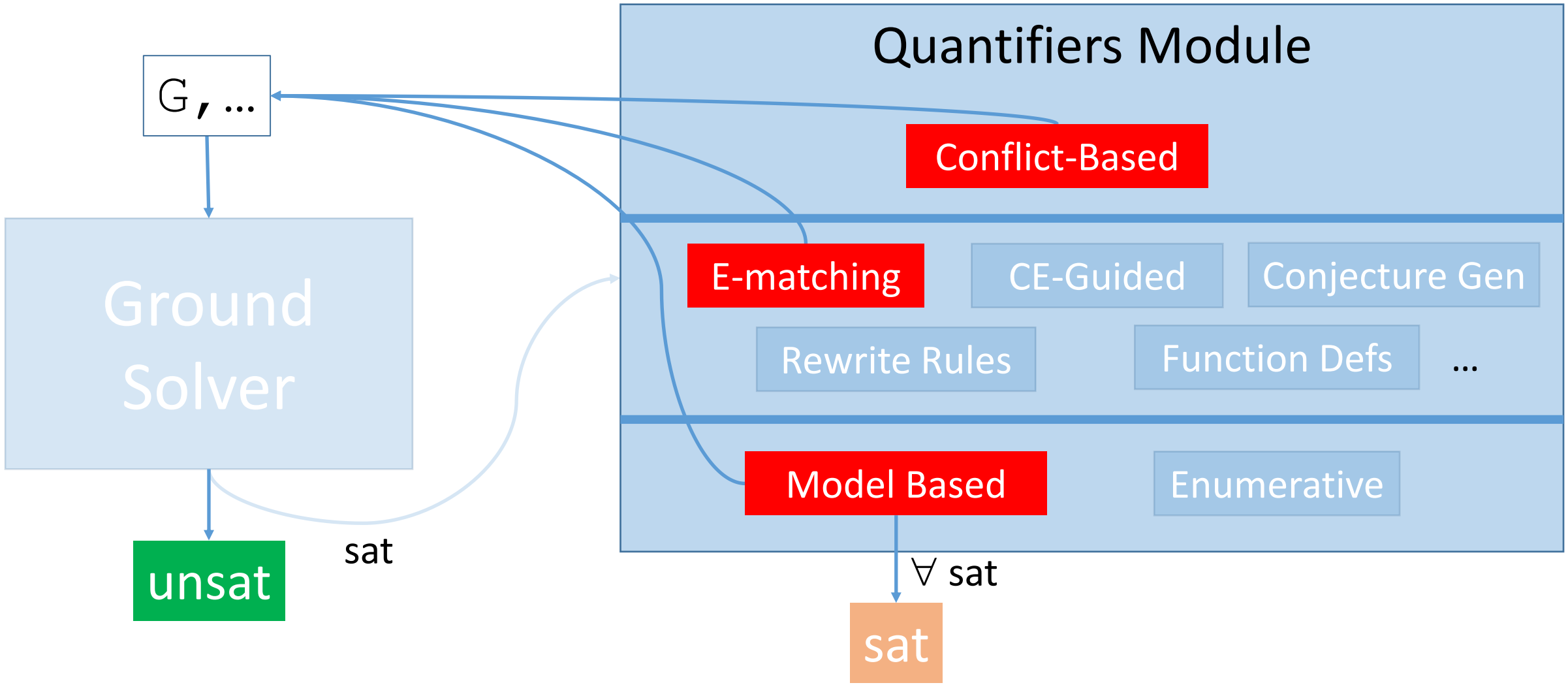
  - Finite Model Finding **[Reynolds et al CADE13]**
  - Instantiation for linear arithmetic

Quantifiers Module of CVC4



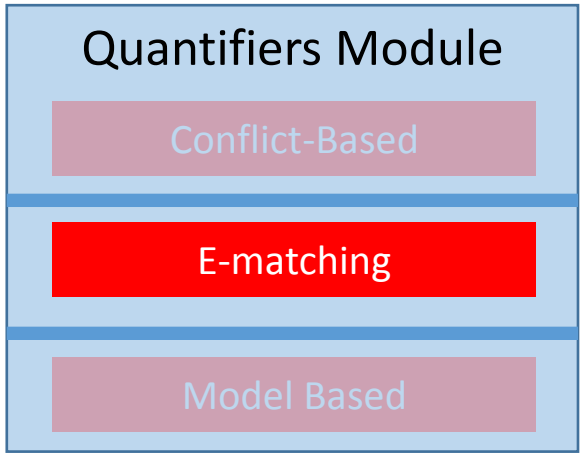
- CVC4's quantifiers module contains numerous strategies and techniques

Quantifiers Module of CVC4



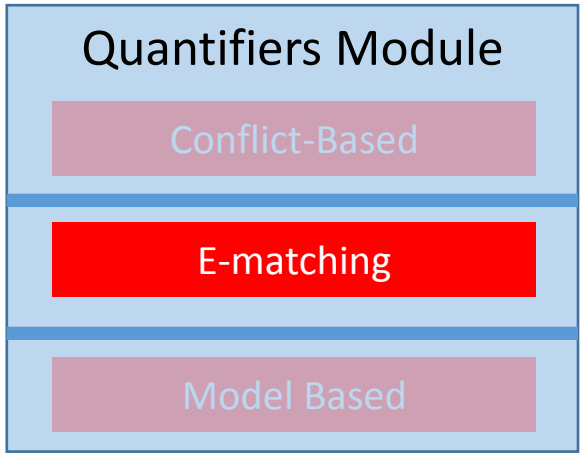
- Core techniques: Conflict-based, Heuristic (e.g. E-matching), Model-based

E-matching



- E-matching:
  - Most widely used and successful technique for quantifiers in SMT
  - Implemented in numerous solvers:
    - Z3, CVC3, CVC4, VeriT, Alt-Ergo, …

E-matching: Example



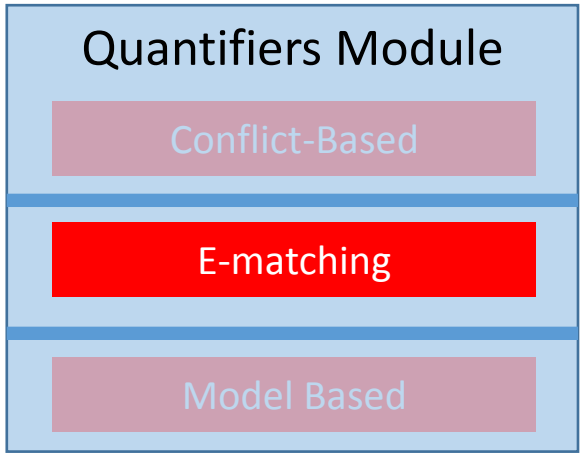
$a,b,c:S$

$f,g:S \rightarrow S$

$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)\neq a$ } E

$\forall x.f(x)=g(x)$ } Q

E-matching: Example



$a,b,c:S$

$f,g:S \rightarrow S$

$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a) \neq a$

$\forall x.\ \mathbf{f(x)} = g(x)$

Pattern

- **Idea**: choose instances based on pattern matching

E-matching: Example

$a,b,c:S$

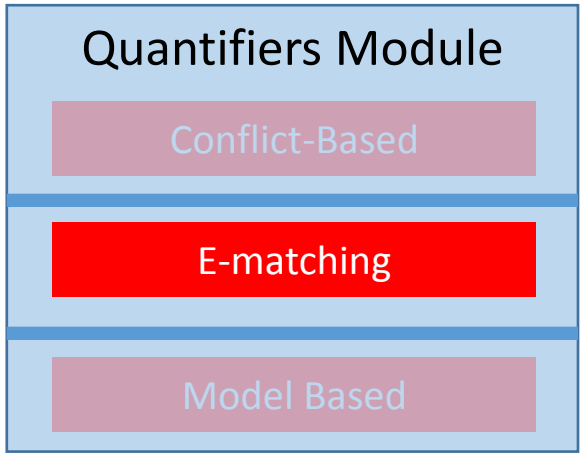
$f,g:S \rightarrow S$

$\mathbf{f(a)}=a,\ \mathbf{f(b)}=b,\ \mathbf{f(c)}=c,\ g(a) \neq a$

$\{x \rightarrow a\}$ $\{x \rightarrow b\}$ $\{x \rightarrow c\}$

$\forall x.\mathbf{f(x)}=g(x)$

E-matching: Example



$a,b,c:S$

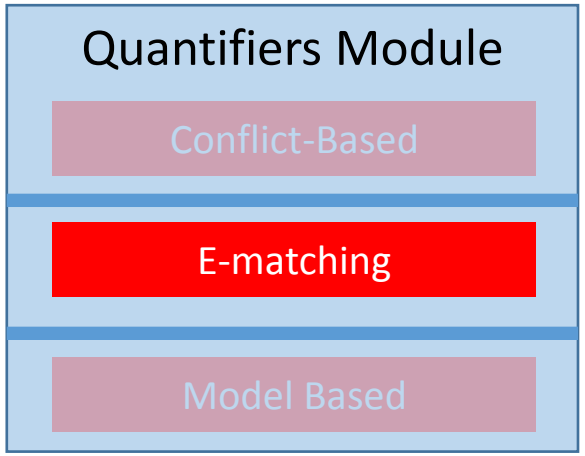
$f,g:S \rightarrow S$

$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a) \neq a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c)$

$\forall x.f(x)=g(x)$

E-matching: Example



$a,b,c:S$

$f,g:S \rightarrow S$

$f(a)=a$, $f(b)=b$, $f(c)=c$, **$g(a) \neq a$**

$f(a)=g(a)$, $f(b)=g(b)$, $f(c)=g(c)$

unsat

$\forall x.f(x)=g(x)$

EXAMPLE…

E-matching: Challenges

- What happens when there too many instances to add?
  - E-matching adds many instances, degrades performance for solver to continue

- What happens when there are no instances to add?
  - E-matching is an incomplete procedure, cannot answer SAT even when saturated

E-matching: Challenges

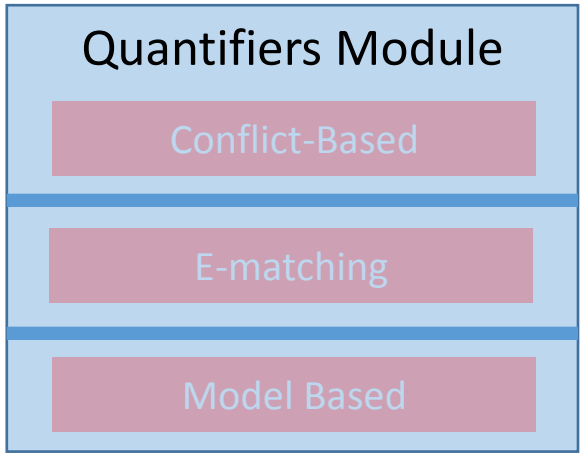
- What happens when there too many instances to add?
  - E-matching adds many instances, degrades performance for solver to continue

  $\Rightarrow$*Use conflict-based instantiation* **[Reynolds/Tinelli/deMoura FMCAD14]**

- What happens when there are no instances to add?
  - E-matching is an incomplete procedure, cannot answer SAT even when saturated

  $\Rightarrow$*Use model-based instantiation* **[Ge/deMoura CAV09]**

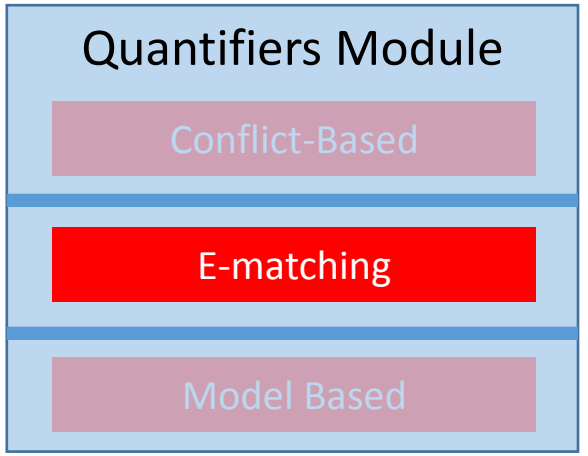
Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ \mathbf{g(a)=a}$

$\forall x.f(x)=g(x)$

Model-based Instantiation: Example



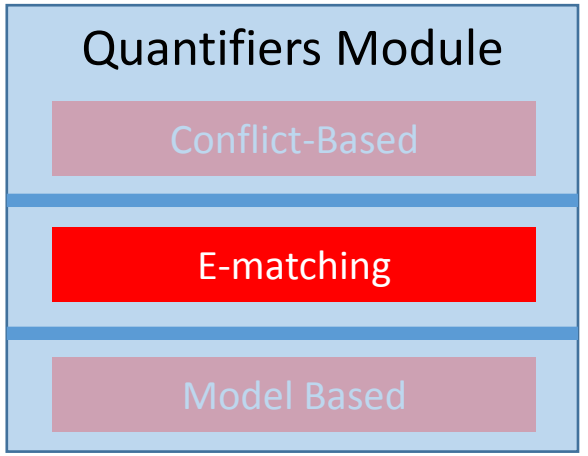
$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c)$

$\forall x.f(x)=g(x)$

- Add instances by E-matching, as before

Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

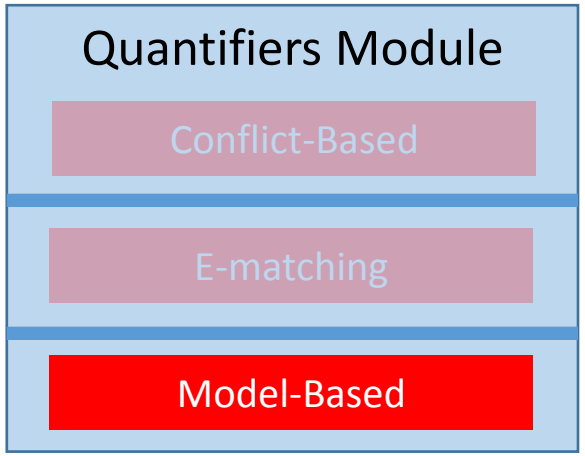
$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c)$

$\forall x.f(x)=g(x)$

sat

- E-matching saturates, but ground constraints are satisfiable
  - Can we check that $\forall x.f(x)=g(x)$ is also satisfiable?

Model-based Instantiation: Example



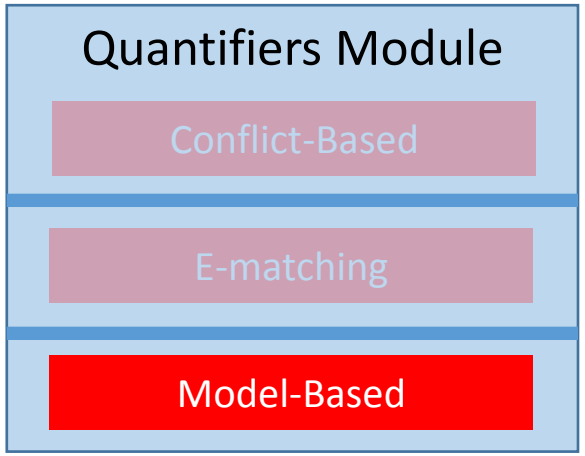
$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c)$

$\forall x.f(x)=g(x)$

- **Idea**: construct candidate model M for functions f and g
  - Check if $\forall x.f(x)=g(x)$ satisfied by M

Model-based Instantiation: Example



f(a)=a, f(b)=b, f(c)=c, g(a)=a

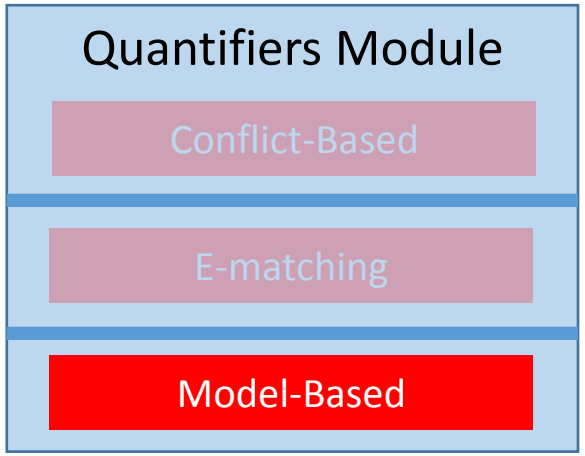
f(a)=g(a), f(b)=g(b), f(c)=g(c)

∀x.f(x)=g(x)

f^M := λx.ite(x=a,a,ite(x=b,b,c))

M

Model-based Instantiation: Example



f(a)=a, f(b)=b, f(c)=c, g(a)=a

f(a)=g(a), f(b)=g(b), f(c)=g(c)

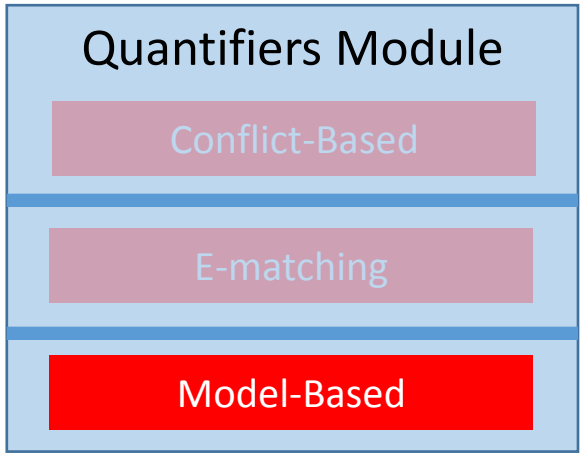
∀x.f(x)=g(x)

f^M := λx.ite(x=a,a,ite(x=b,b,c))

g^M := λx.ite(x=a,a,ite(x=b,b,c))

M

Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c)$

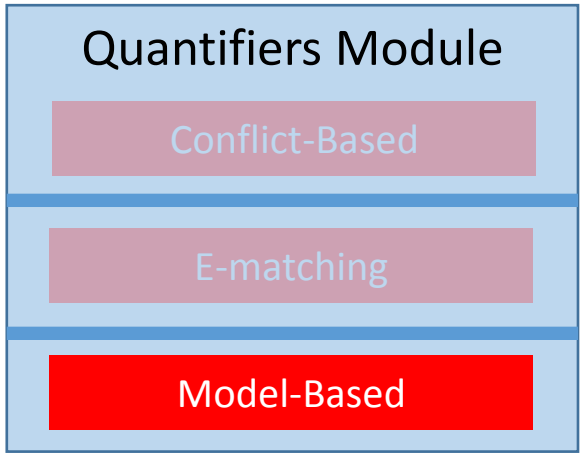
$\forall x.f(x)=g(x)$

$$f^{M} := \lambda x.ite(x=a,a,ite(x=b,b,c))$$
$$g^{M} := \lambda x.ite(x=a,a,ite(x=b,b,c))$$

M

- Does M satisfy $\forall x.f(x)=g(x)$?

Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c)$

$\forall x.f(x)=g(x)$

$f^{M} := \lambda x.ite(x=a,a,ite(x=b,b,c))$

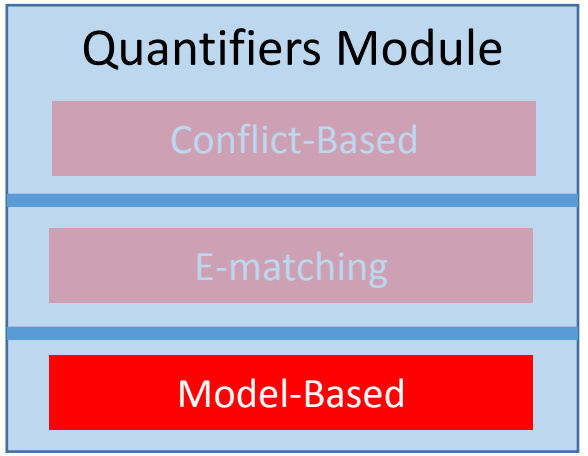
$g^{M} := \lambda x.ite(x=a,a,ite(x=b,b,c))$

M

- Does M satisfy $\forall x.f(x)=g(x)$?

$\Rightarrow$ If $\exists x.f^{M}(x) \neq g^{M}(x)$ is unsat, then yes

Model-based Instantiation: Example



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} M

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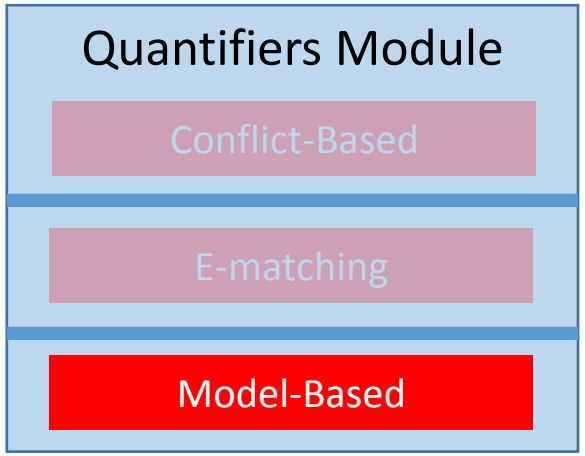
$ite(x=a,a,ite(x=b,b,c)) \neq ite(x=a,a,ite(x=b,b,c))$ } unsat

Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c)$

$\forall x.f(x)=g(x)$

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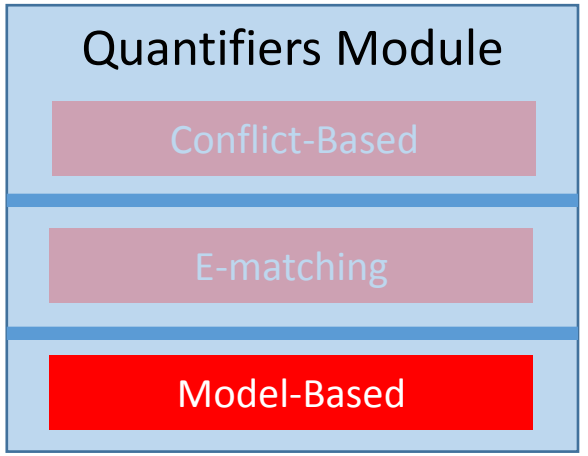
M

- Does M satisfy $\forall x.f(x)=g(x)$?

  $\Rightarrow$ Yes, return sat with model M

Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c),\ \mathbf{d \notin \{a,b,c\}}$

$\forall x.f(x)=g(x)$

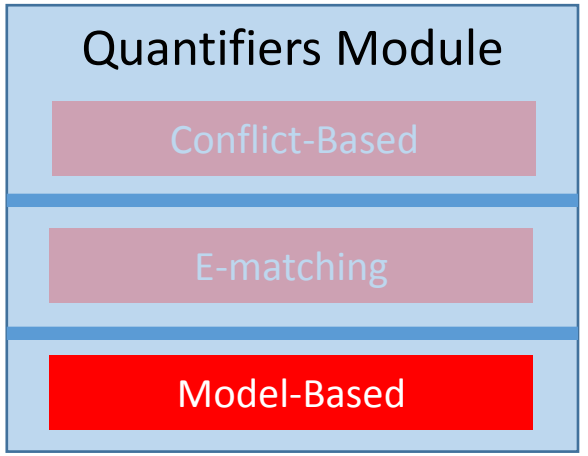
$f^M := \lambda x.ite(x=a,a,ite(x=b,b,c))$

$g^M := \lambda x.ite(x=a,a,ite(\mathbf{x=c,c,b}))$

} M

- If M does not satisfy $\forall x.f(x)=g(x)$,

Model-based Instantiation: Example



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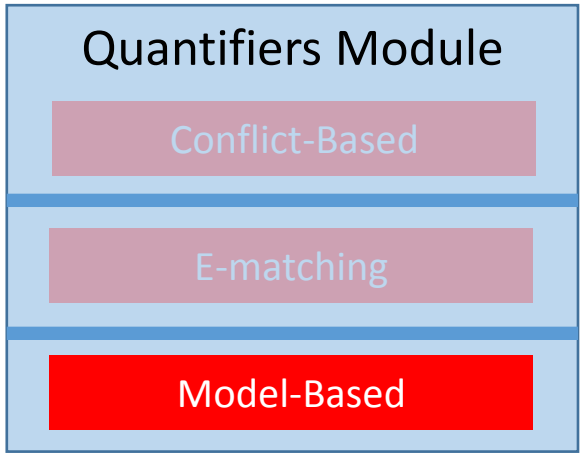
} M

- If M does not satisfy $\forall x.f(x)=g(x)$,

$ite(x=a,a,ite(x=b,b,c)) \neq ite(x=a,a,ite(x=c,c,b))$ is unsat?

Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c),\ d \notin \{a,b,c\}$

$\forall x.f(x)=g(x)$

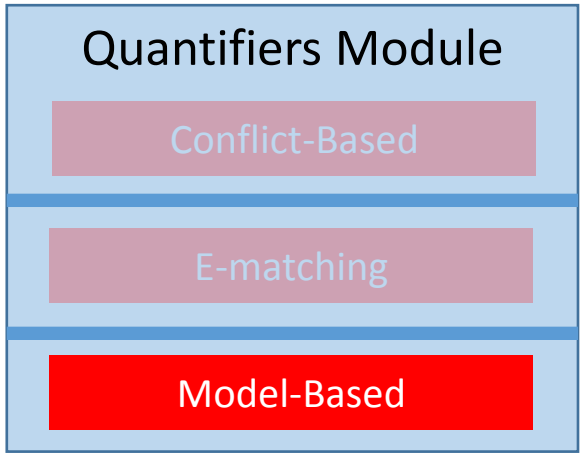
$$f^{M} := \lambda x.ite(x=a,a,ite(x=b,b,c))$$
$$g^{M} := \lambda x.ite(x=a,a,ite(x=c,c,b))$$

M

- If M does not satisfy $\forall x.f(x)=g(x)$,

$ite(d=a,a,ite(d=b,b,c)) \neq ite(d=a,a,ite(d=c,c,b))$ Take **x=d**

Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c),\ d \notin \{a,b,c\}$

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$g^M := \lambda x.ite(x=a,a,ite(x=c,c,b))$

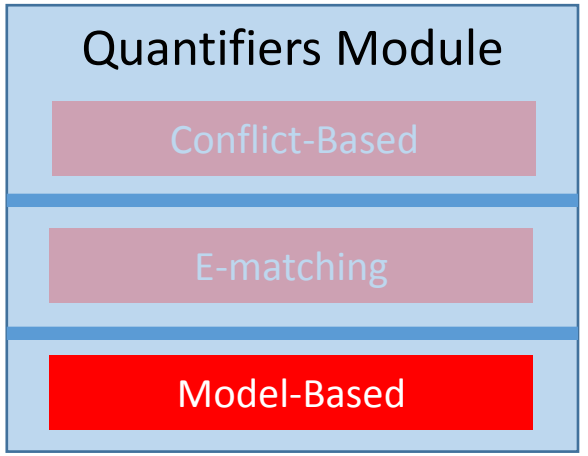
} M

- If M does not satisfy $\forall x.f(x)=g(x)$,

$c \neq b$

} sat, where $x=d$

Model-based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)=a$

$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c),\ d \notin \{a,b,c\},$

$\forall x.f(x)=g(x)$ **$f(d)=g(d)$**

$f^M := \lambda x.ite(x=a,a,ite(x=b,b,c))$

$g^M := \lambda x.ite(x=a,a,ite(x=c,c,b))$

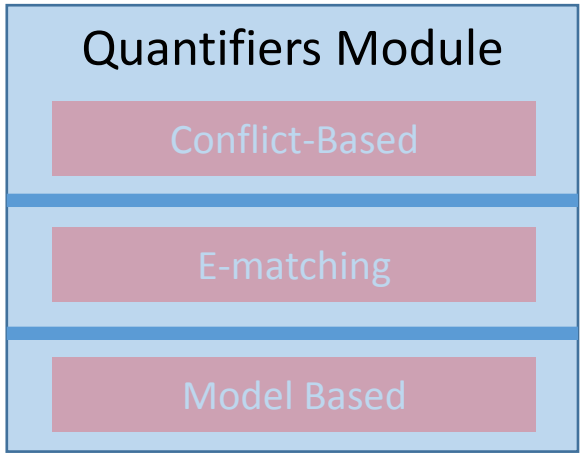
} M

- If M does not satisfy $\forall x.f(x)=g(x)$,

  $\Rightarrow$Add instance **$f(d)=g(d)$**, will refine model

EXAMPLE…

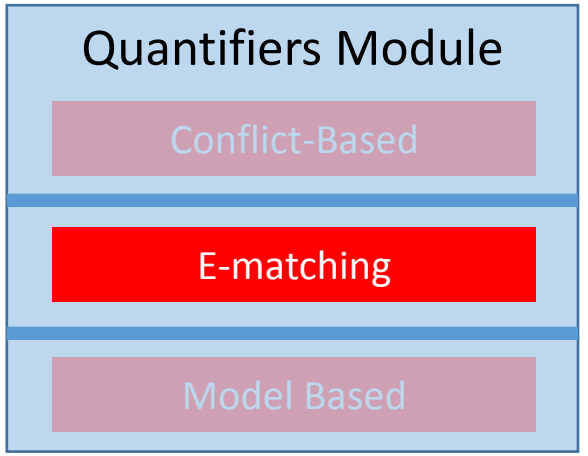
Conflict-Based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)\neq a$

$\forall x.f(x)=g(x)$

Conflict-Based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)\neq a$

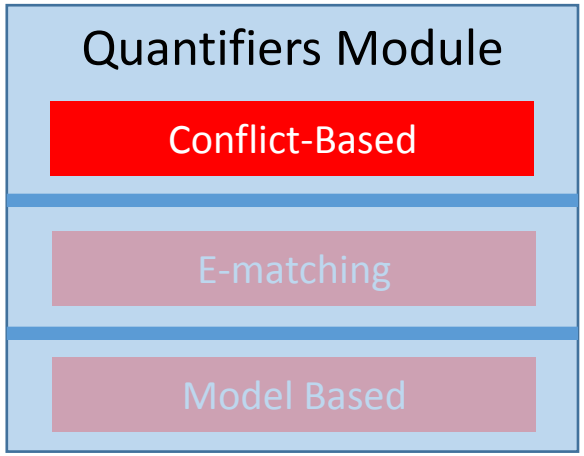
$f(a)=g(a),\ f(b)=g(b),\ f(c)=g(c),\ \ldots$

$\forall x.f(x)=g(x)$

- E-matching may return with many ground instances
  - In practice, 1000+ instances per invocation

    $\Rightarrow$Degrades solver performance

Conflict-Based Instantiation: Example

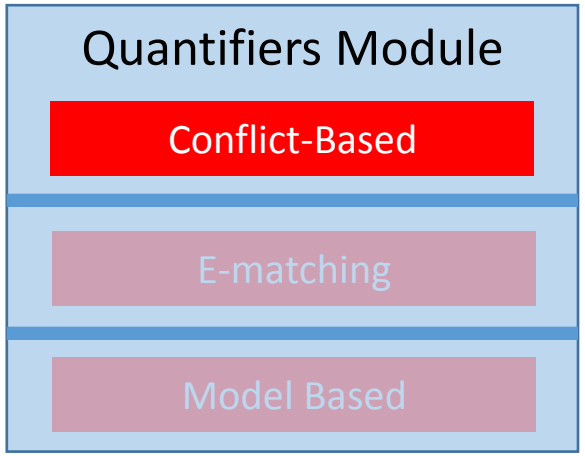


$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)\neq a$

$\forall x.f(x)=g(x)$

- **Idea**: find an instance of $\forall x.f(x)=g(x)$ that is conflicting with ground constraints
  - If so, add *only* that instance

Conflict-Based Instantiation: Example



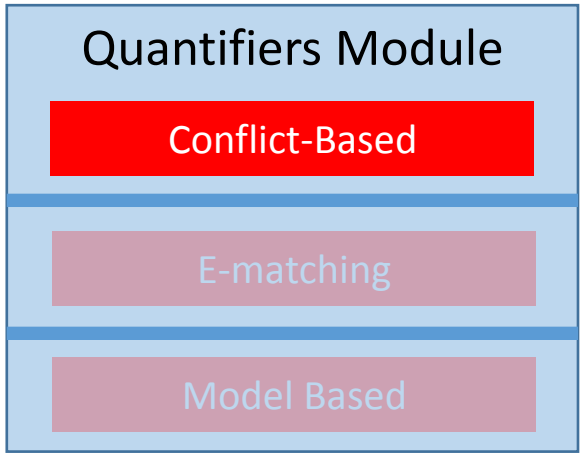
f(a)=a, f(b)=b, f(c)=c, **g(a)≠a**

∀x.f(x)=g(x)

- **Idea**: find an instance of ∀x.f(x)=g(x) that is conflicting with ground constraints

  ⇒ **f(a)=a, g(a)≠a ⊨ f(x)≠g(x){x→a}**

Conflict-Based Instantiation: Example



$f(a)=a,\ f(b)=b,\ f(c)=c,\ g(a)\neq a$

$f(a)=g(a)$

$\forall x.f(x)=g(x)$

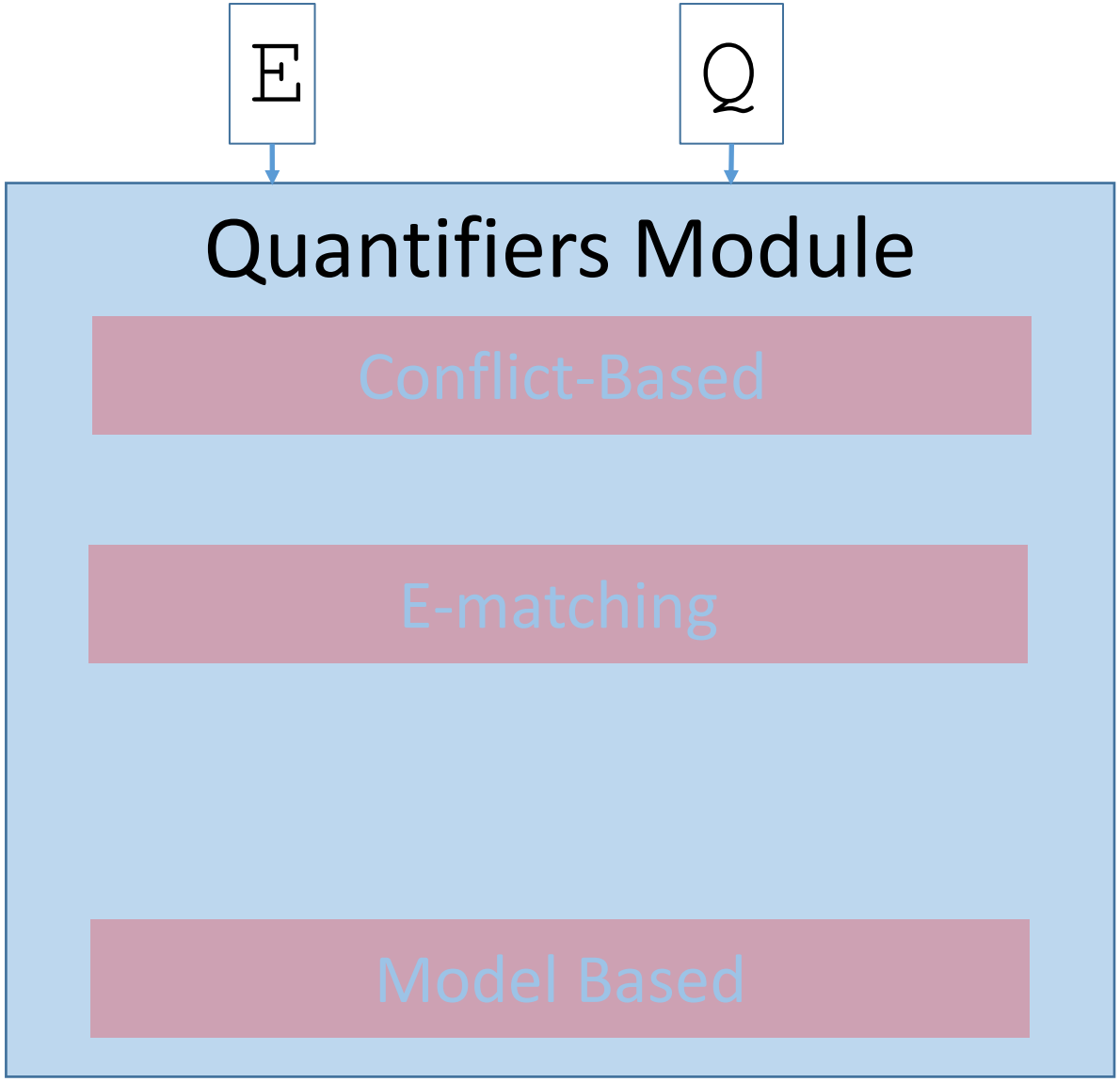
unsat

- **Idea**: find an instance of $\forall x.f(x)=g(x)$ that is conflicting with ground constraints

$$\Rightarrow f(a)=a,\ g(a)\neq a \models f(x)\neq g(x)\{x\rightarrow \mathbf{a}\}$$

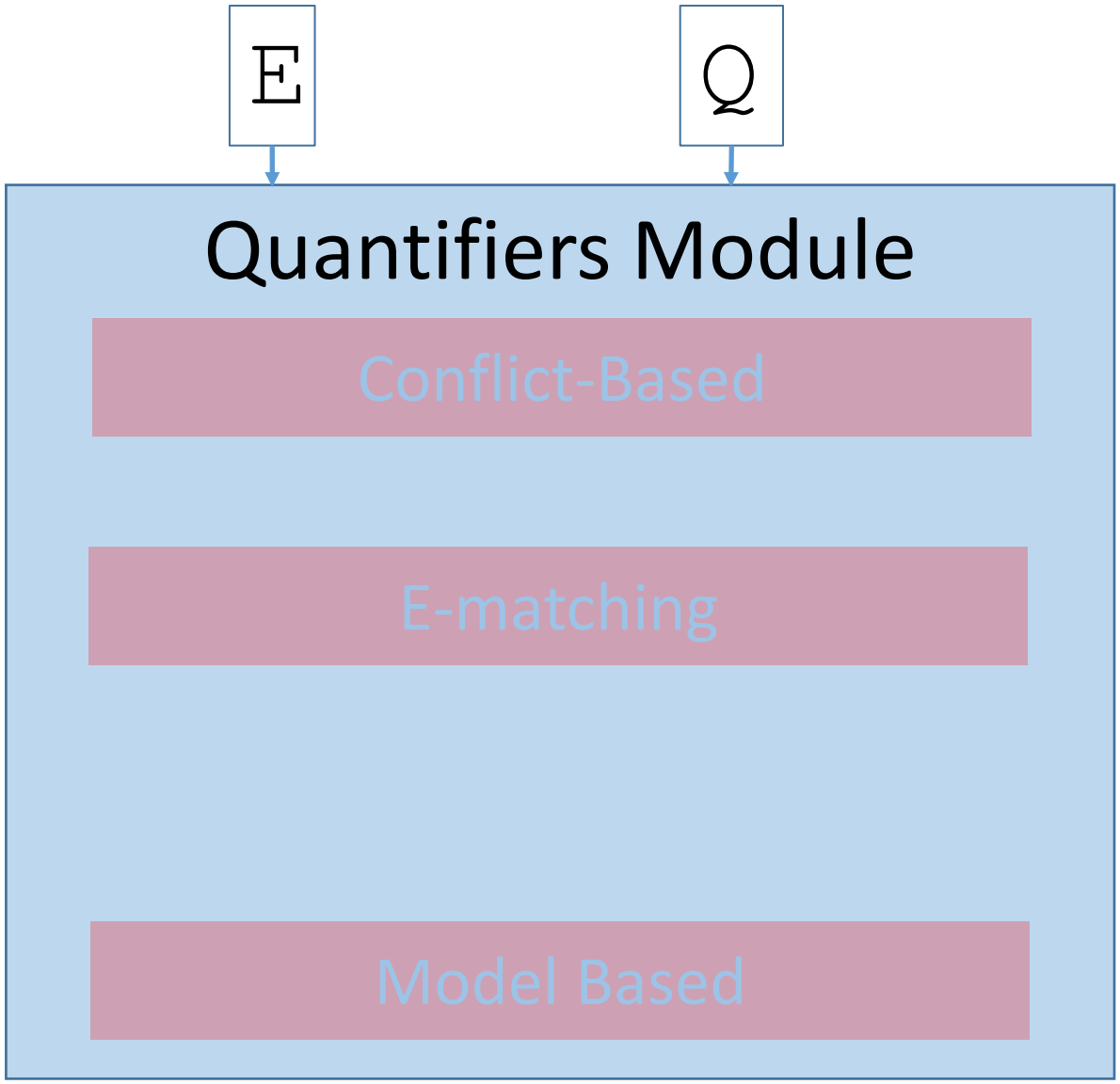
EXAMPLE…

Putting it Together

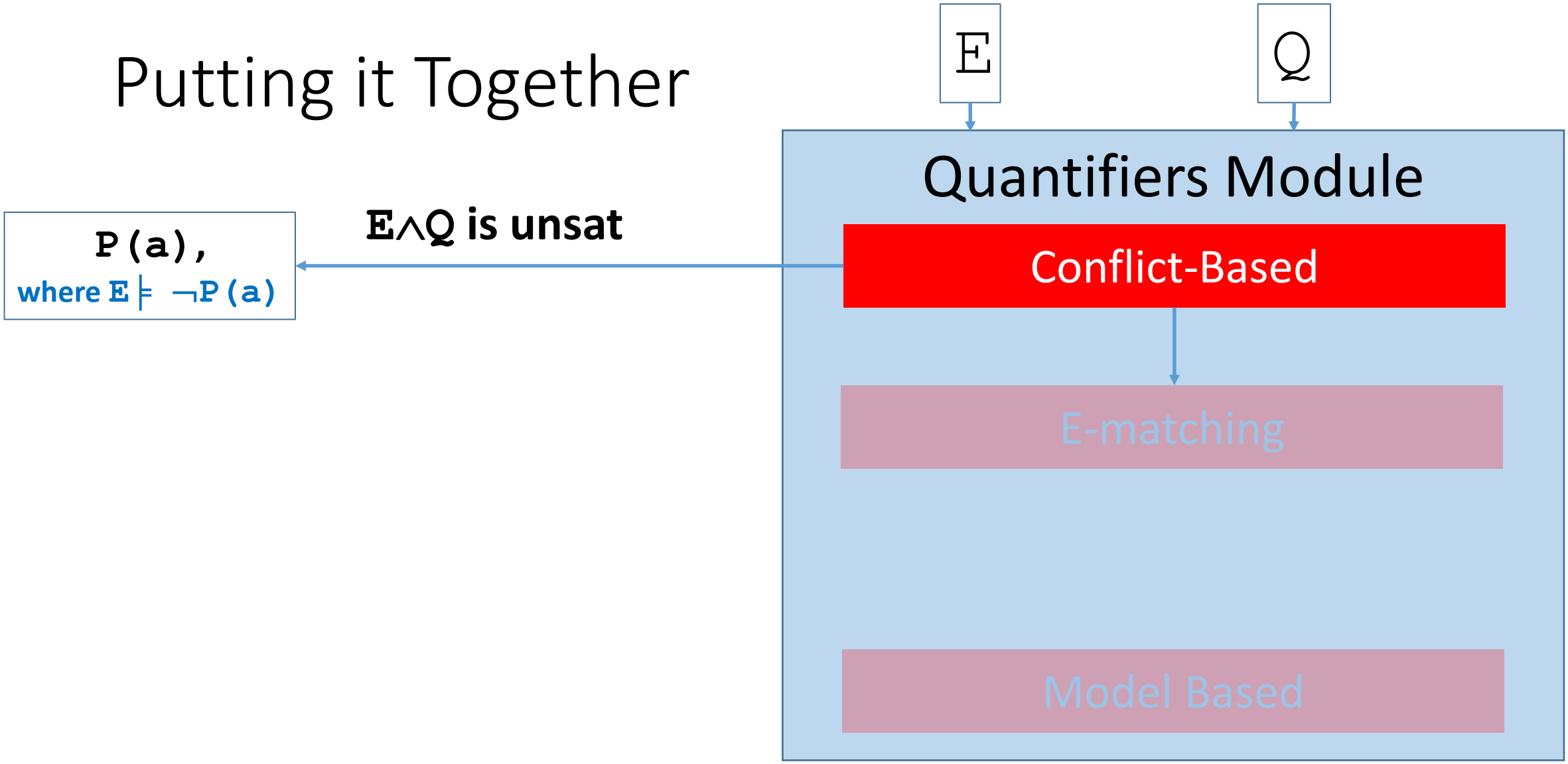


Putting it Together

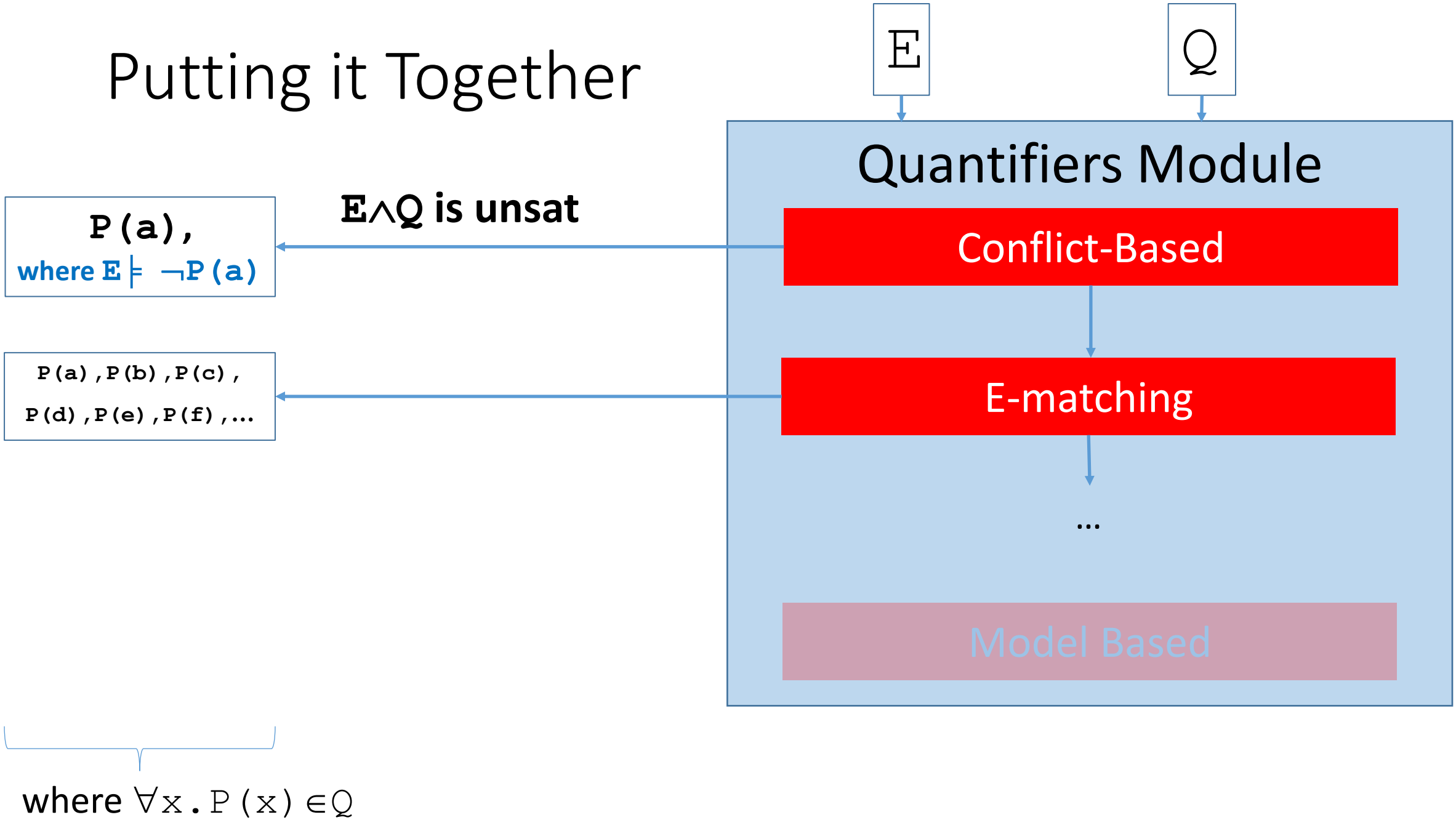
- Input:
  - Ground literals $\mathbb{E}$
  - Quantified formulas $\mathbb{Q}$



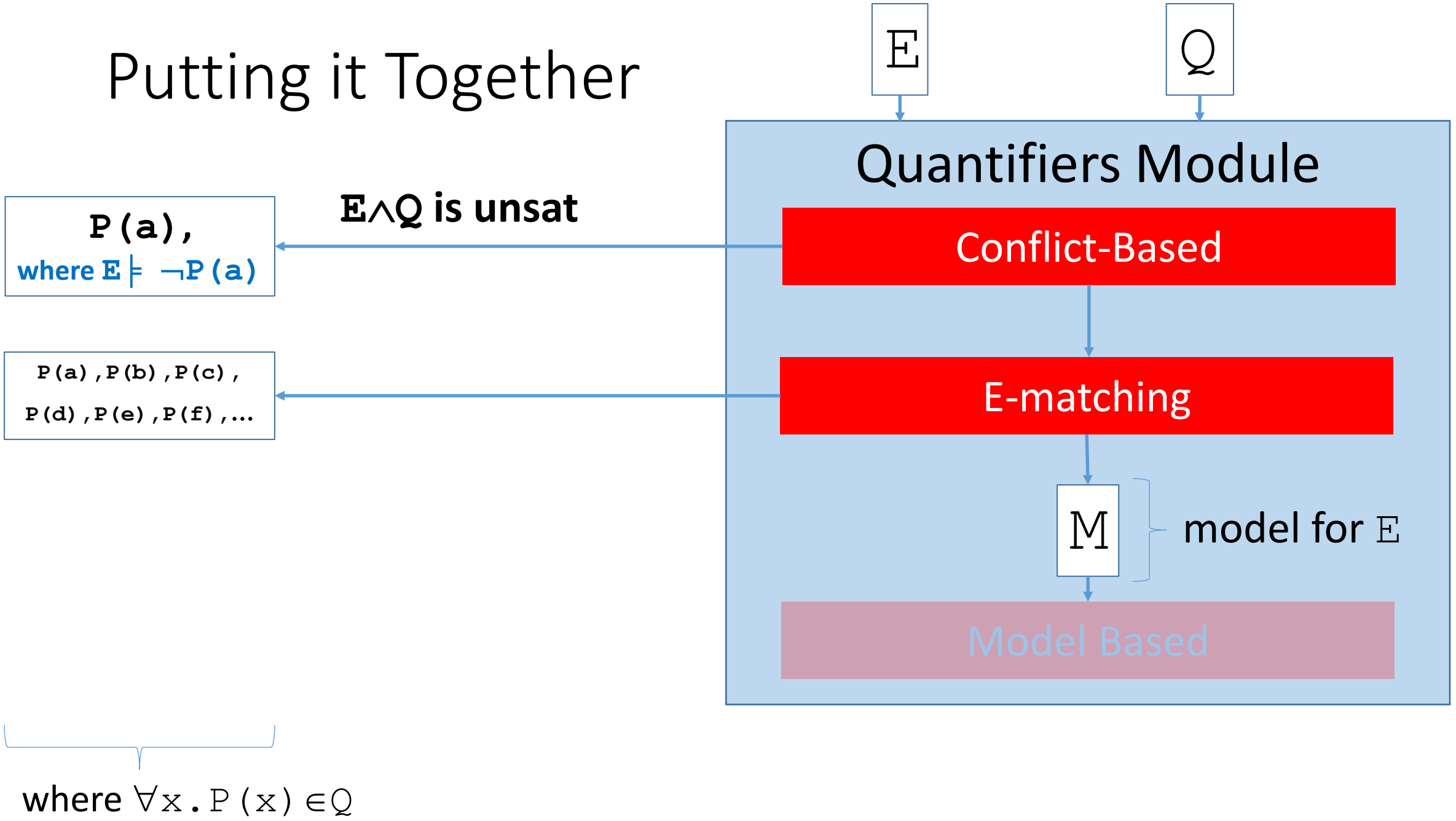
Putting it Together



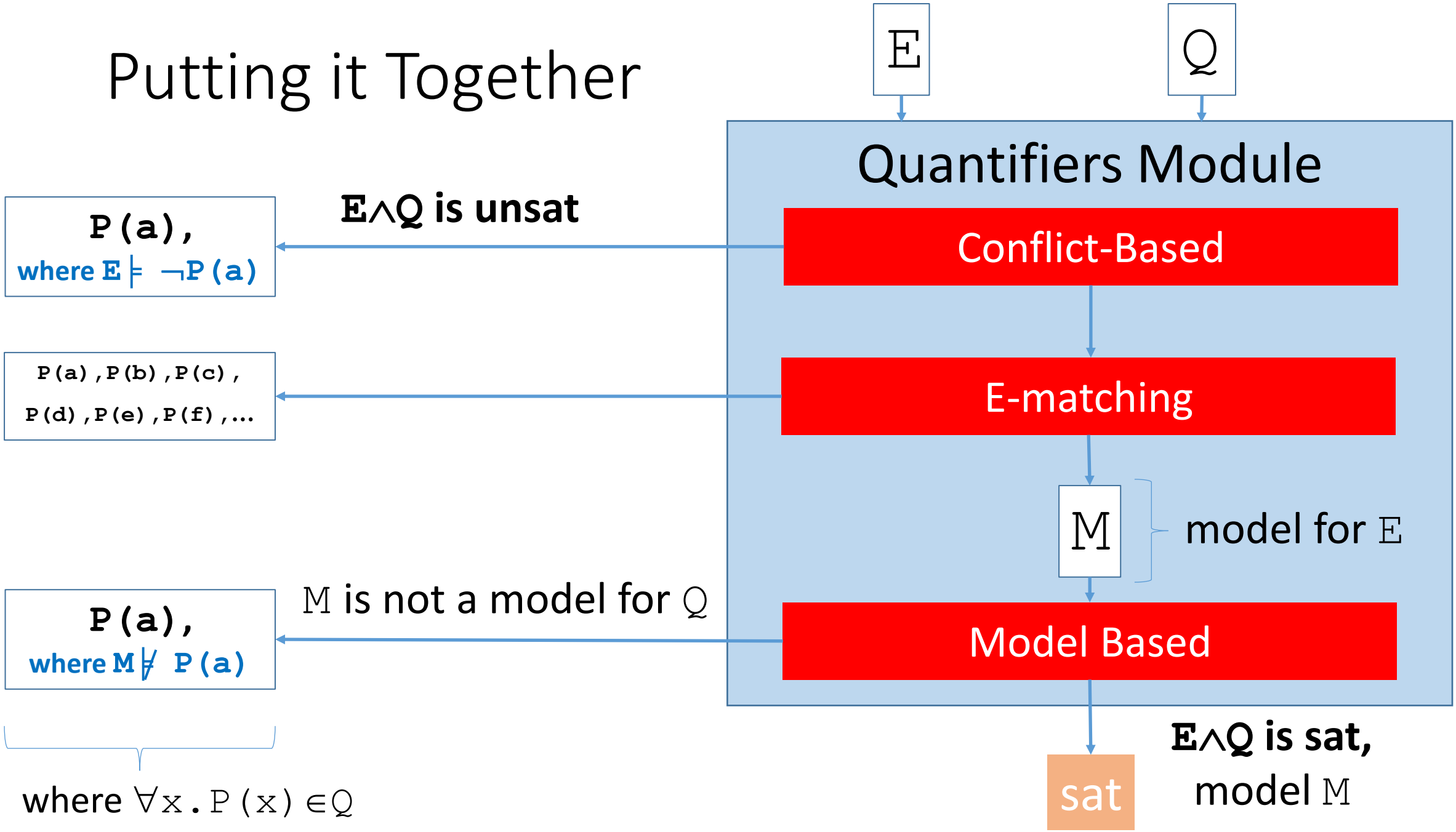
Putting it Together



Putting it Together



Putting it Together



Other techniques for Quantified Formulas

- Advanced techniques in CVC4:
  - Rewrite Rules
  - Automated Induction **[Reynolds/Kuncak VMCAI15]**
  - Finite Model Finding **[Reynolds et al CADE13]**
  - Synthesis **[Reynolds et al CAV15]**

  $\Rightarrow$ Each target a particular type of quantified formulas

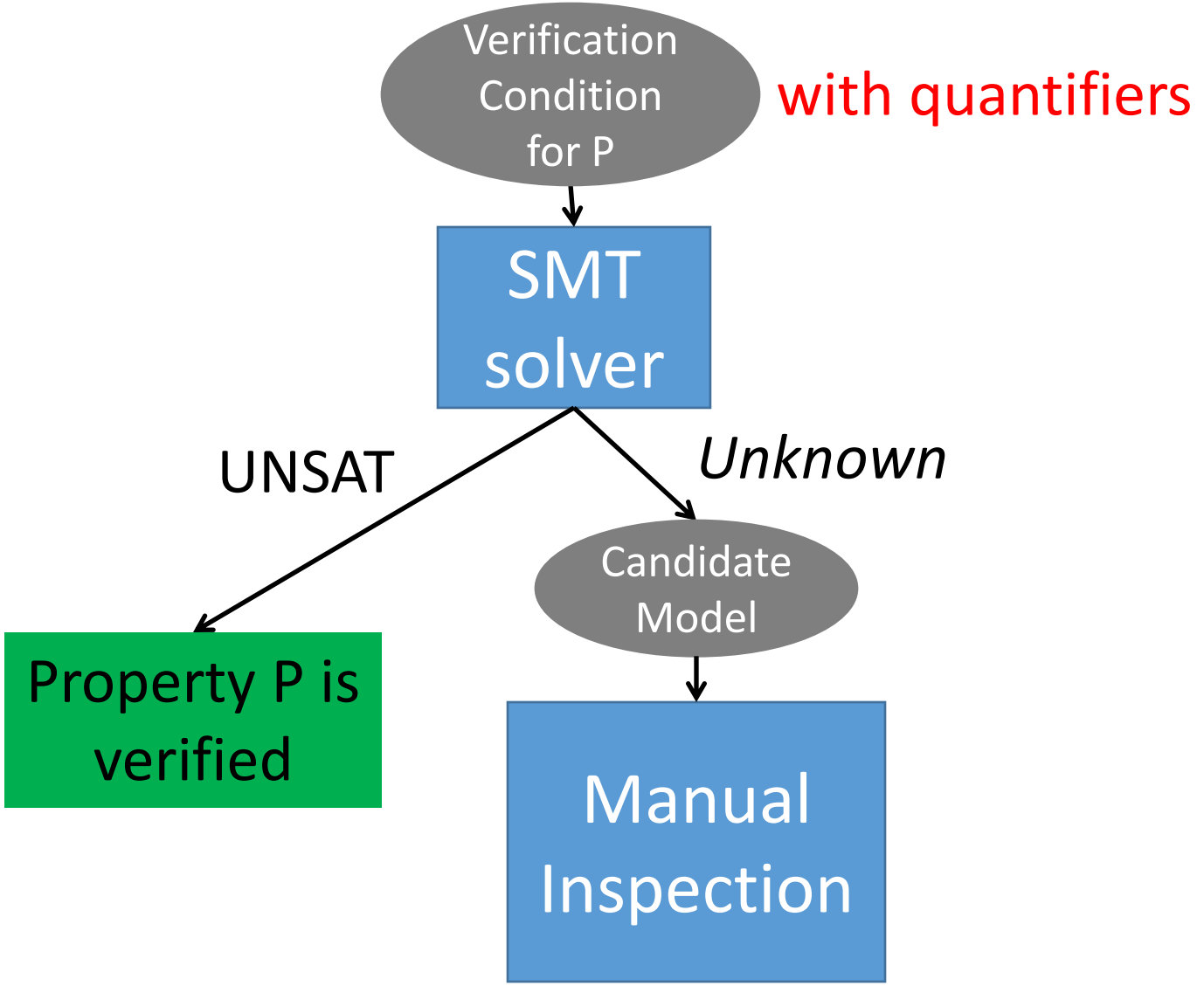
Other techniques for Quantified Formulas

- Advanced techniques in CVC4:
  - Rewrite Rules
  - Automated Induction **[Reynolds/Kuncak VMCAI15]**
  - Finite Model Finding **[Reynolds et al CADE13]**
  - Synthesis **[Reynolds et al CAV15]**

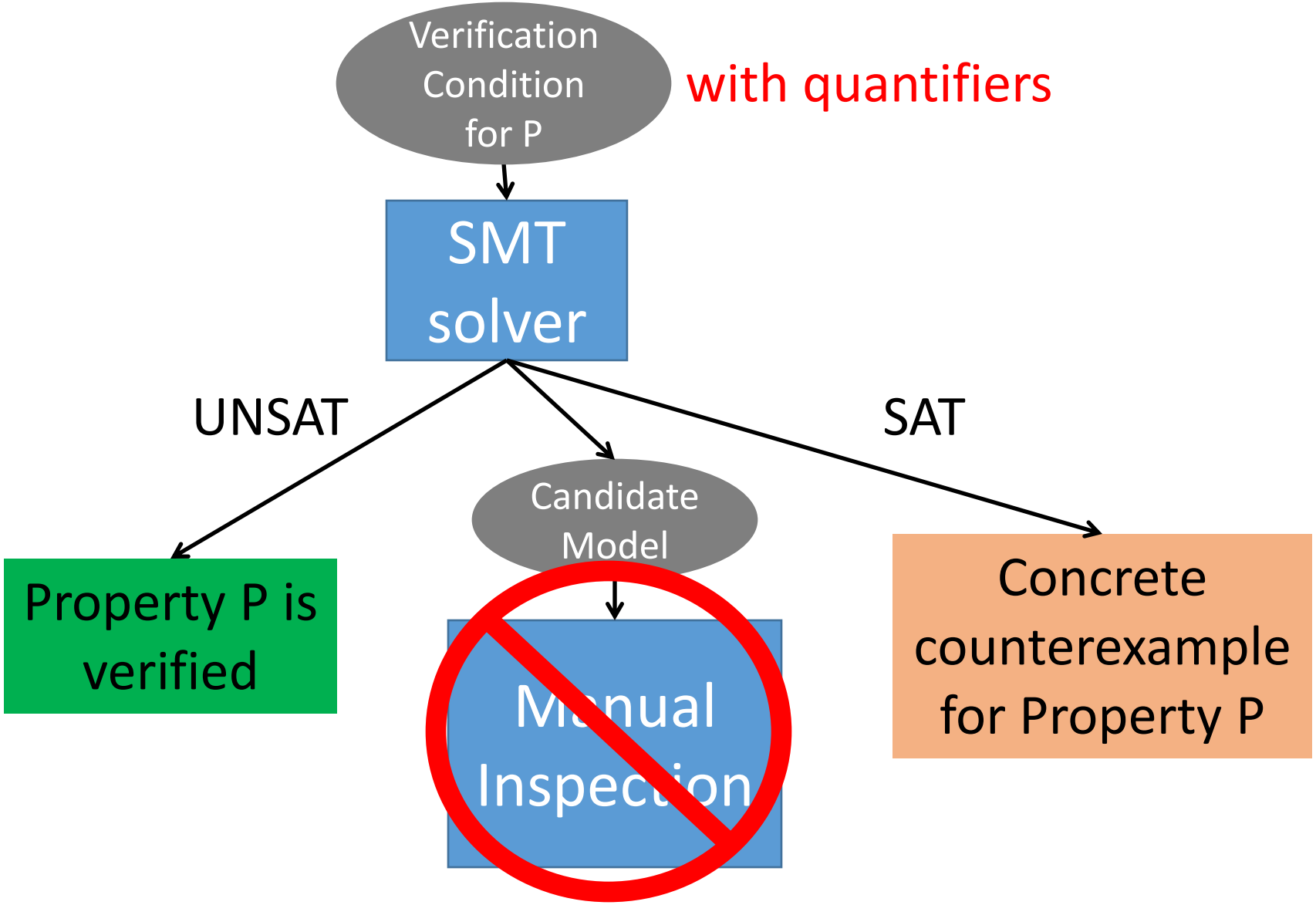
  } Focus of the remainder

  $\Rightarrow$ Each target a particular type of quantified formulas

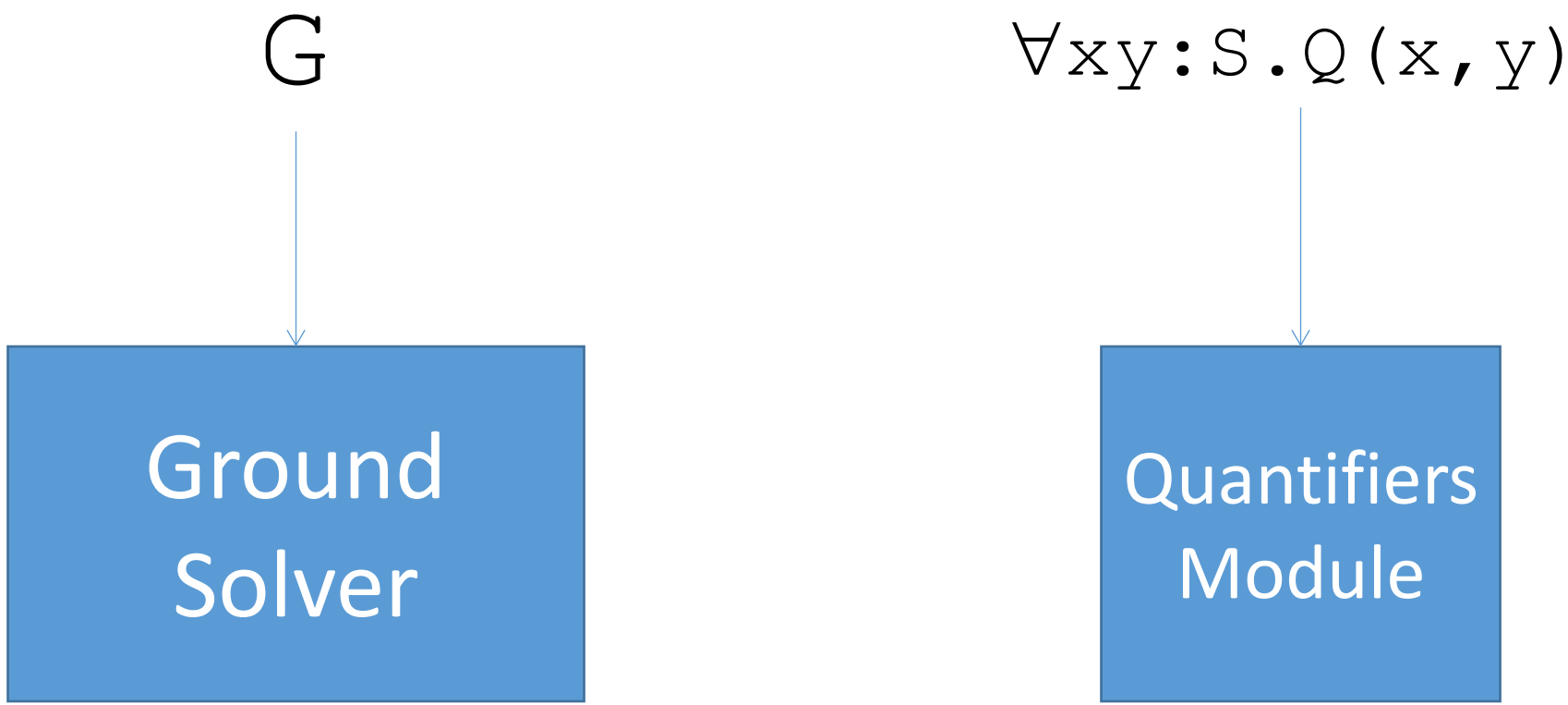
Finite Model Finding : Motivation



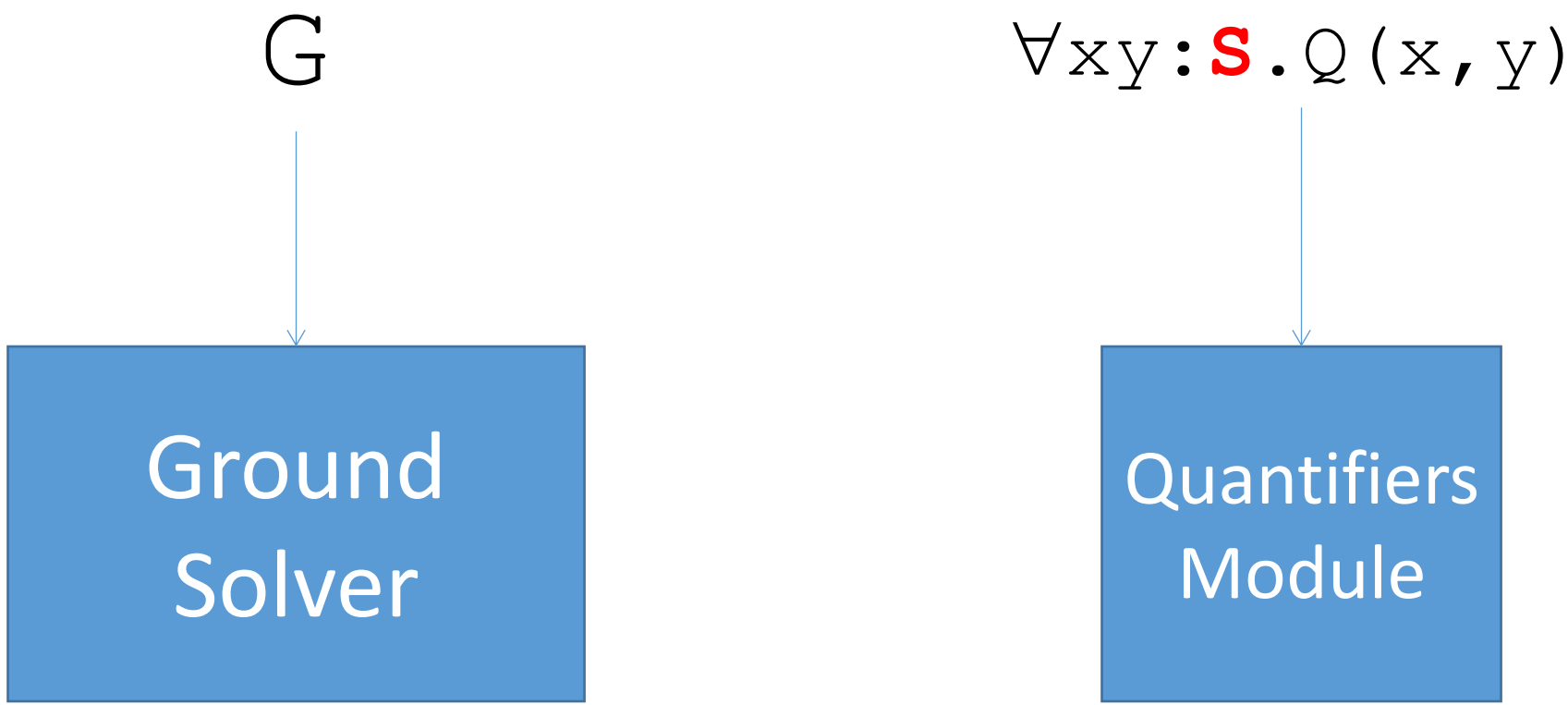
Finite Model Finding : Motivation



Finite Model Finding in SMT



Finite Model Finding in SMT

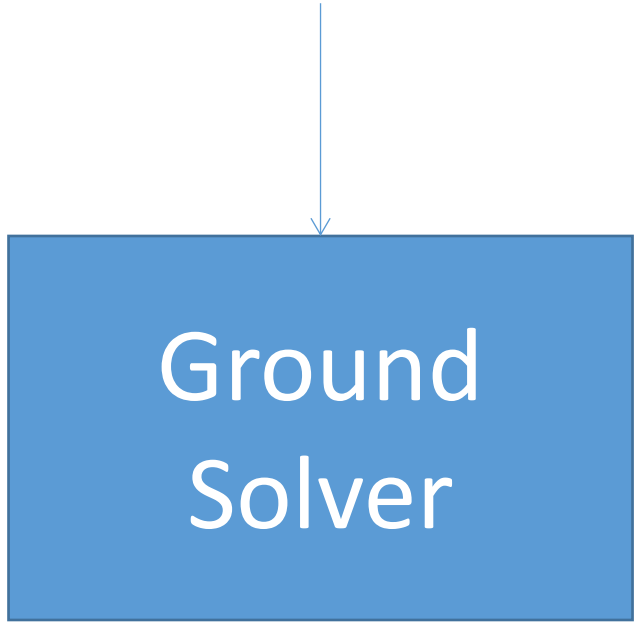


⇒*If* ***S*** *has finite interpretation,*

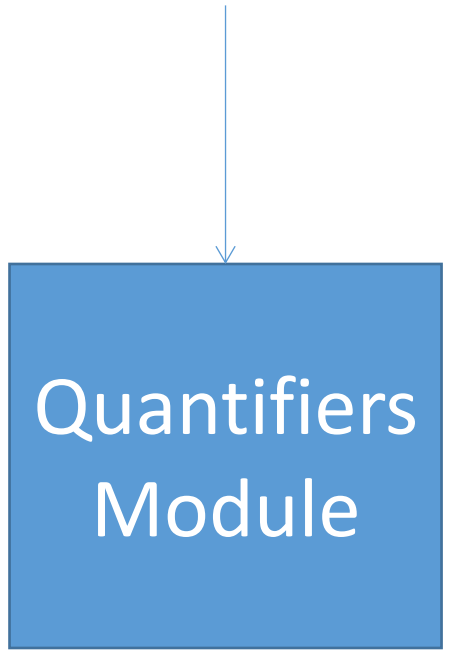
- *use finite model finding*

Finite Model Finding in SMT

G

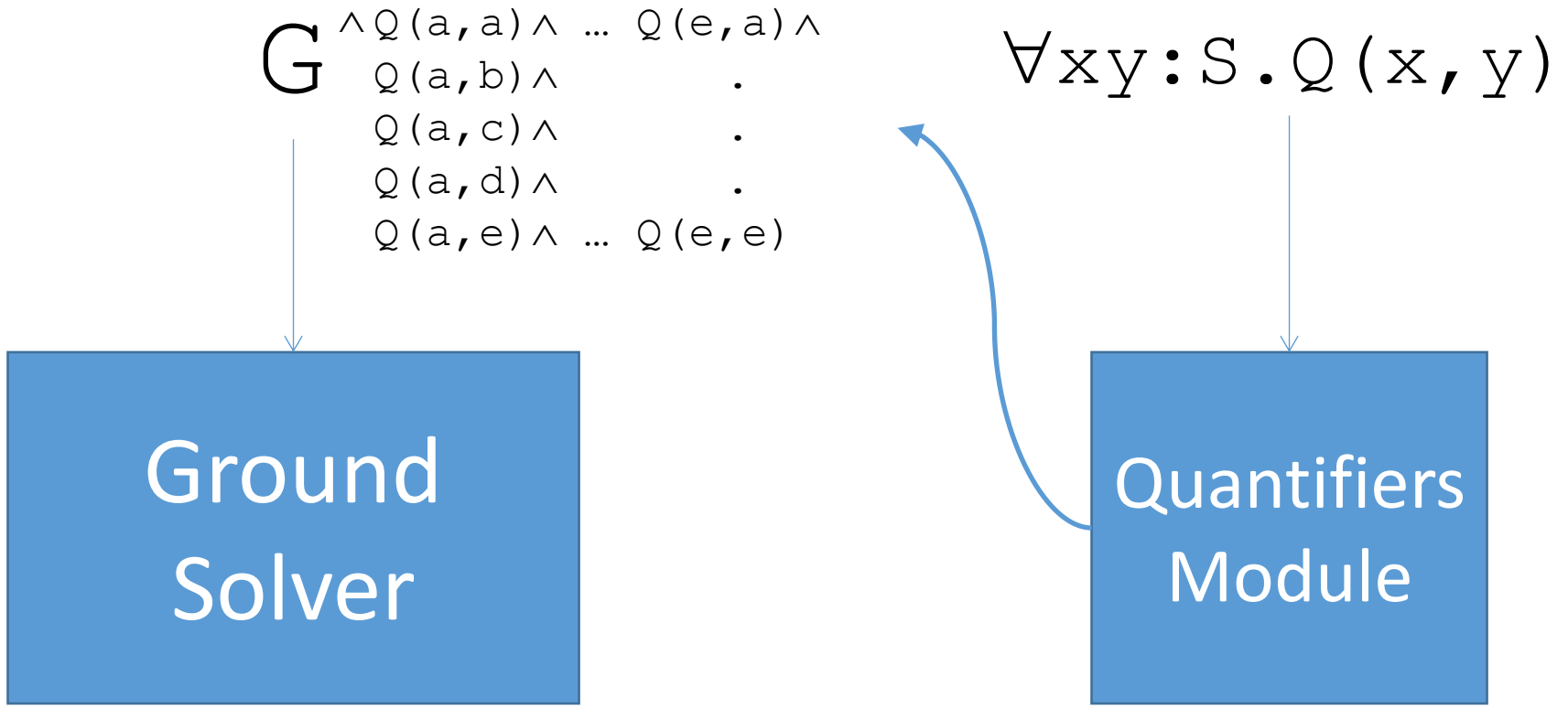


$\forall xy:S.Q(x,y)$



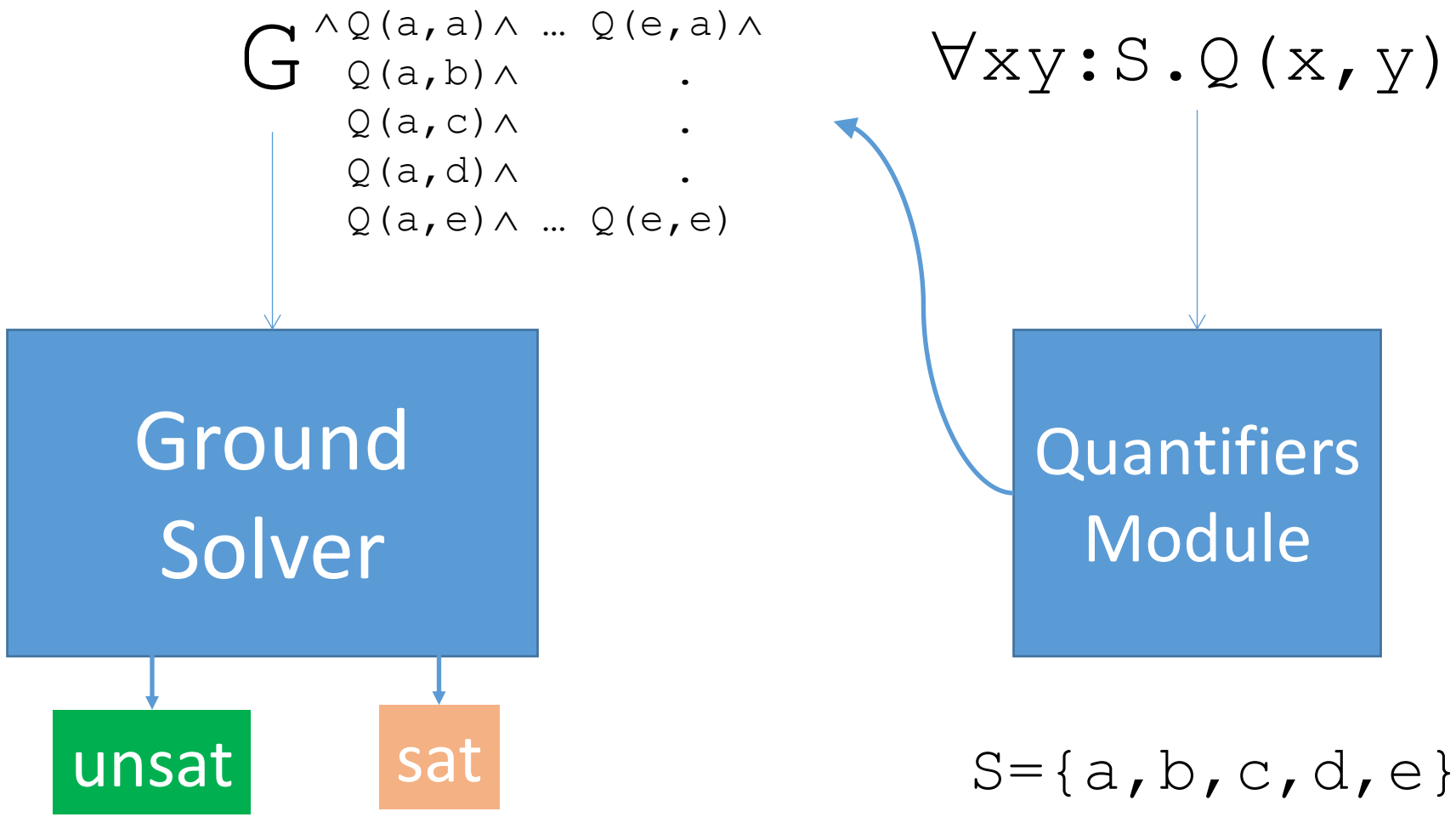
S={a,b,c,d,e}

Finite Model Finding in SMT



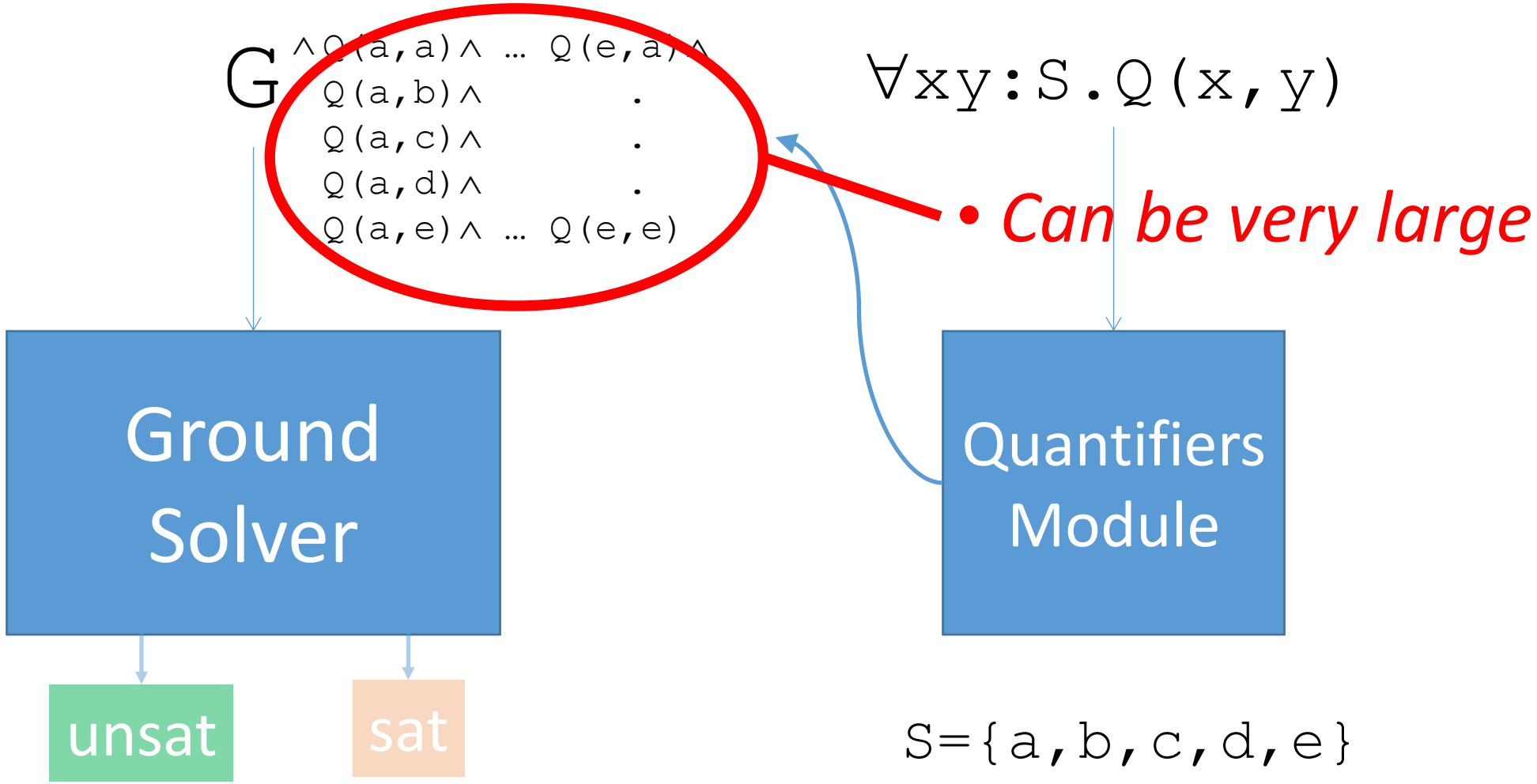
- Reduction of quantified formulas to ground formulas

Finite Model Finding in SMT



⇒ Ability to answer SAT, assuming decision procedure for $G \wedge Q(a,a) \wedge \ldots \wedge Q(e,e)$

Finite Model Finding in SMT



Finite Model Finding: Example

$a,b,c,d,e:S$

$P,R:(S,S)\rightarrow Bool$

$a\neq b,\ b\neq c,\ c\neq d,\ d\neq e,\ e\neq a$

$\neg P(a,b),\ \neg R(a,c)$

$\forall xy.P(x,y)\vee R(x,y)$

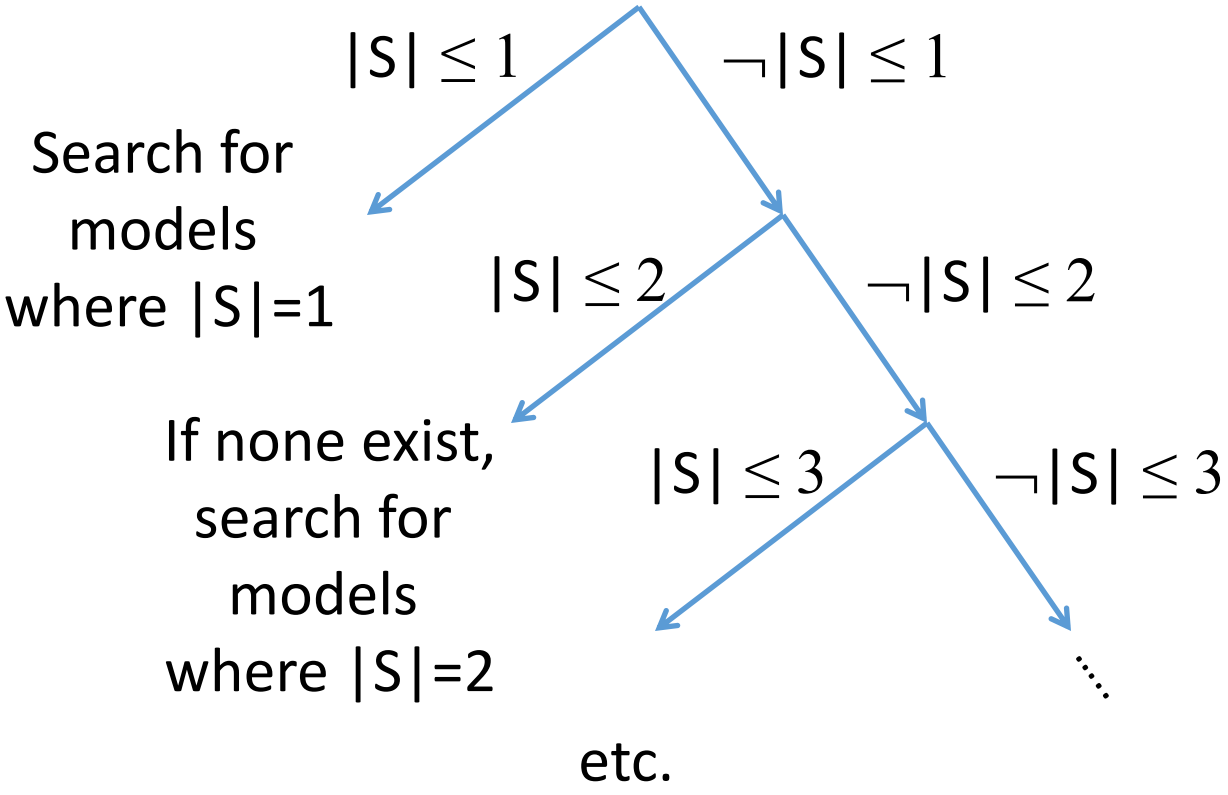
EXAMPLE…

Finite Model Finding in SMT

- Address large # instantiations by:
  1. Minimizing model sizes **[Reynolds et al CAV13]**
     - Find interpretation that minimizes the #elements in S
  2. Only add instantiations that refine model **[Reynolds et al CADE13]**
     - Model-based quantifier instantiation [Ge/deMoura CAV 2009]

Finite Model Finding : Minimizing Model Sizes

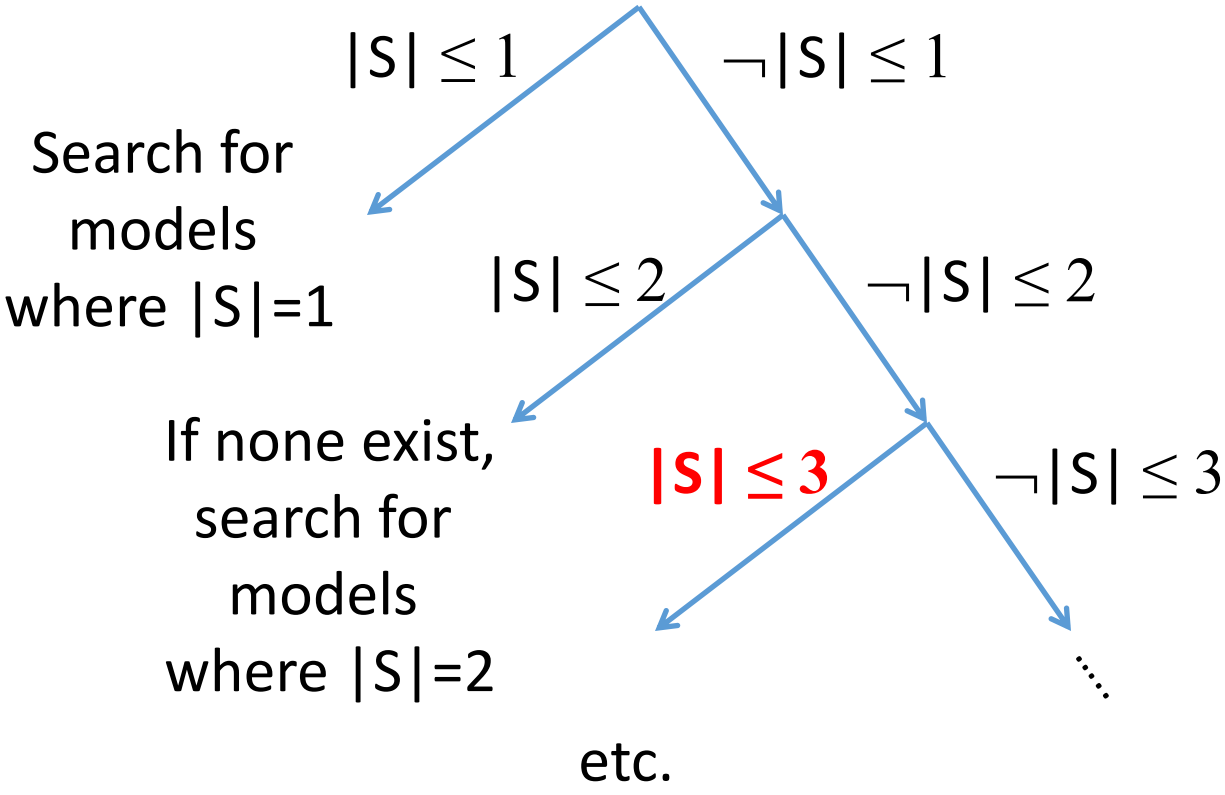
- Minimize model sizes using a theory solver for cardinality constraints



[Reynolds/Tinelli/Goel/Krstic CAV13]

Finite Model Finding : Minimizing Model Sizes

- Minimize model sizes using a theory solver for cardinality constraints



$\Rightarrow$ If model exists where $|S| \leq 3$, only need 3*3=9 instances instead of 5*5=25 instances

[Reynolds/Tinelli/Goel/Krstic CAV13]

FMF: Example

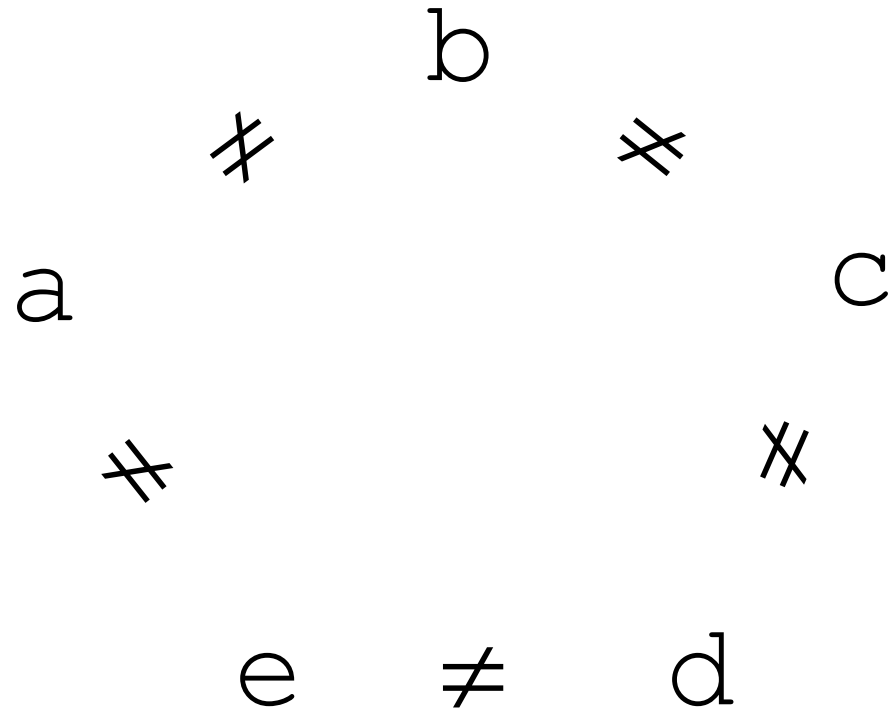
$a,b,c,d,e:S$

$P,R:(S,S)\to Bool$

$a\neq b,\ b\neq c,\ c\neq d,\ d\neq e,\ e\neq a$

$\neg P(a,b),\ \neg R(a,c)$

$\forall xy.P(x,y)\vee R(x,y)$



FMF: Example

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$P,R:(S,S)\rightarrow Bool$

$a\neq b,\ b\neq c,\ c\neq d,\ d\neq e,\ e\neq a$

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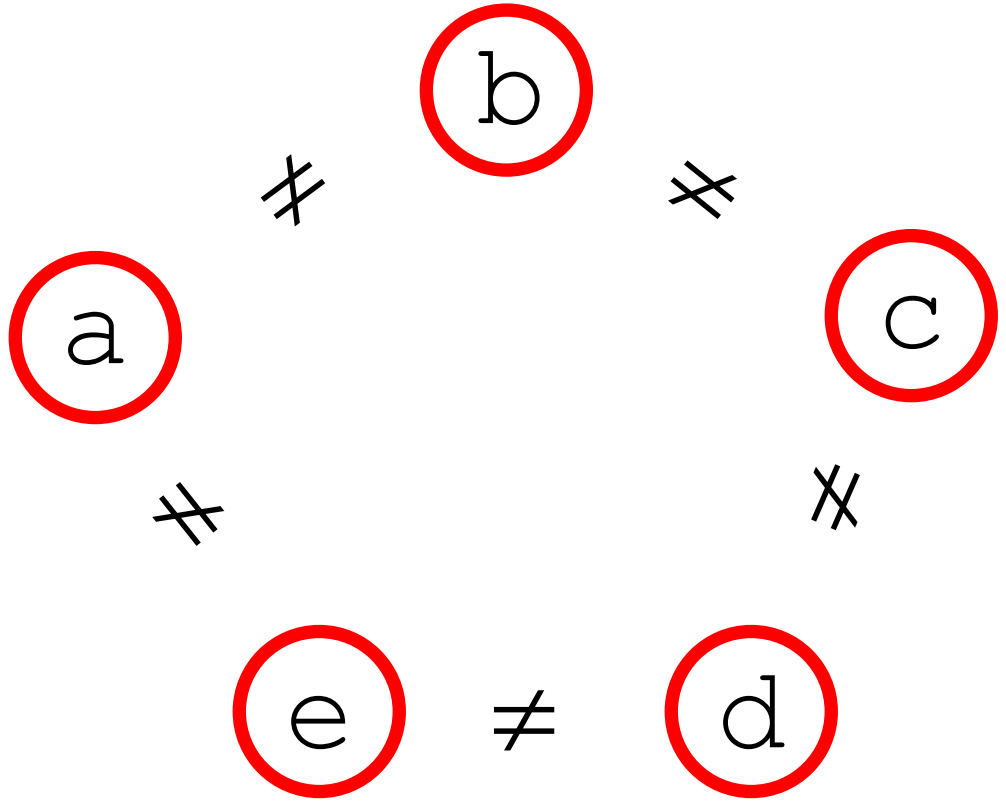
$\forall xy.P(x,y)\vee R(x,y)$



FMF: Example

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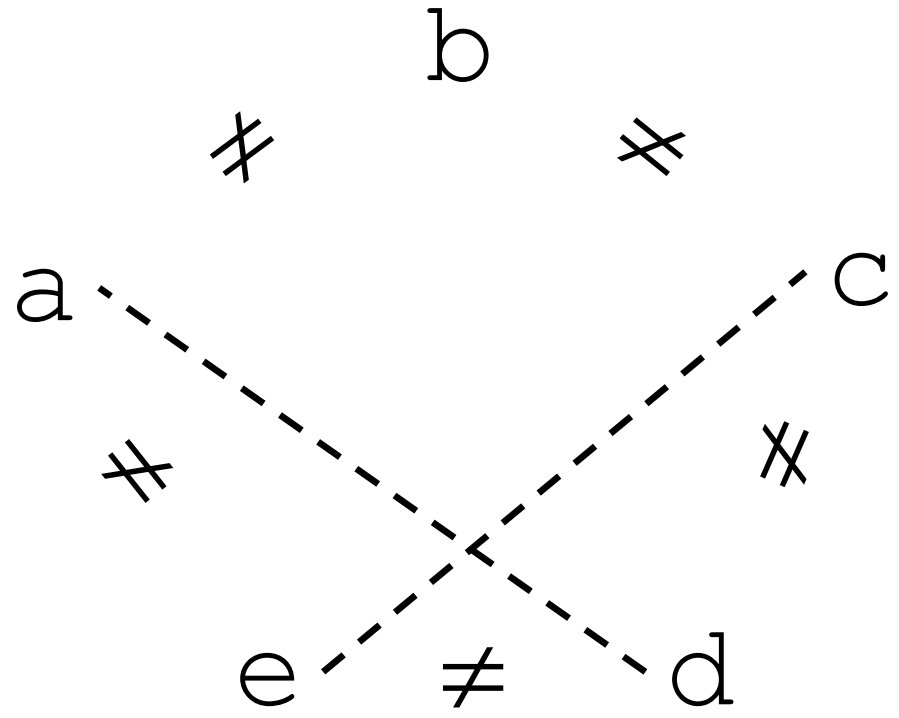


FMF: Example

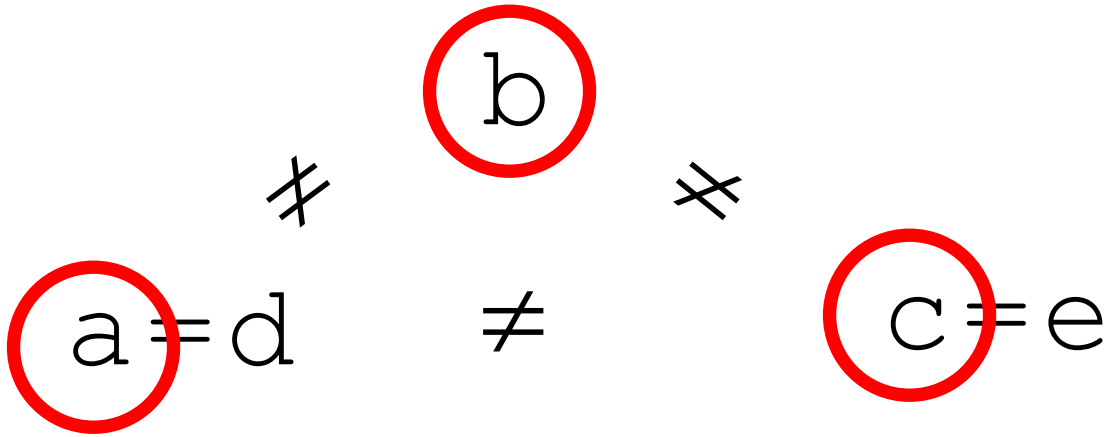
a,b,c,d,e:S

P,R:(S,S)→Bool

a≠b, b≠c, c≠d, d≠e, e≠a

¬P(a,b), ¬R(a,c), **a=d**, **c=e**

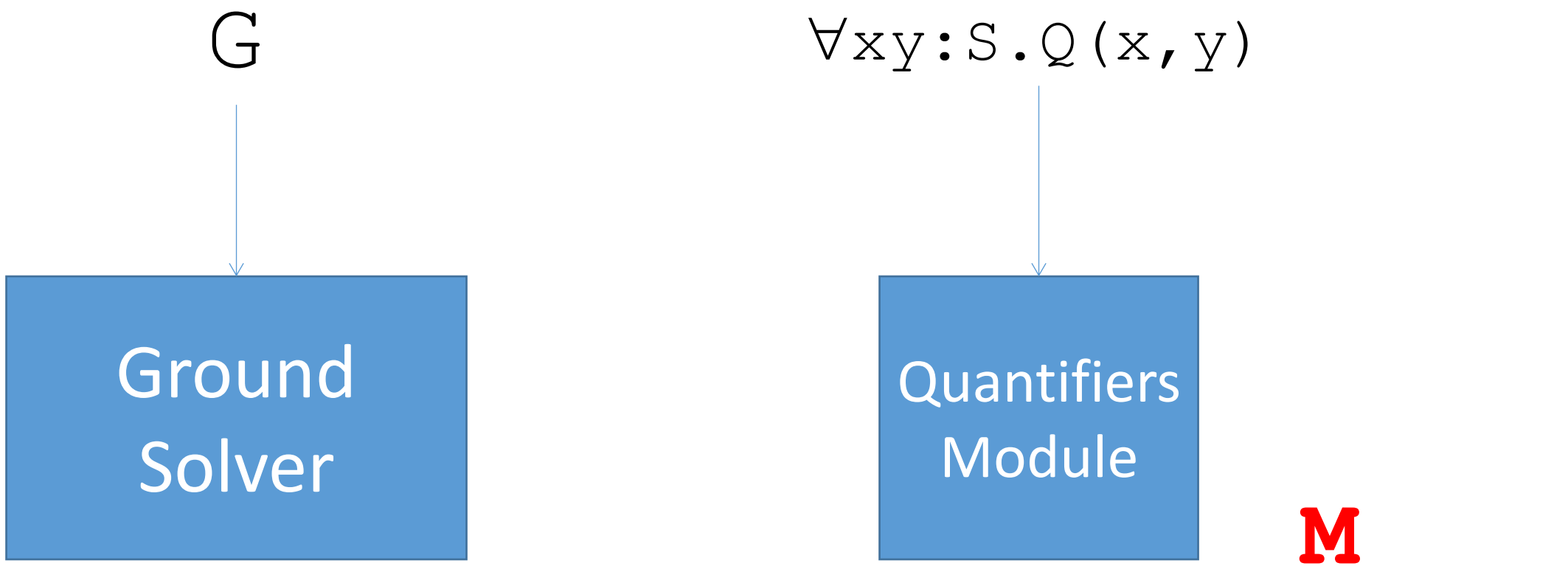
∀xy.P(x,y)∨R(x,y)



S={a,b,c}

EXAMPLE…

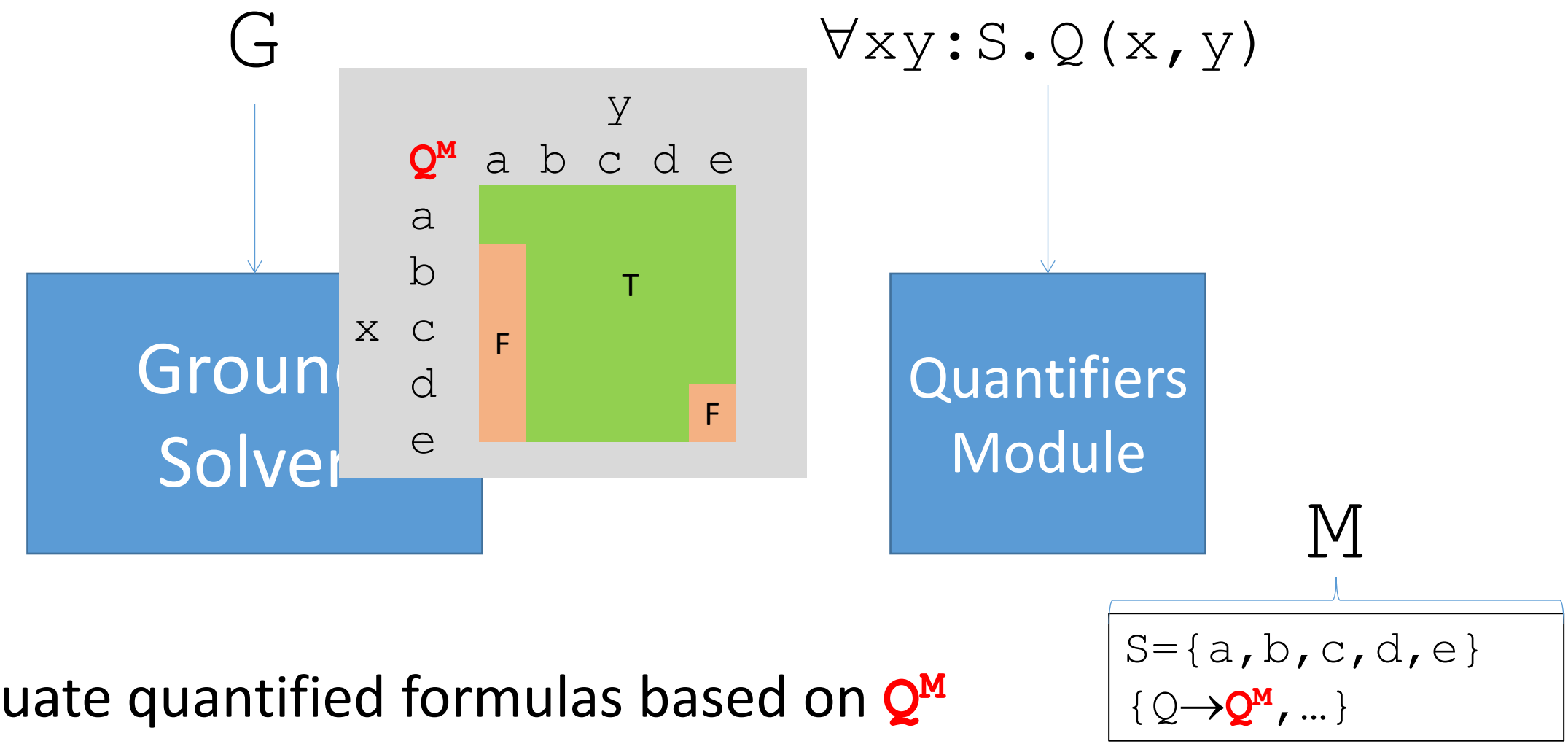
Finite Model Finding : Model-Based Instantiation



- Construct candidate model **M**

[Reynolds/Tinelli/Goel/Krstic/Barrett/Deters CADE13]

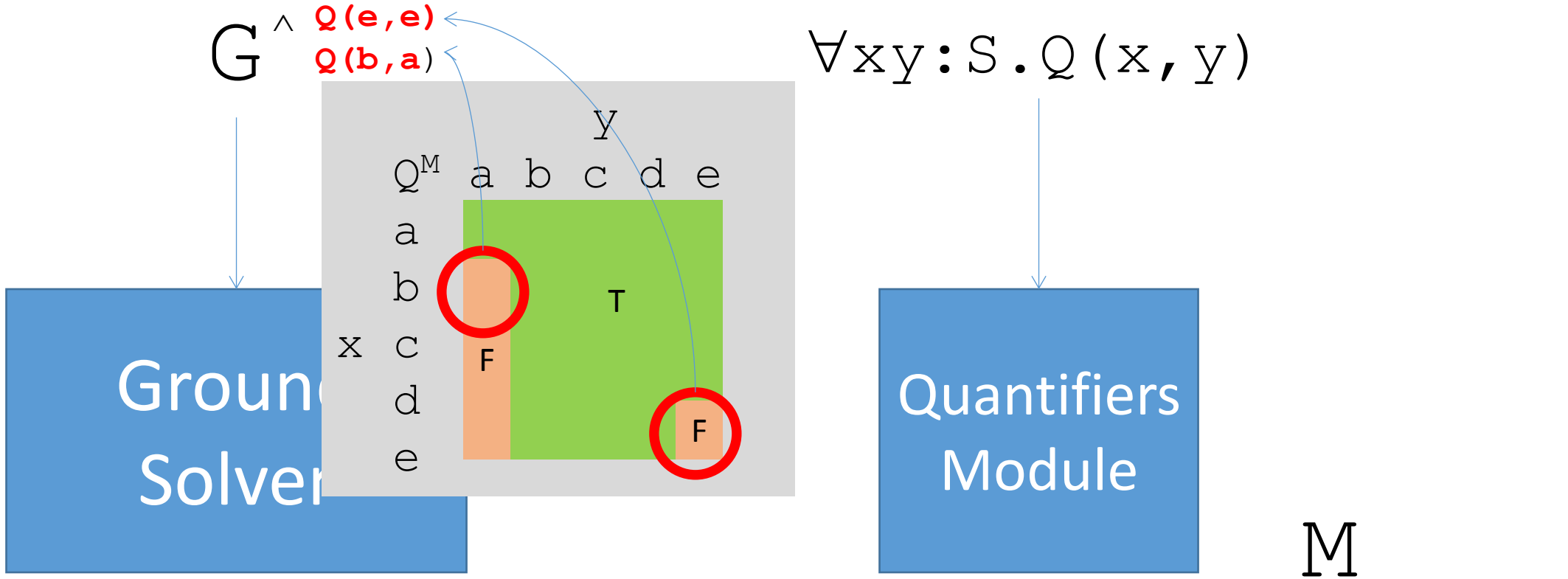
Finite Model Finding : Model-Based Instantiation



- Evaluate quantified formulas based on Q^M

[Reynolds/Tinelli/Goel/Krstic/Barrett/Deters CADE13]

Finite Model Finding : Model-Based Instantiation



- Only add instances that evaluate to F in Q^M

  $\Rightarrow$ Significantly increased scalability

[Reynolds/Tinelli/Goel/Krstic/Barrett/Deters CADE13]

FMF: Example

$a,b,c,d,e:S$

$P,R:(S,S)\to Bool$

$a\neq b,\ b\neq c,\ c\neq d,\ d\neq e,\ e\neq a$

$\neg P(a,b),\ \neg R(a,c)$

$\forall xy.P(x,y)\vee R(x,y)$

FMF: Example

a,b,c,d,e:S

P,R:(S,S)→Bool

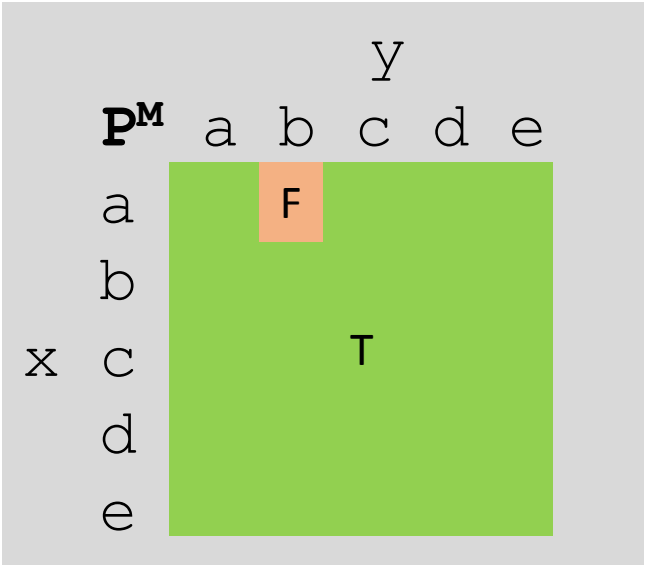
a≠b, b≠c, c≠d, d≠e, e≠a

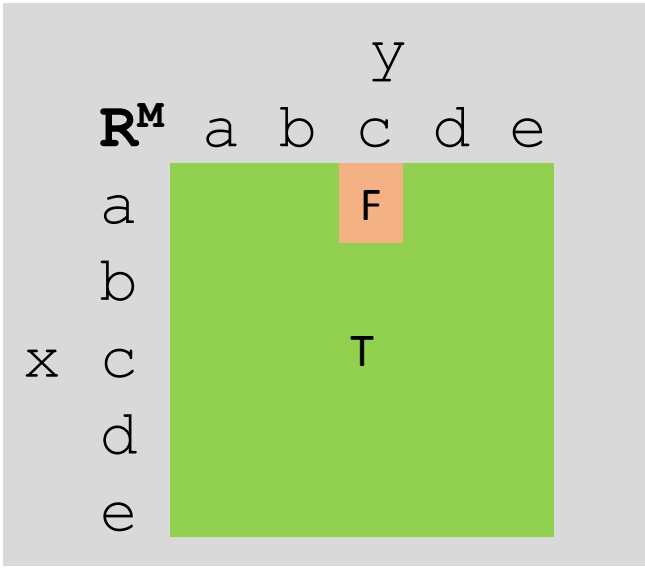
¬P(a,b), ¬R(a,c)

∀xy.P(x,y)∨R(x,y)

P := λxy.(x≠a ∨ y≠b)

R := λxy.(x≠a ∨ y≠c)





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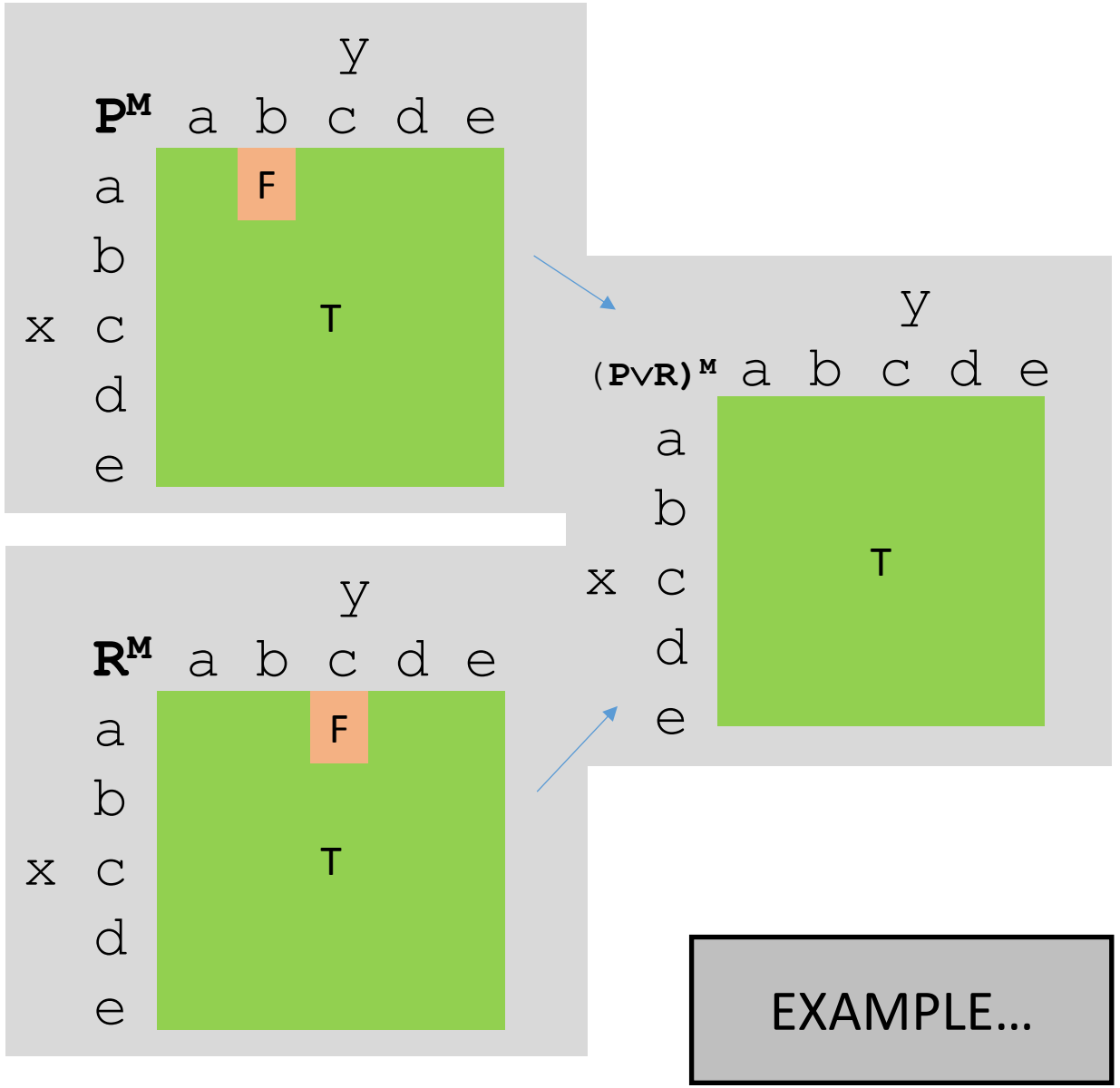
$\neg P(a,b),\ \neg R(a,c)$

$\forall xy.P(x,y)\vee R(x,y)$

$P := \lambda xy.(x\neq a \vee y\neq b)$

$R := \lambda xy.(x\neq a \vee y\neq c)$

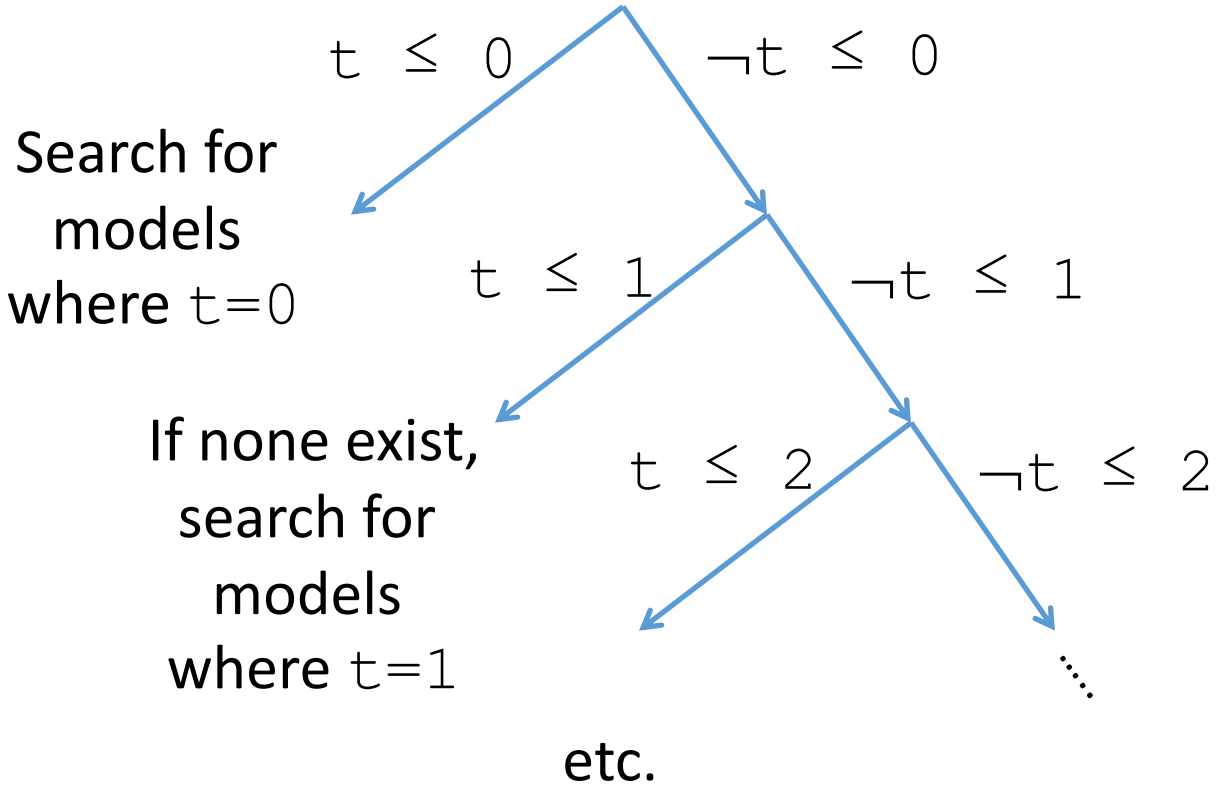


Finite Model Finding in CVC4

- **Sound** for both “sat” and “unsat”
- **Finite-model complete**
  - If there is a finite model, CVC4 will eventually find it
    (when all quantification is over sorts that are interpreted as finite)
- Refutationally **incomplete** in general
  - But regardless, is often able to answer “unsat”

Extension: Bounded Integer Quantification

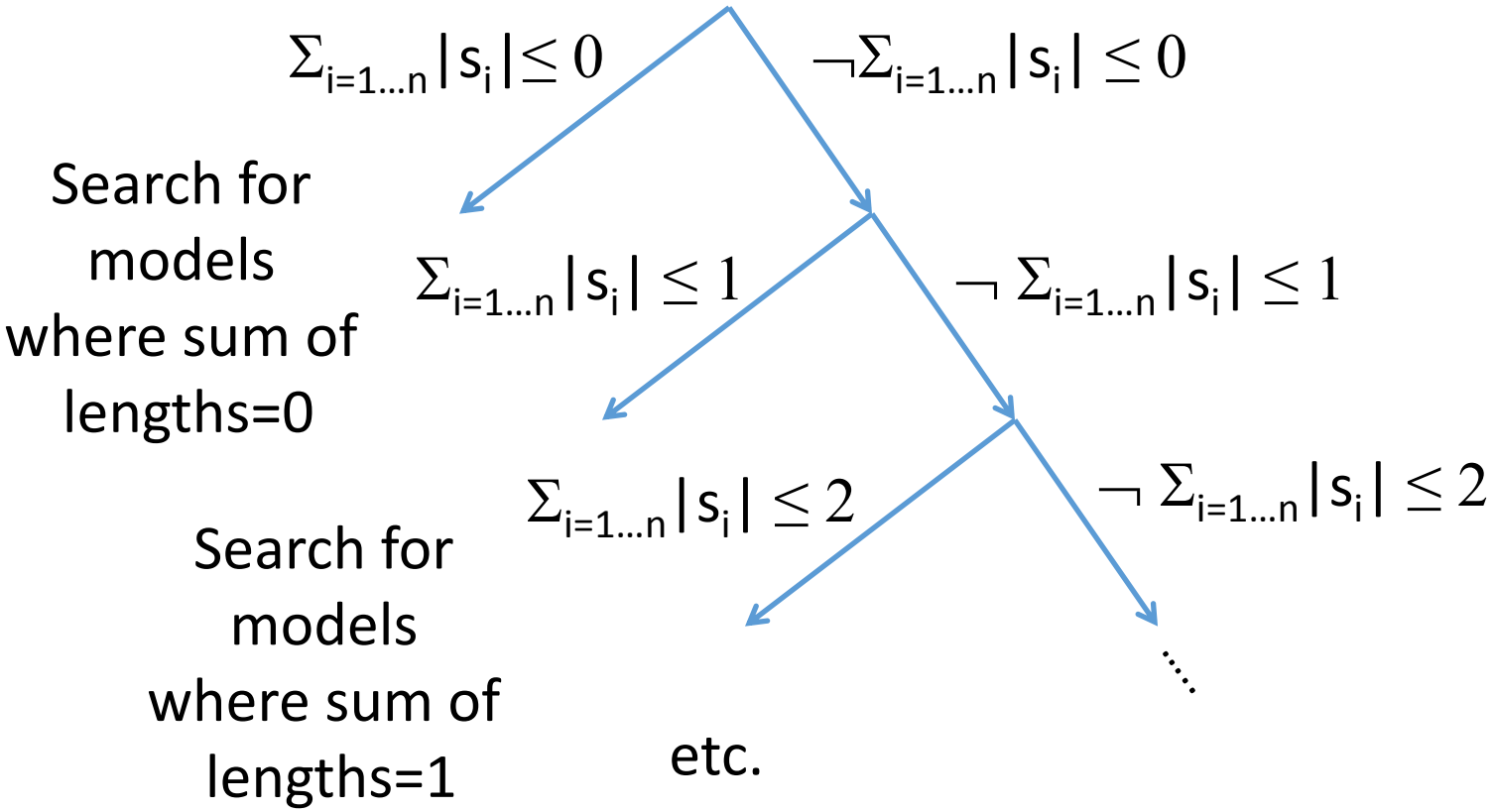
- $\forall x{:}\mathrm{Int}.\ 0 \leq x < t \Rightarrow P(x)$



EXAMPLE…

Extension: Bounded Length Strings

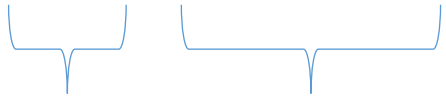
- Given input $F[s_1,\ldots,s_n]$ for strings $s_1 \ldots s_n$:



EXAMPLE…

Synthesis: Motivation

- Synthesis Problem : $\exists f . \forall x . P(f, x)$



There exists a function f such that for all x, $P(f, x)$

- Most existing approaches for synthesis
  - Rely on specialized solver that makes subcalls to an SMT Solver
- *CVC4 has approach for synthesis, which is entirely inside SMT solver*

Example : Max of Two Integers

$$\exists f.\forall xy.(f(x,y) \geq x \wedge f(x,y) \geq y \wedge (f(x,y) = x \vee f(x,y) = y))$$

- Specifies that f computes the maximum of integers x and y
- Solution:

$$f := \lambda xy.ite(x \geq y, x, y)$$

How does an SMT solver handle Max example?

$$\exists f.\forall xy.(f(x,y)\geq x \wedge f(x,y)\geq y \wedge (f(x,y)=x \vee f(x,y)=y))$$

- Challenge: quantification over function **f**
  - No SMT solvers directly support second-order quantification

How does an SMT solver handle Max example?

$$\mathbf{f} : \mathbf{Int} \times \mathbf{Int} \rightarrow \mathbf{Int}$$

$$\forall xy.(\mathbf{f}(x,y) \geq x \wedge \mathbf{f}(x,y) \geq y \wedge (\mathbf{f}(x,y) = x \vee \mathbf{f}(x,y) = y))$$

- Direct approach:
  - Treat **f** as an *uninterpreted function*
  - Succeed if SMT solver can find correct interpretation of **f**

    $\Rightarrow$*This is challenging*
    - How does the solver know the right interpretation for **f** to pick?

How does an SMT solver handle Max example?

$$\exists f.\forall xy.(f(x,y) \geq x \wedge f(x,y) \geq y \wedge (f(x,y) = x \vee f(x,y) = y))$$

How does an CVC4 handle Max example?

$$\exists f.\forall xy.(\mathbf{f(x,y)} \geq x \wedge \mathbf{f(x,y)} \geq y \wedge (\mathbf{f(x,y)} = x \vee \mathbf{f(x,y)} = y))$$

- Alternative:
  - This property is **single invocation**
    - All occurrences of **f** are of the form **f(x,y)**

  … and thus, can be converted to a first-order quantification
    - Introduce first-order variable **g**
    - Push quantification downwards “anti-skolemization”

[Reynolds/Deters/Kuncak/Tinelli/Barrett CAV15]

How does an CVC4 handle Max example?

$$\exists f.\forall xy.(\mathbf{f(x,y)} \geq x \wedge \mathbf{f(x,y)} \geq y \wedge (\mathbf{f(x,y)} = x \vee \mathbf{f(x,y)} = y))$$

Convert to first-order

$$\forall xy.\ \exists \mathbf{g}.(\mathbf{g} \geq x \wedge \mathbf{g} \geq y \wedge (\mathbf{g} = x \vee \mathbf{g} = y))$$

[Reynolds/Deters/Kuncak/Tinelli/Barrett CAV15]

How does an CVC4 handle Max example?

$$\exists f.\forall xy.(f(x,y)\geq x \wedge f(x,y)\geq y \wedge (f(x,y)=x \vee f(x,y)=y))$$

Convert to first-order

$$\forall xy.\ \exists g.(g\geq x \wedge g\geq y \wedge (g=x \vee g=y))$$

- Problem is now:
  - First-order, linear (integer) arithmetic, with one quantifier alternation

    $\Rightarrow$ CVC4 has specialized instantiation procedure

[Reynolds/Deters/Kuncak/Tinelli/Barrett CAV15]

Max Example

$$\forall xy.\ \exists g.(g \geq x \wedge g \geq y \wedge (g = x \vee g = y))$$

Ground Solver

Quantifiers Module

Max Example

$\forall xy.\ \exists g.isMax(g,x,y)$

Ground Solver

Quantifiers Module

Max Example

```
∀xy. ∃g.isMax(g,x,y)
```

- Goal: show the above formula is sat

Max Example

$\exists xy.\ \forall g.\neg isMax(g,x,y)$

- Since F is LIA-sat if and only if $\neg F$ is LIA-unsat,
  $\Rightarrow$ Suffices to show that negation is unsat

Max Example

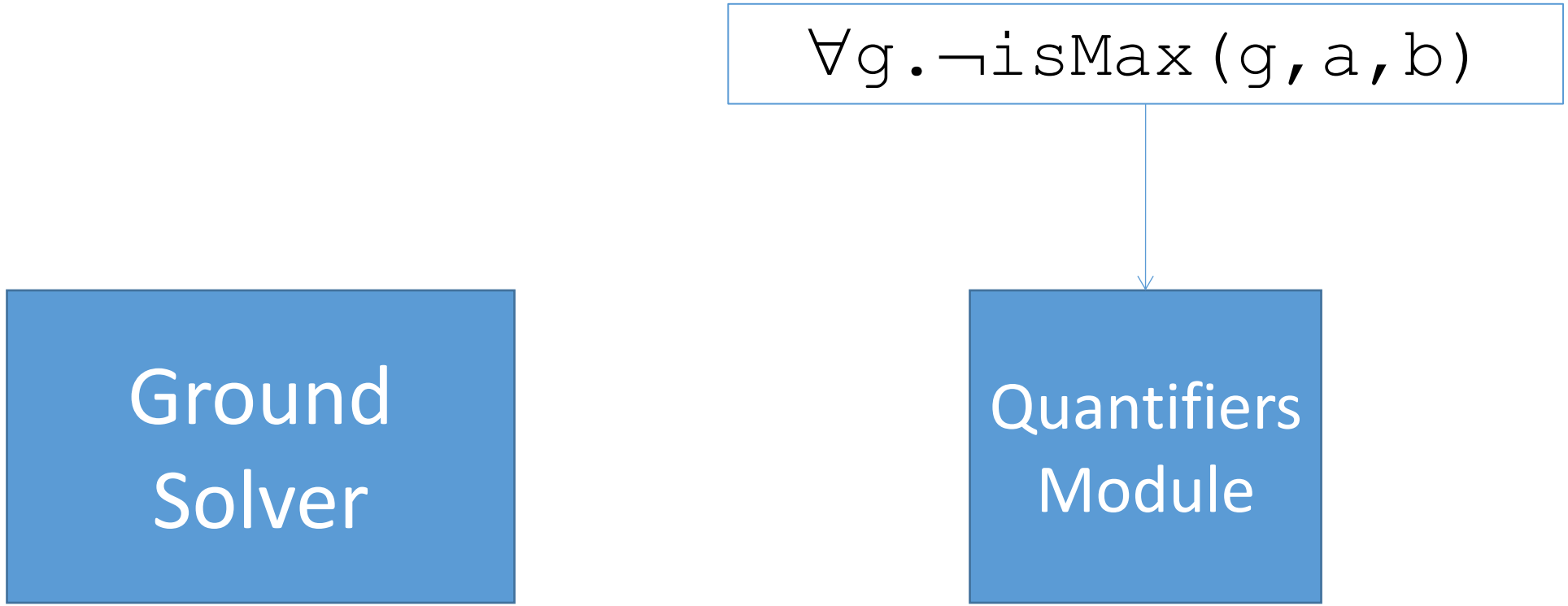
$\forall g.\neg isMax(g, \mathbf{a}, \mathbf{b})$

Ground Solver

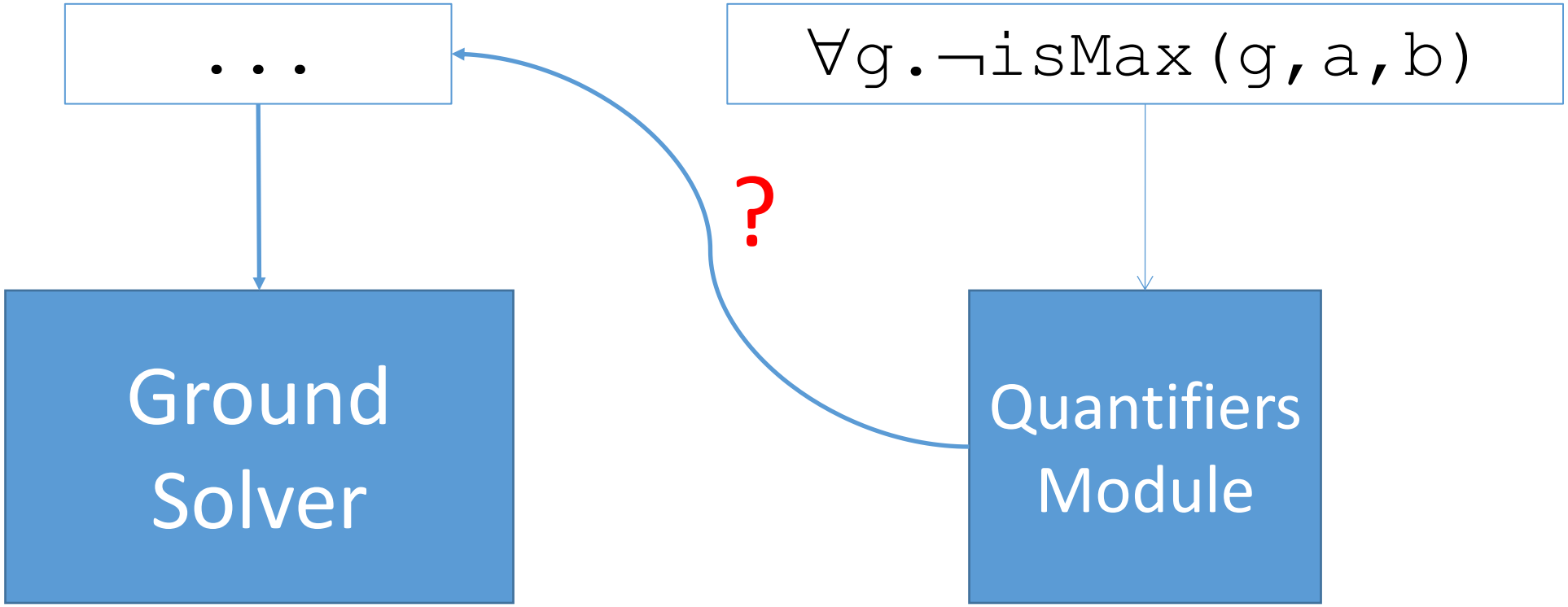
Quantifiers Module

- Skolemize, for fresh constants **a** and **b**

Max Example

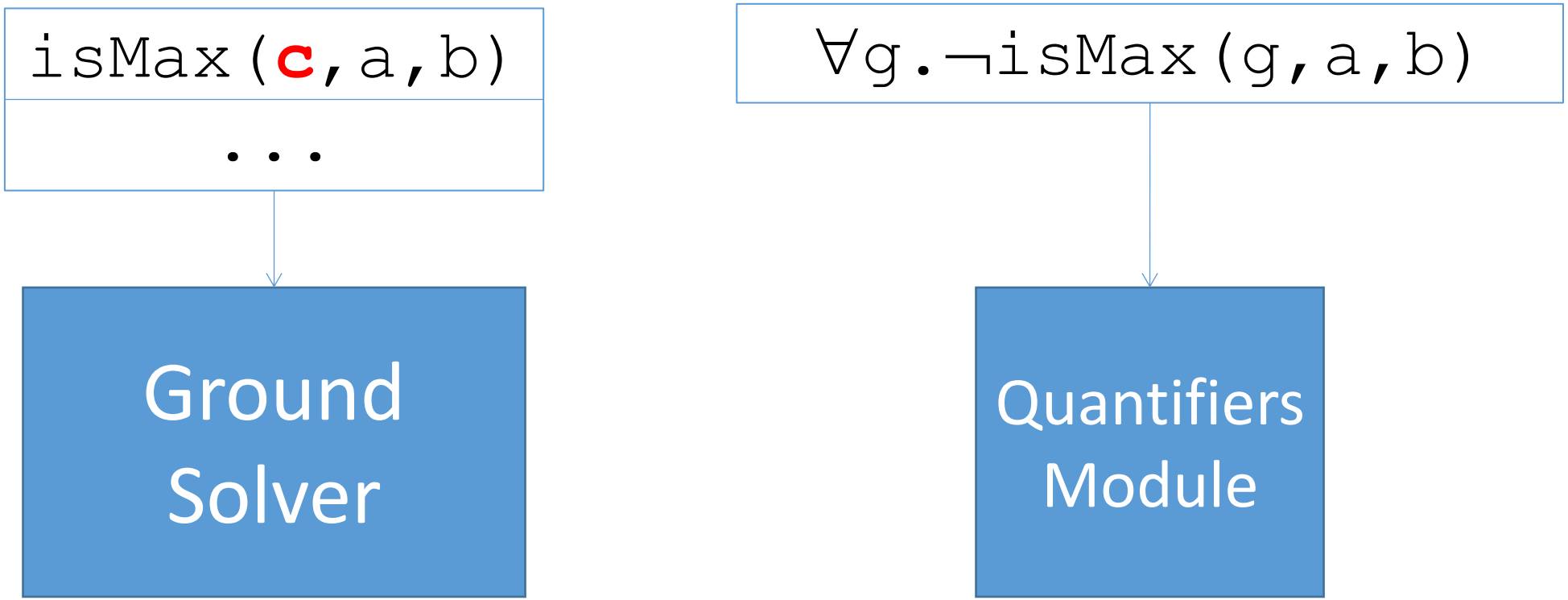


Max Example



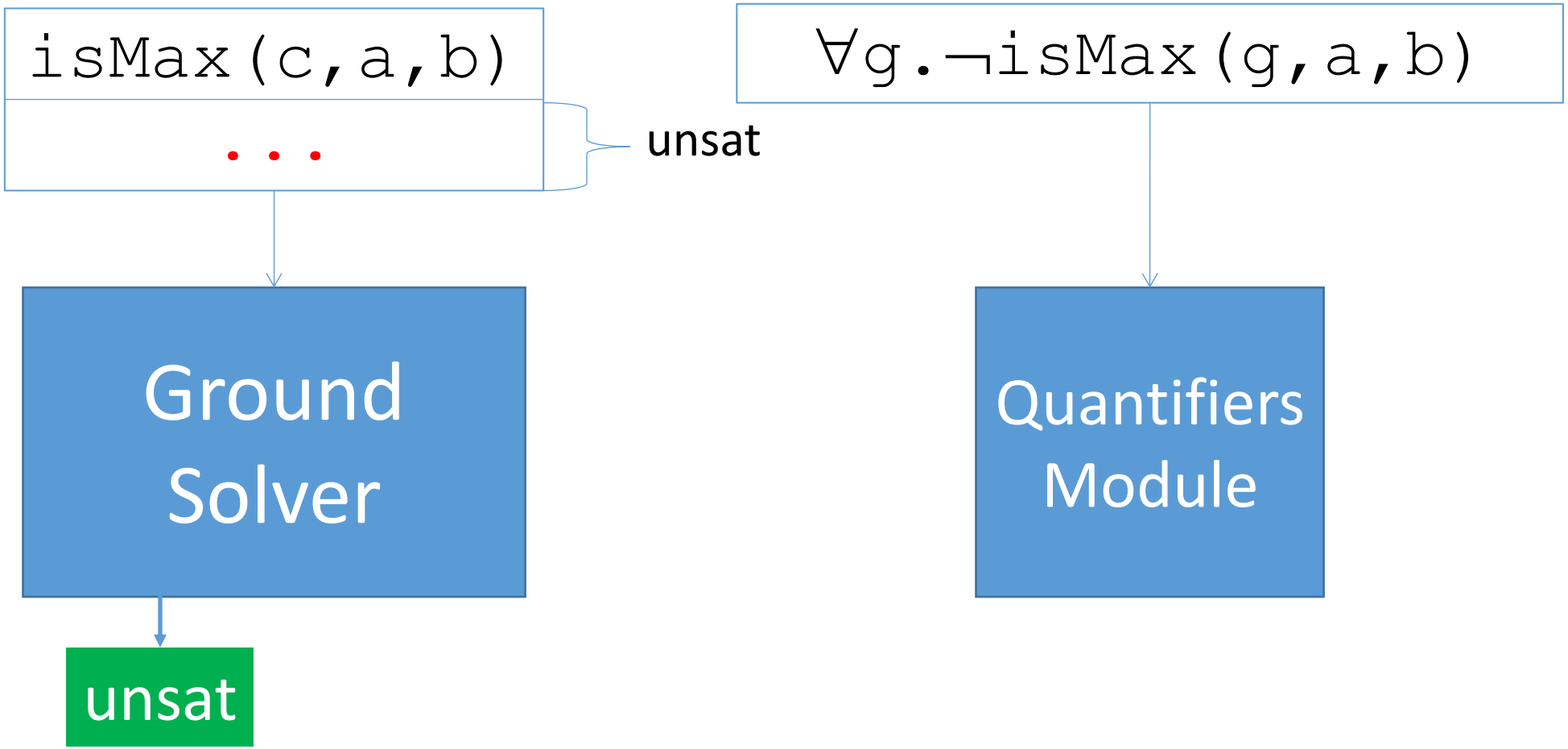
- Which instances of `∀g.¬isMax(g,a,b)` do we consider?

Counterexample-Guided Instantiation



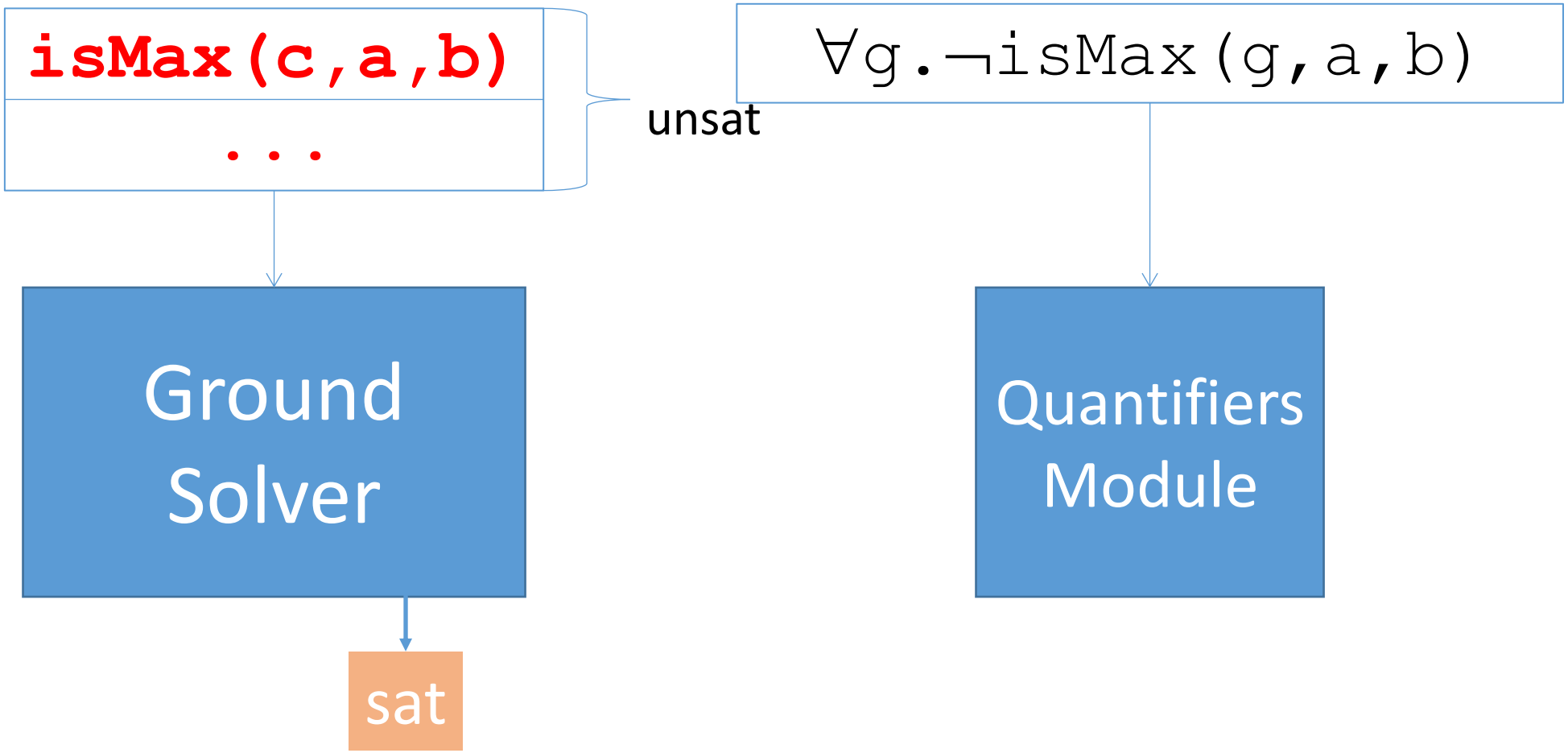
- **Idea**: choose instances of $\forall g.\neg isMax(g,a,b)$ based on models for "counterexample" fresh constant c

Counterexample-Guided Instantiation



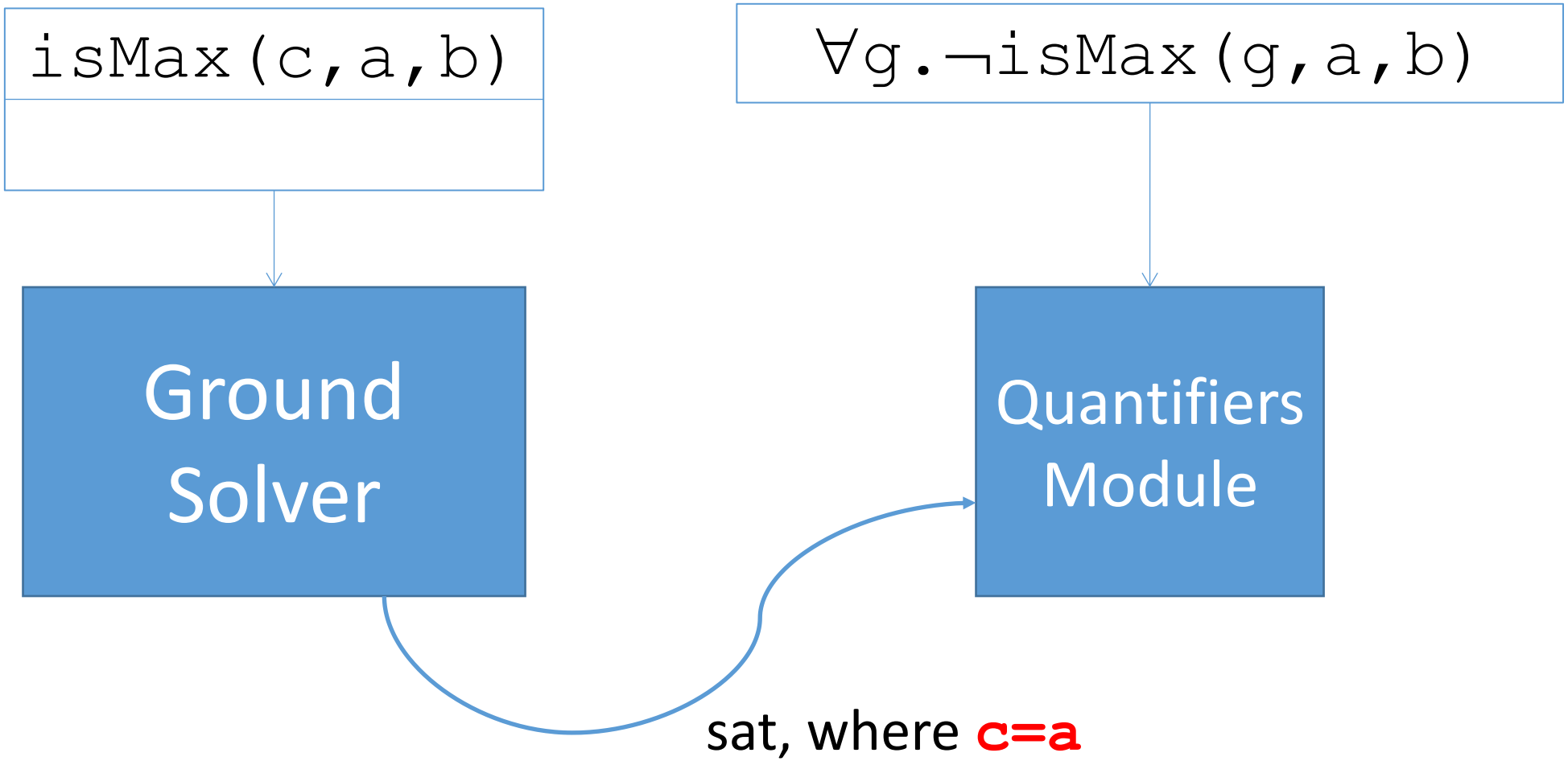
- If ground constraints without CE is unsat, answer “unsat”

Counterexample-Guided Instantiation

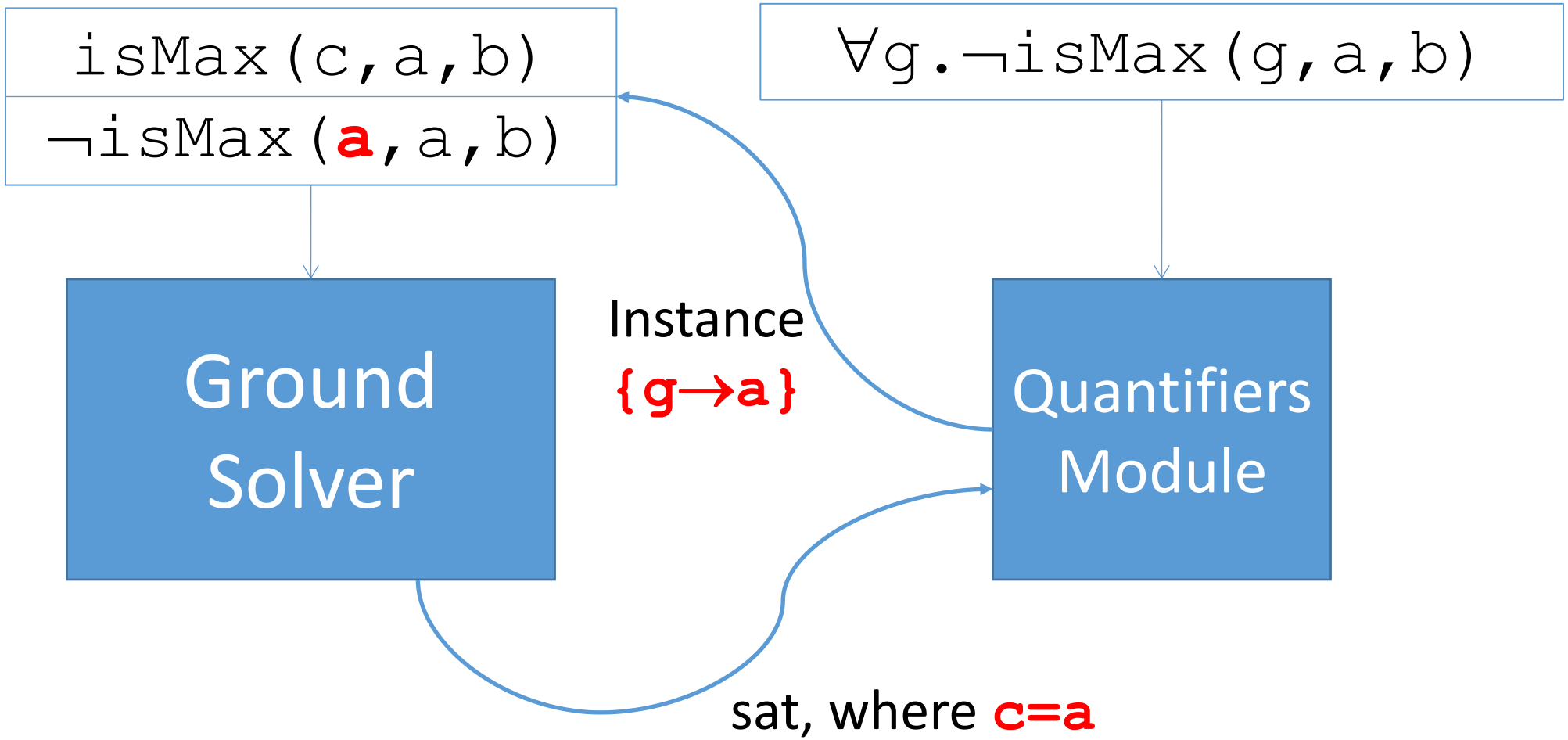


- Else, if ground constraints with CE is unsat, answer “sat”

Counterexample-Guided Instantiation



Counterexample-Guided Instantiation



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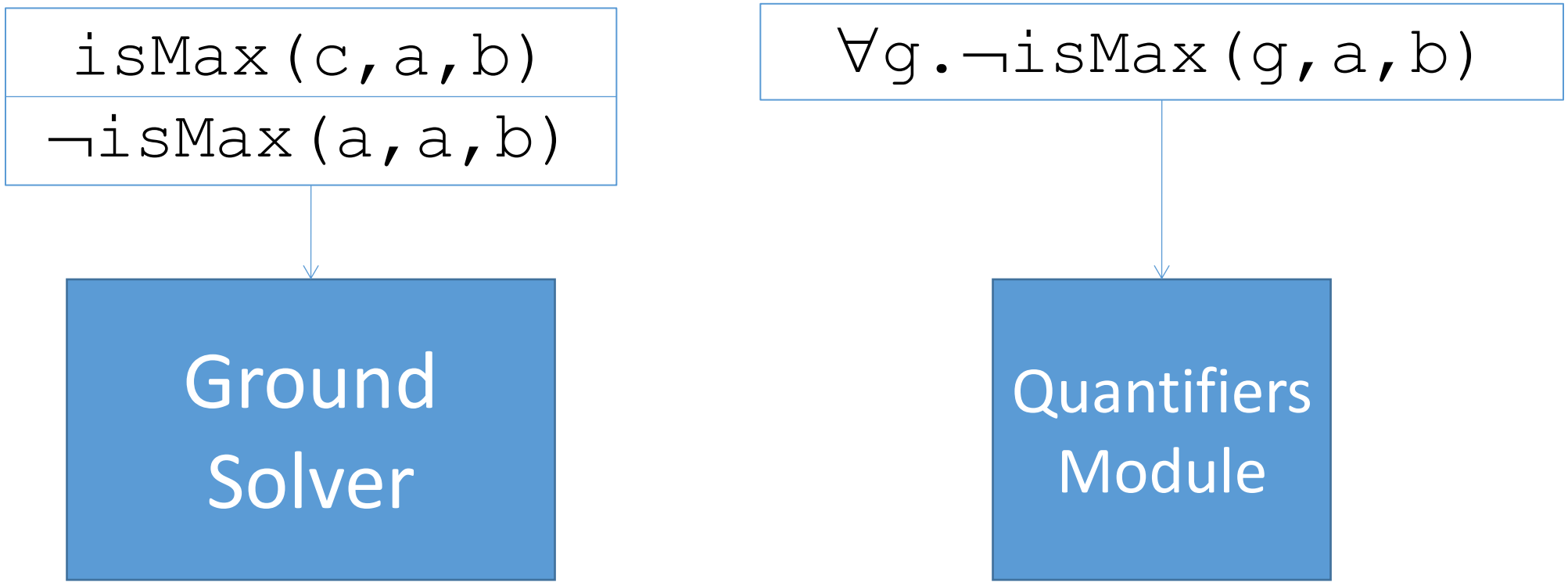

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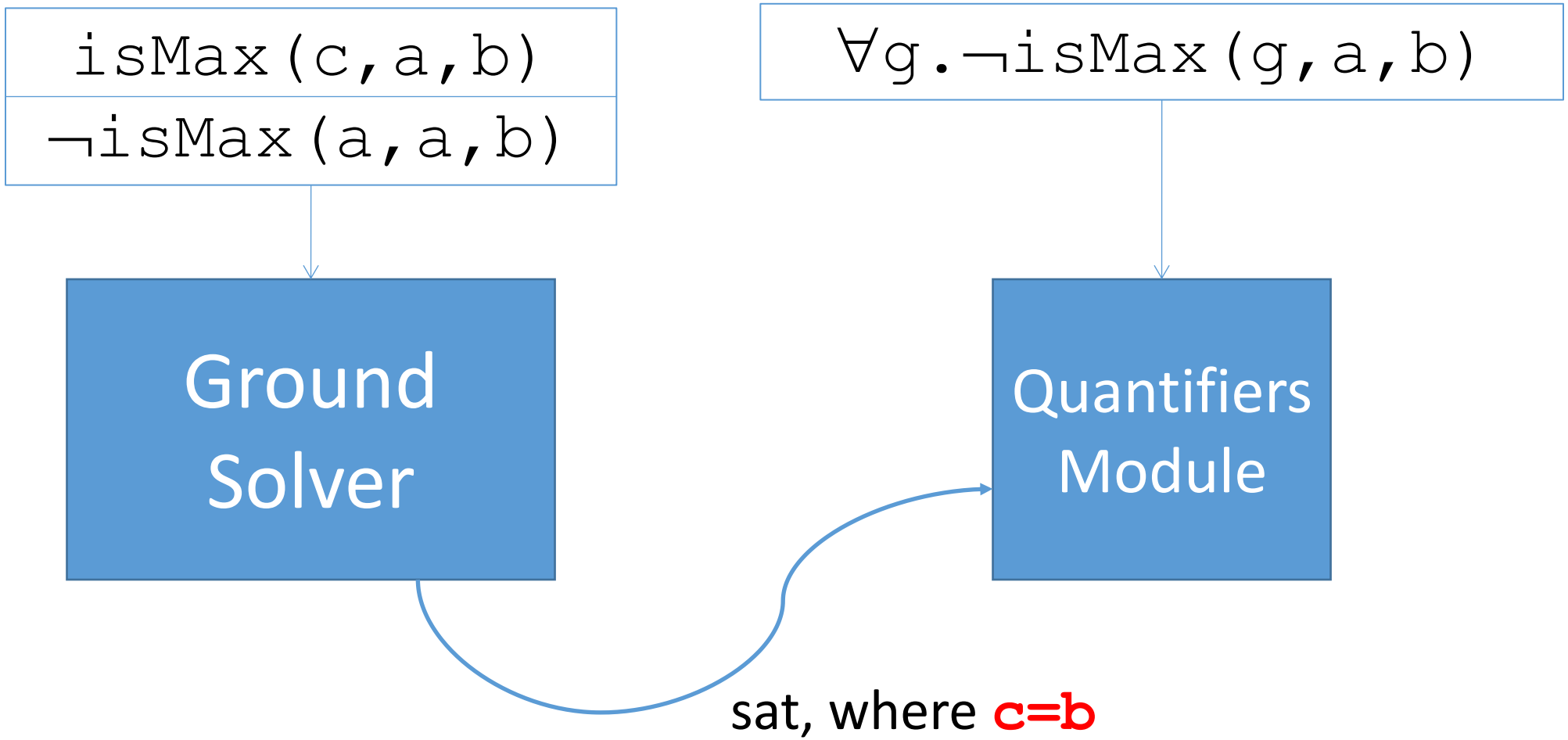

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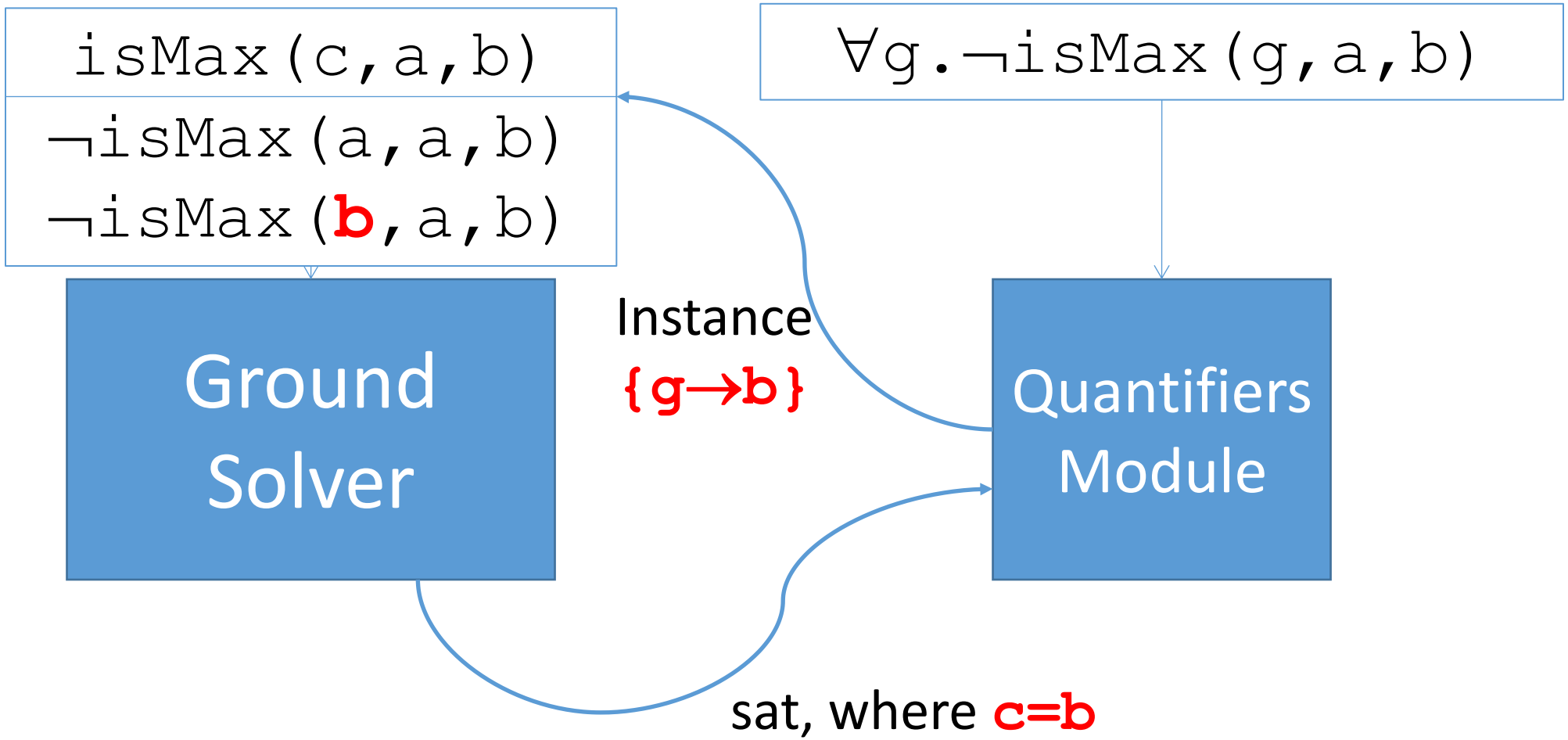
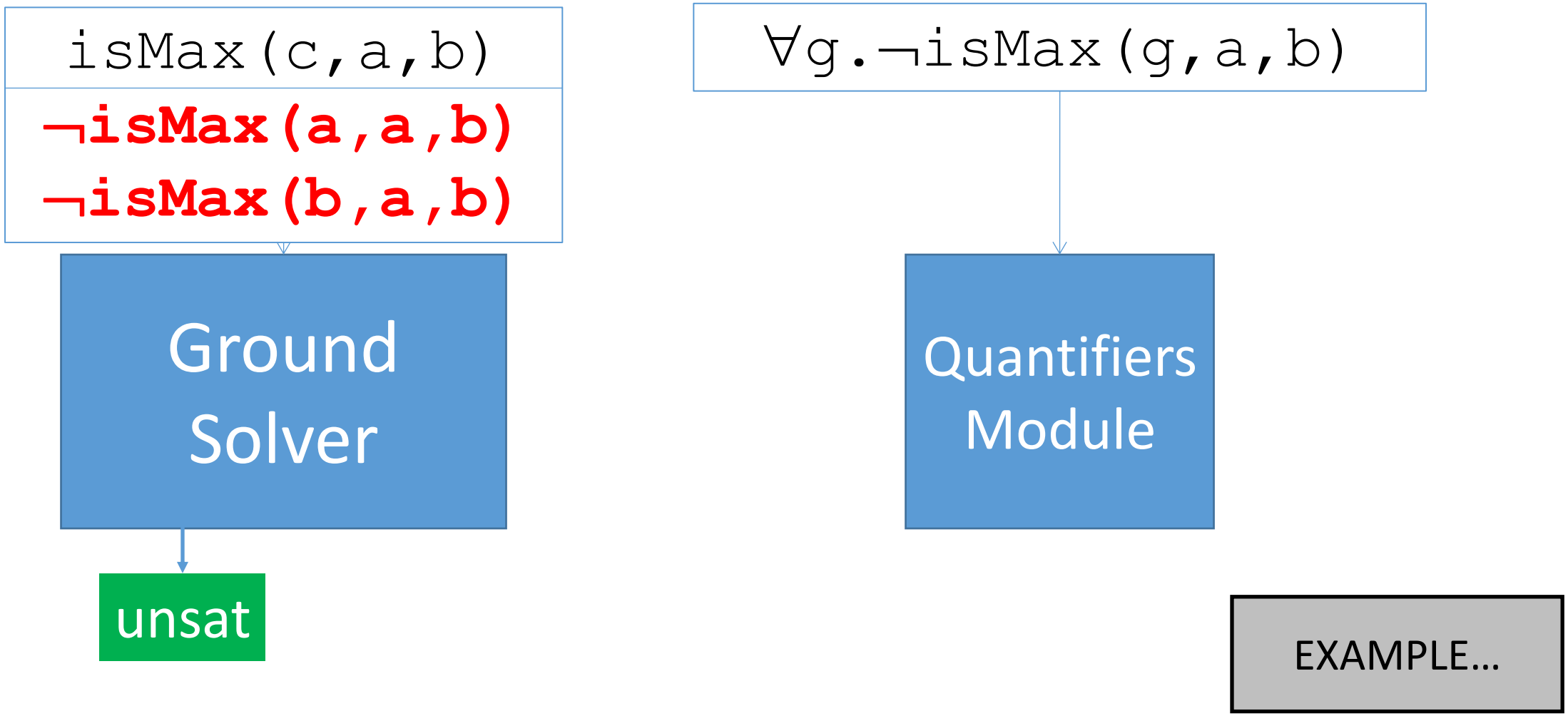
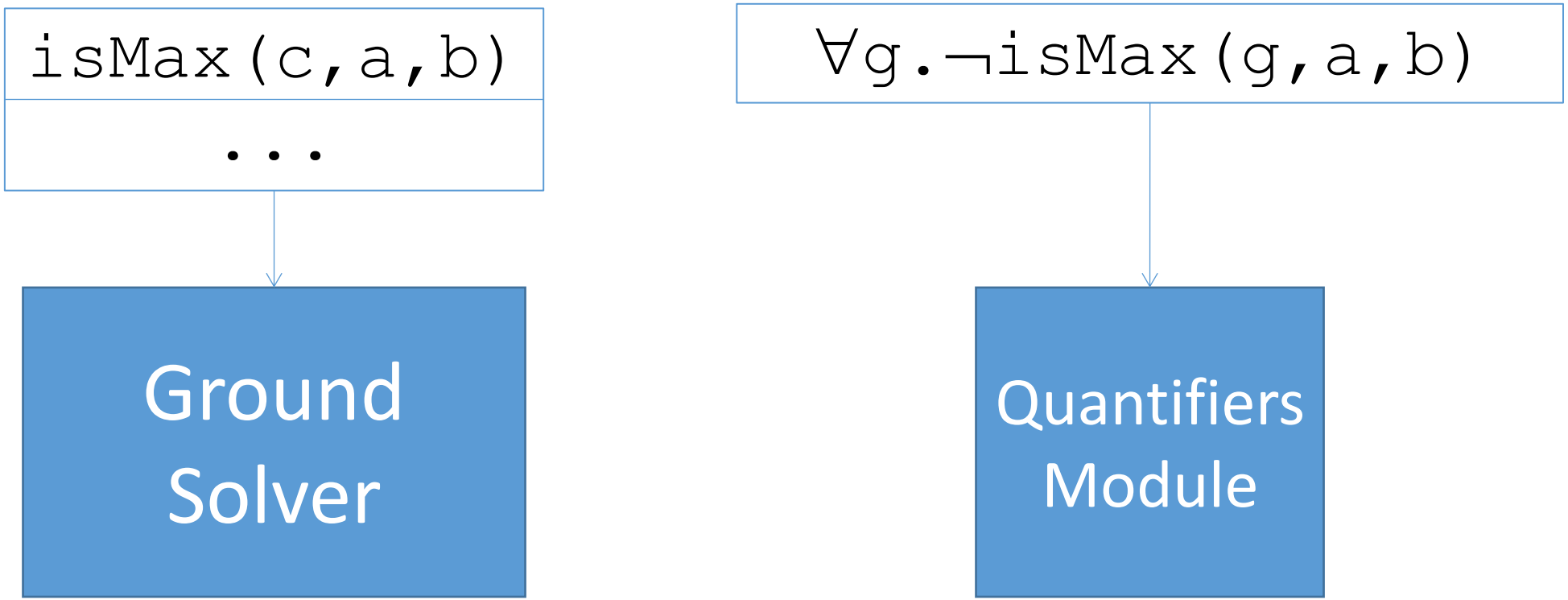


Counterexample-Guided Instantiation



Counterexample-Guided Instantiation



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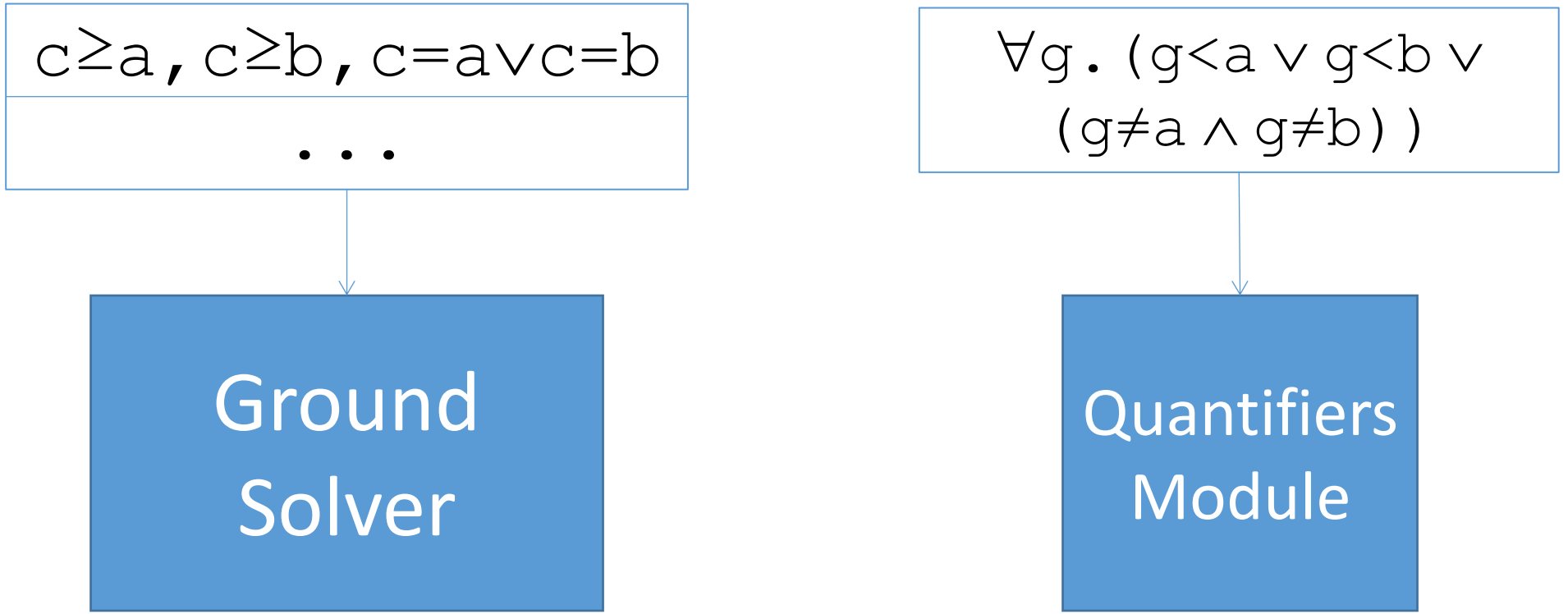

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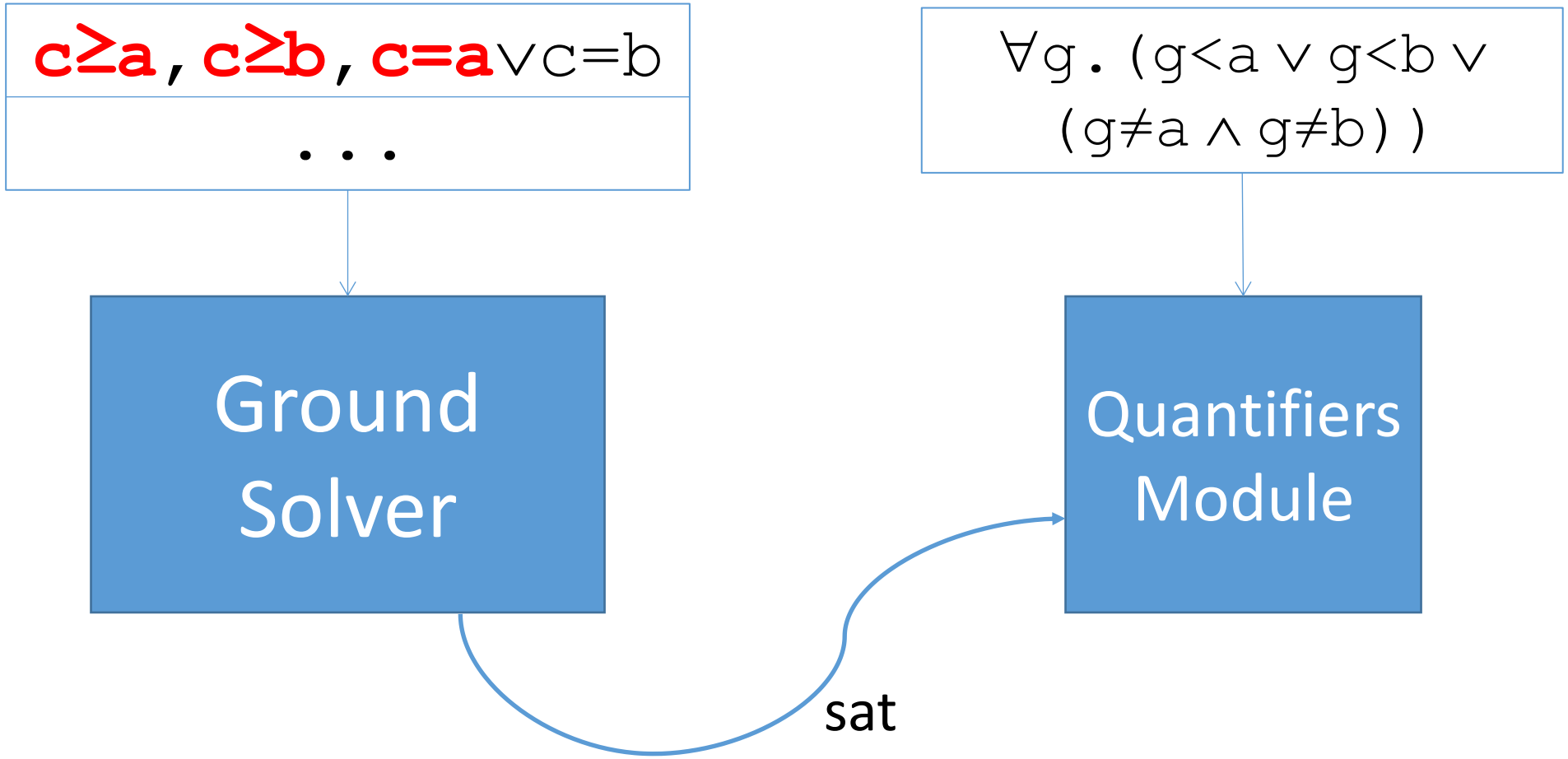

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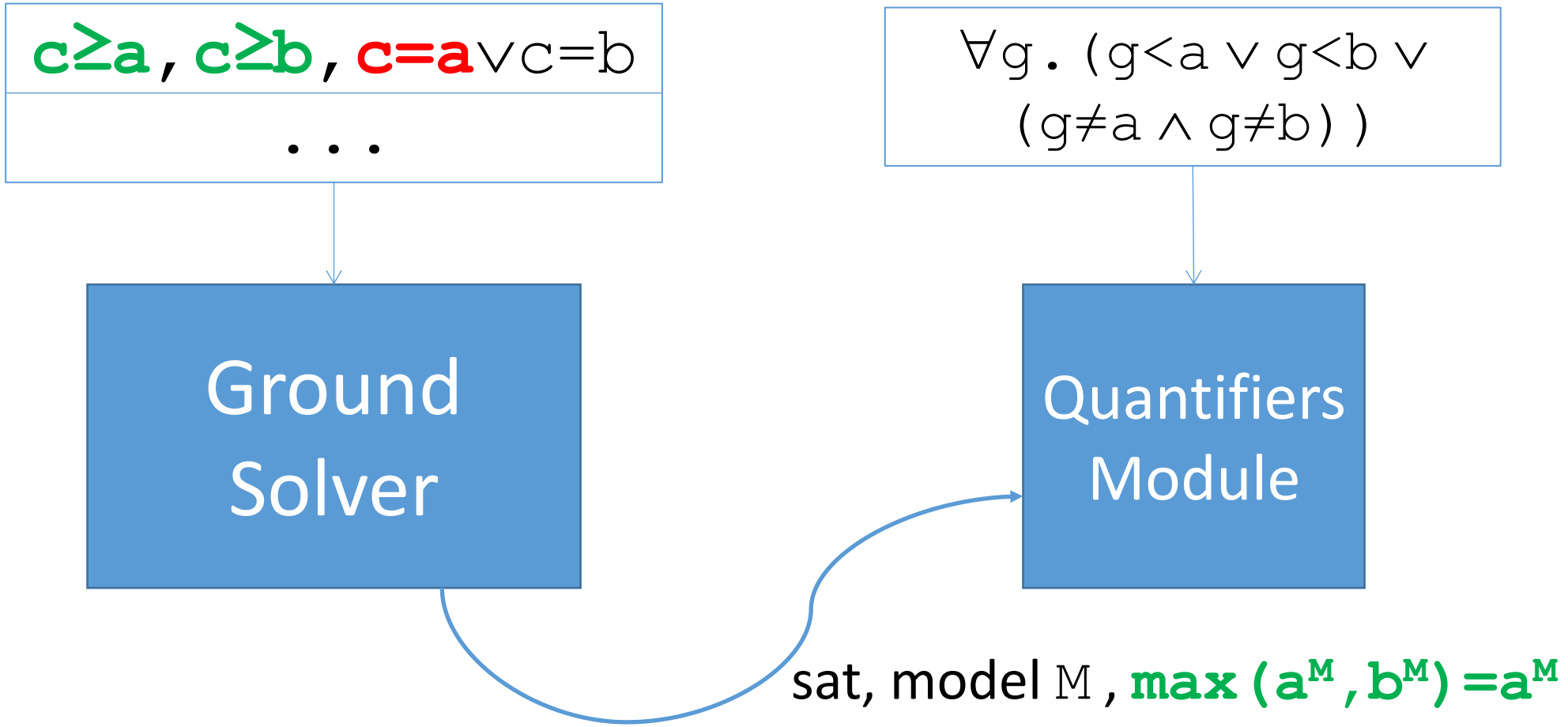
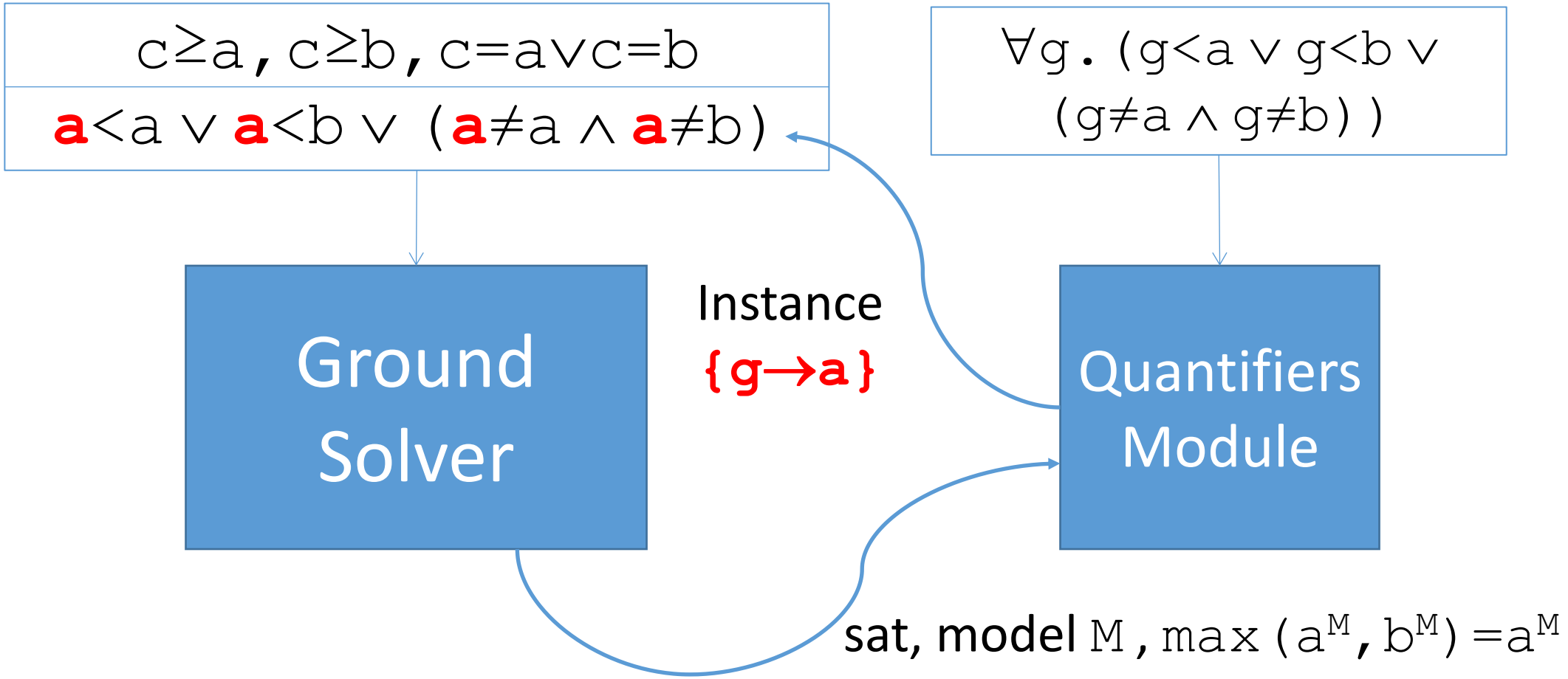
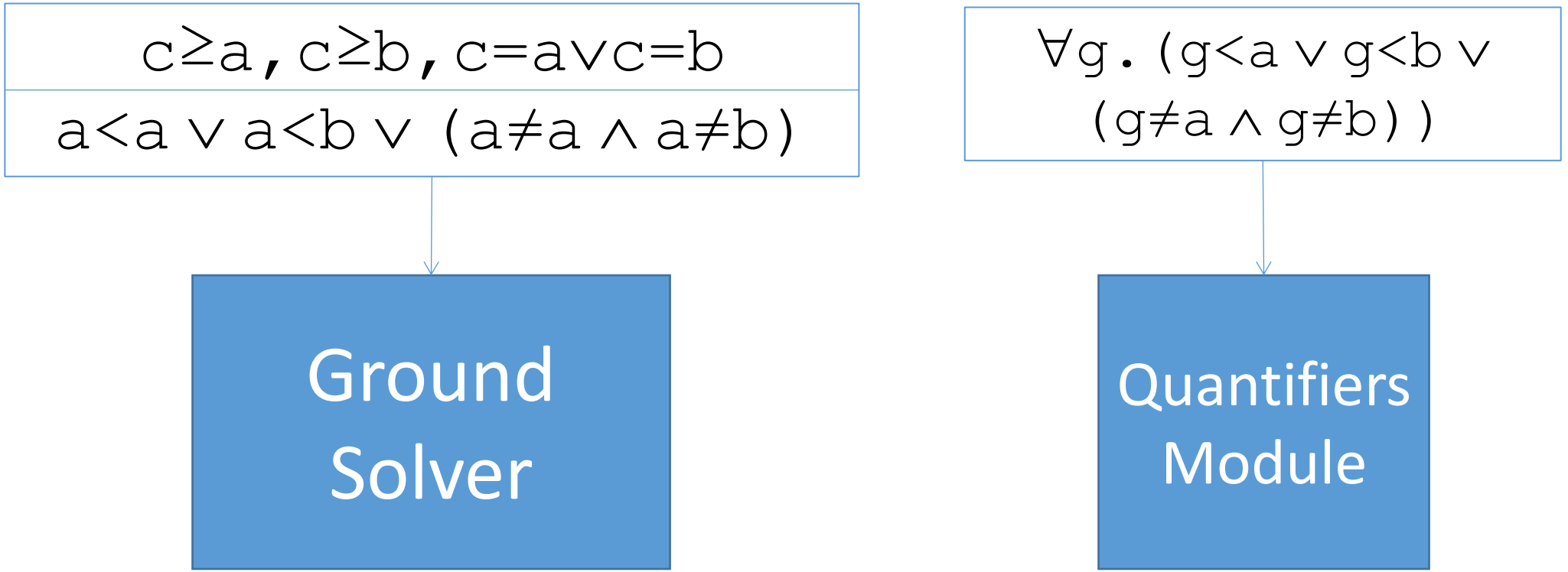


- Take maximal lower bound for c in model M

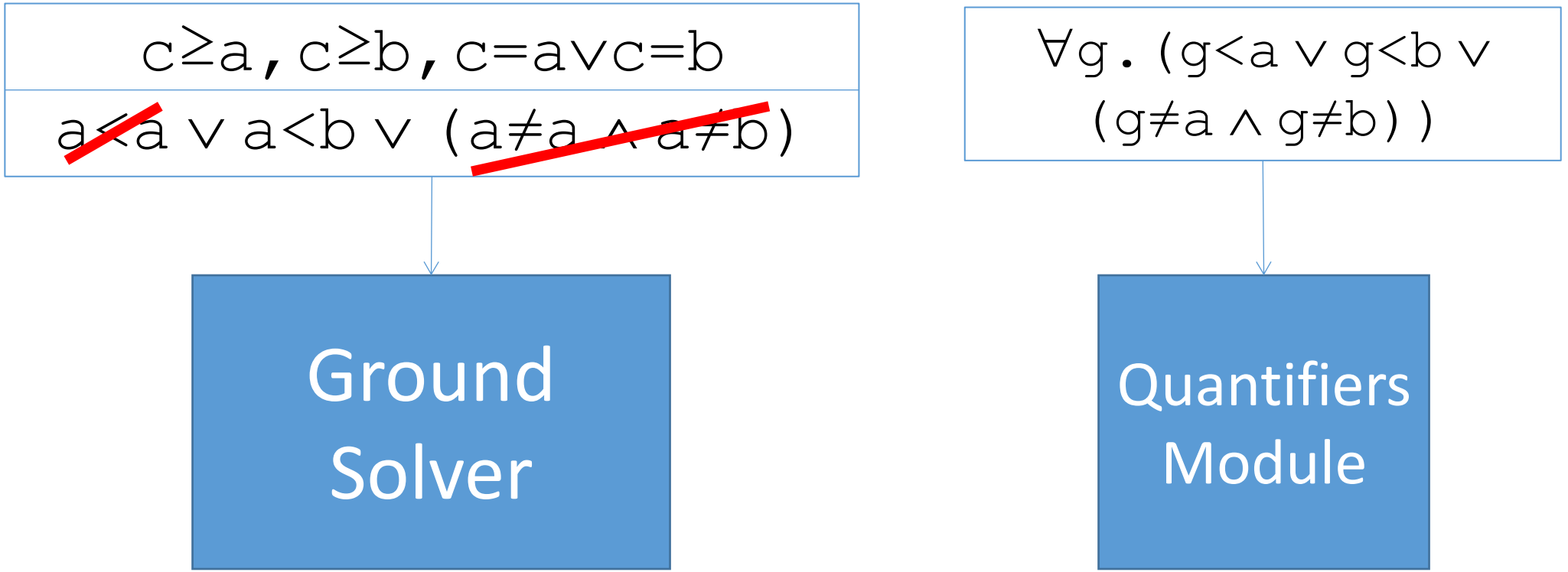
Counterexample-Guided Instantiation



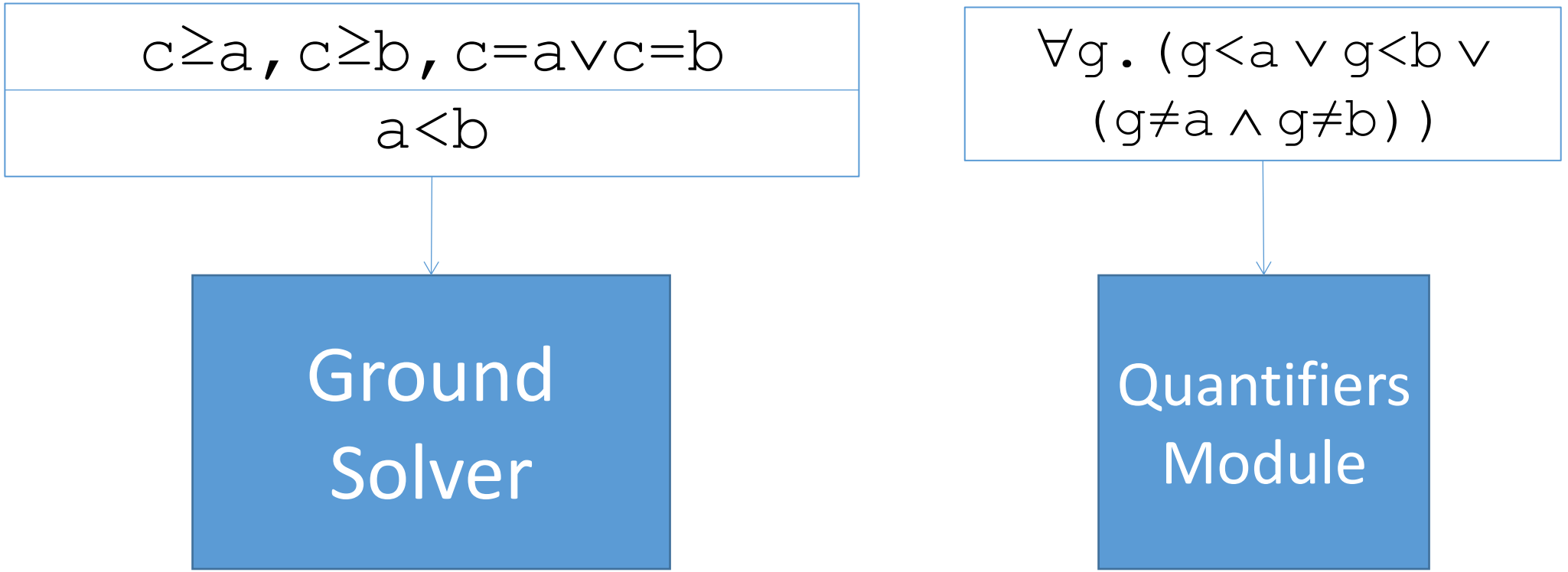
Counterexample-Guided Instantiation



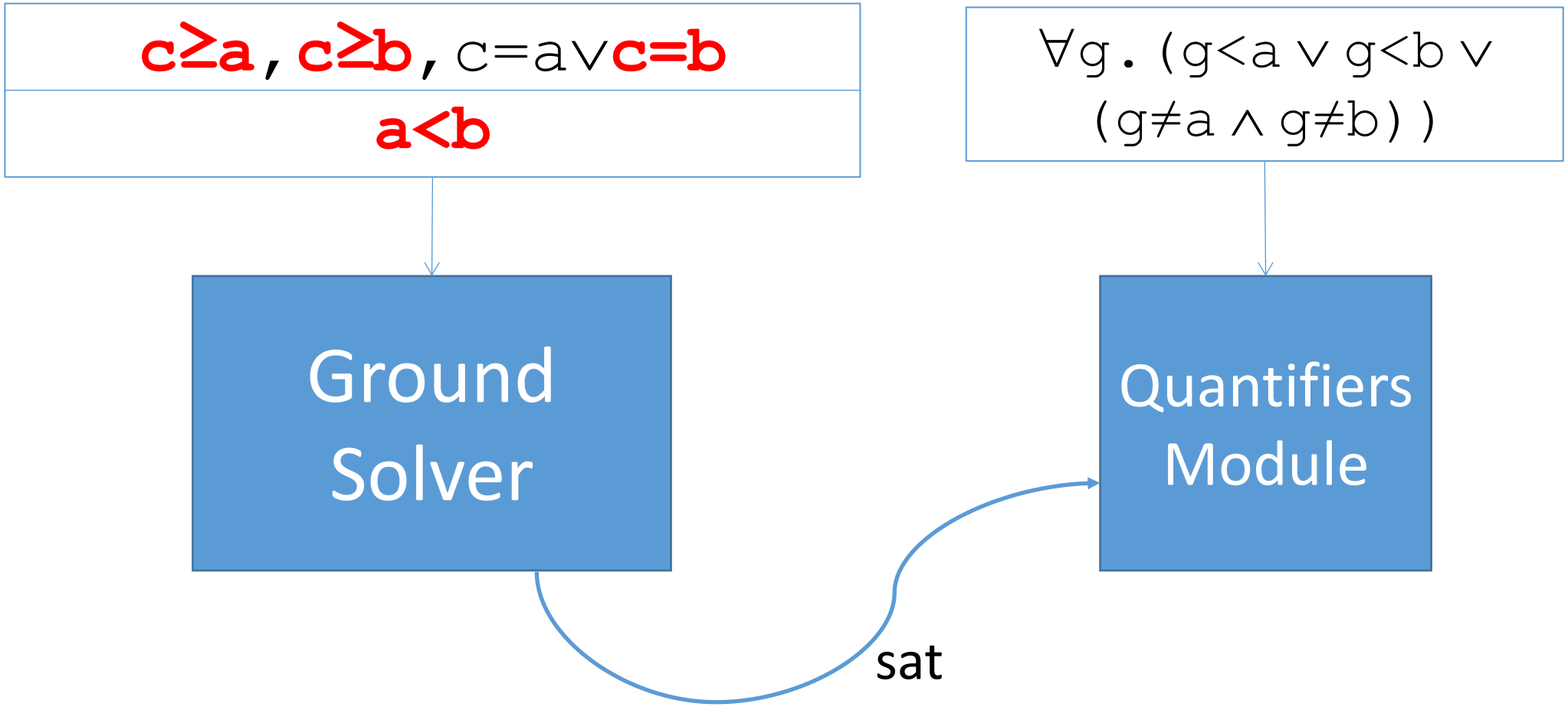
Counterexample-Guided Instantiation



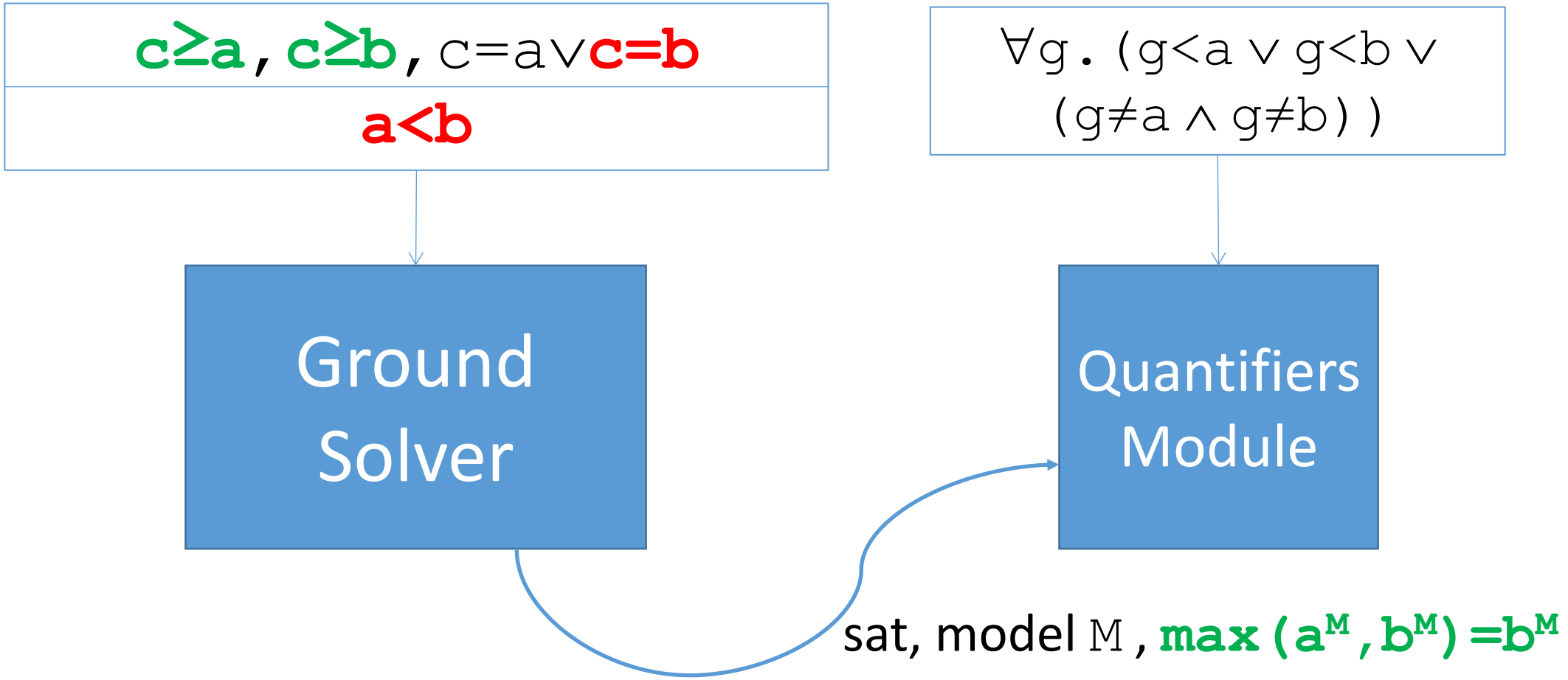
Counterexample-Guided Instantiation



Counterexample-Guided Instantiation

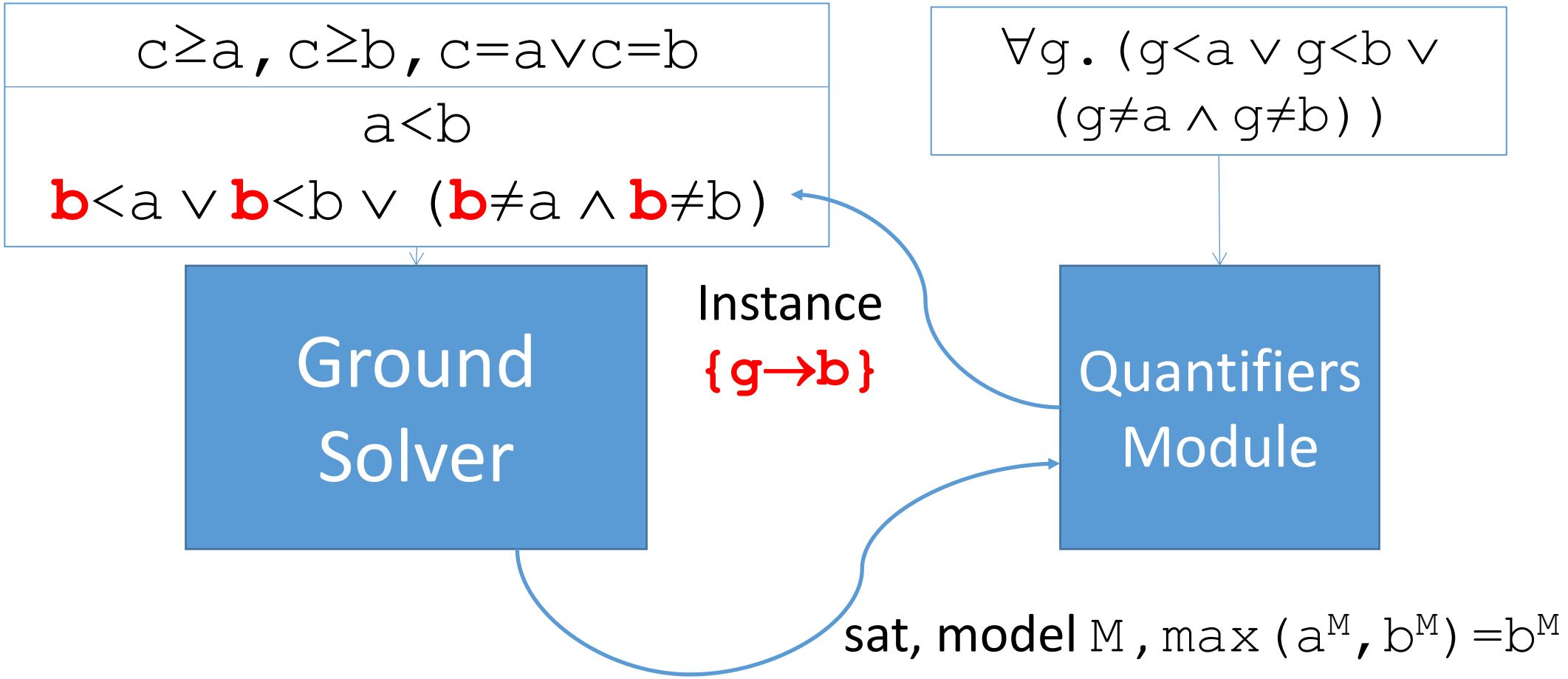


Counterexample-Guided Instantiation

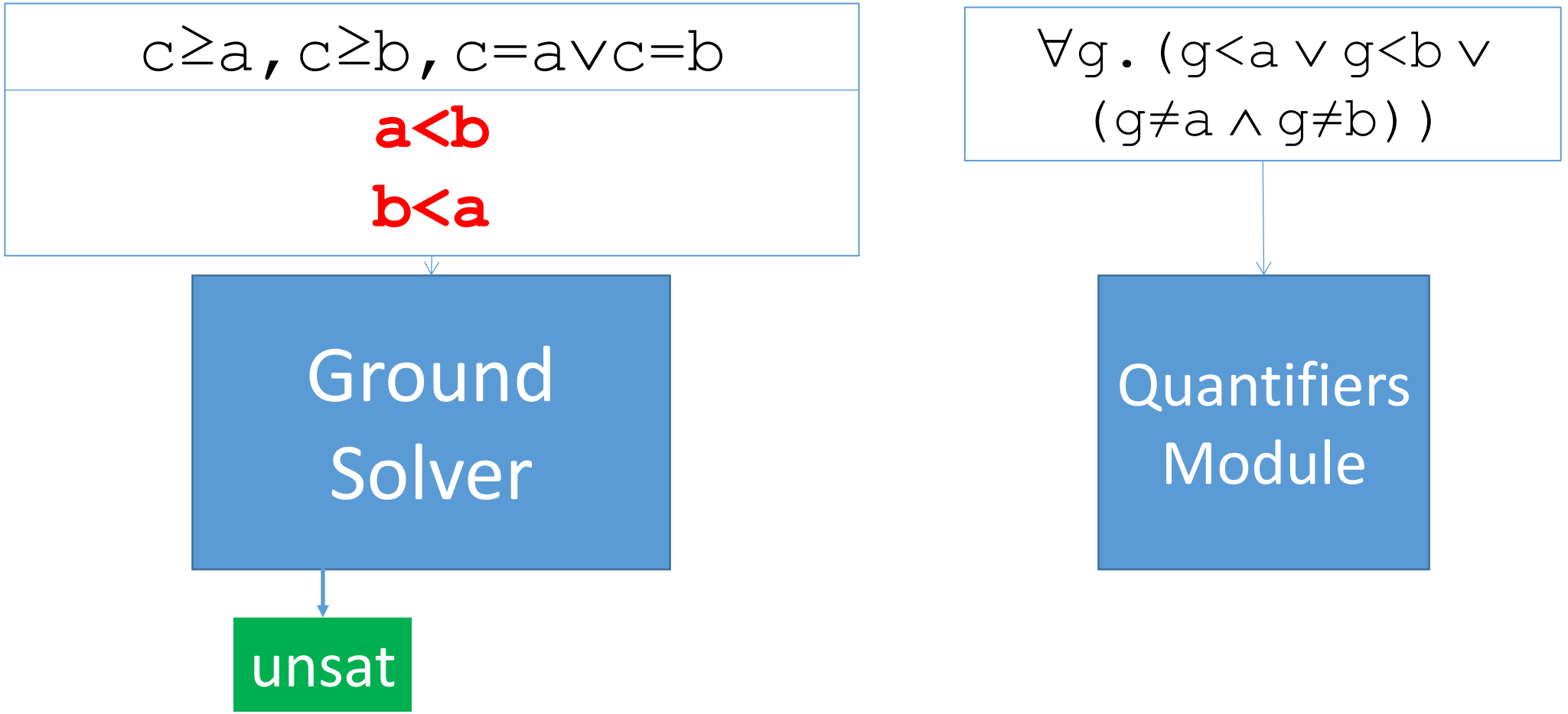


- Take maximal lower bound for c in model M

Counterexample-Guided Instantiation



Counterexample-Guided Instantiation



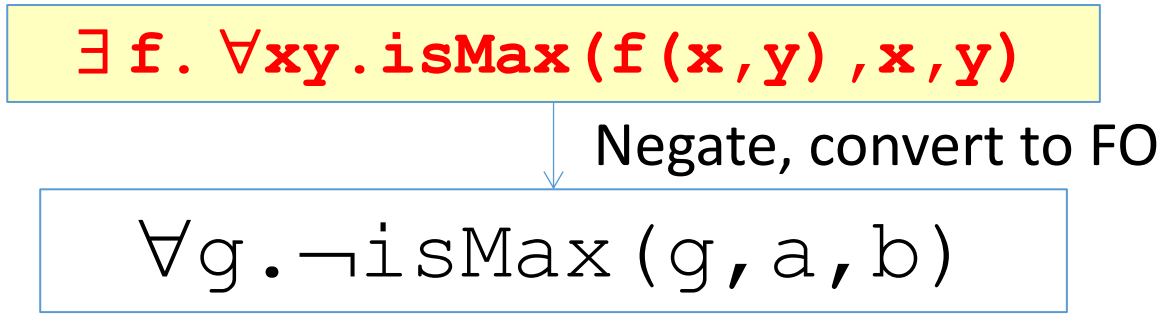
Synthesis: Solutions

$\exists f.\ \forall xy.isMax(f(x,y),x,y)$

Ground Solver

Quantifiers Module

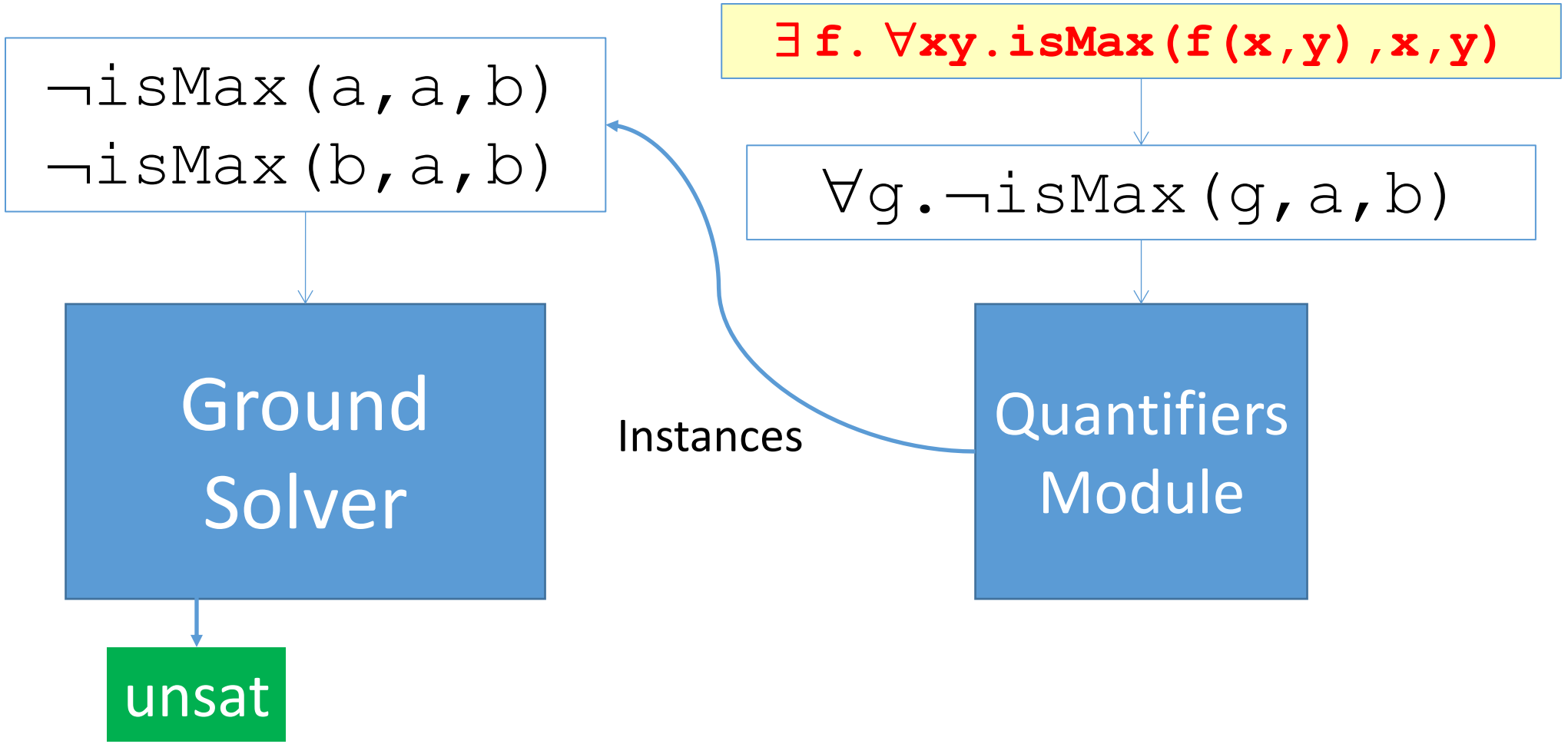
Synthesis: Solutions



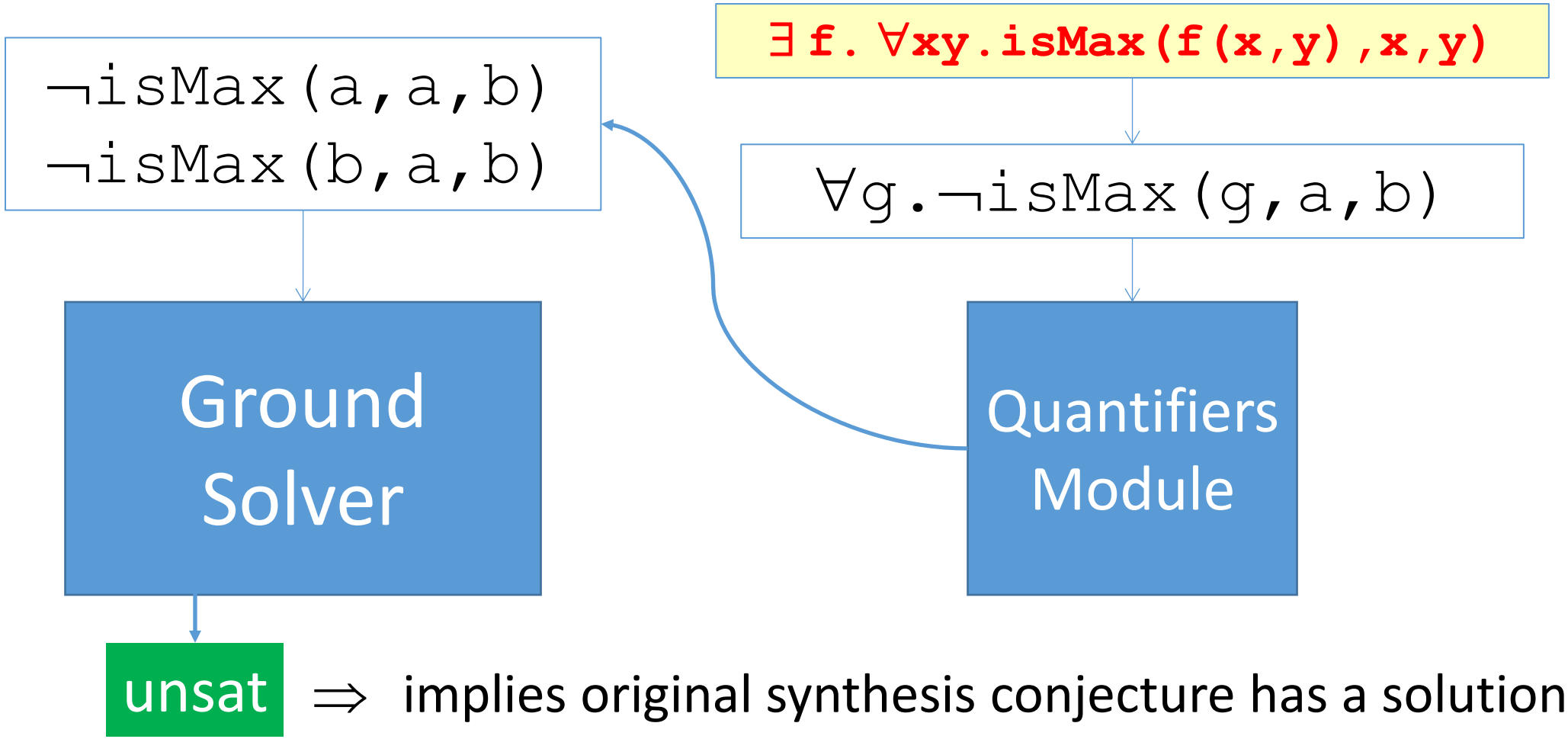
Ground Solver

Quantifiers Module

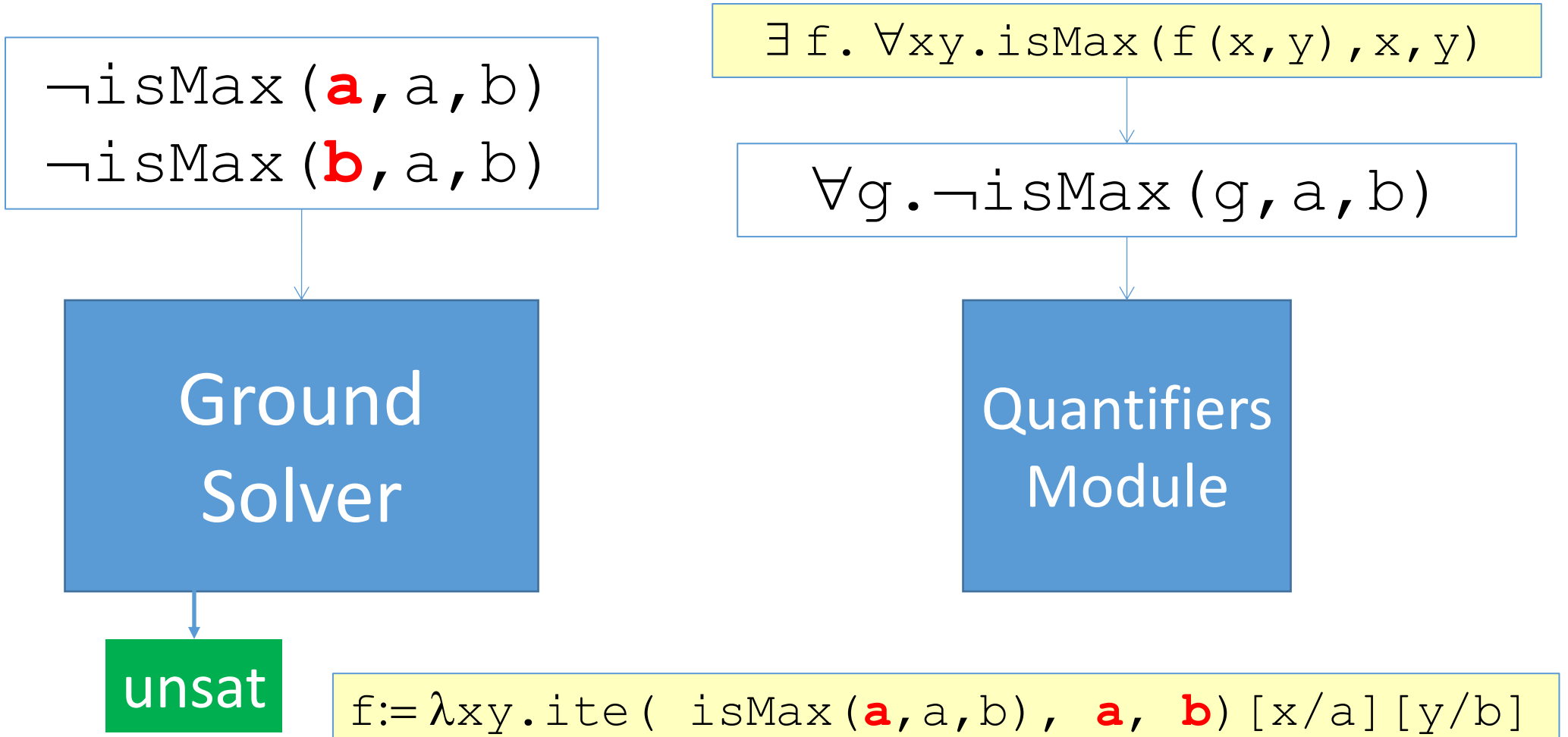
Synthesis: Solutions



Synthesis: Solutions

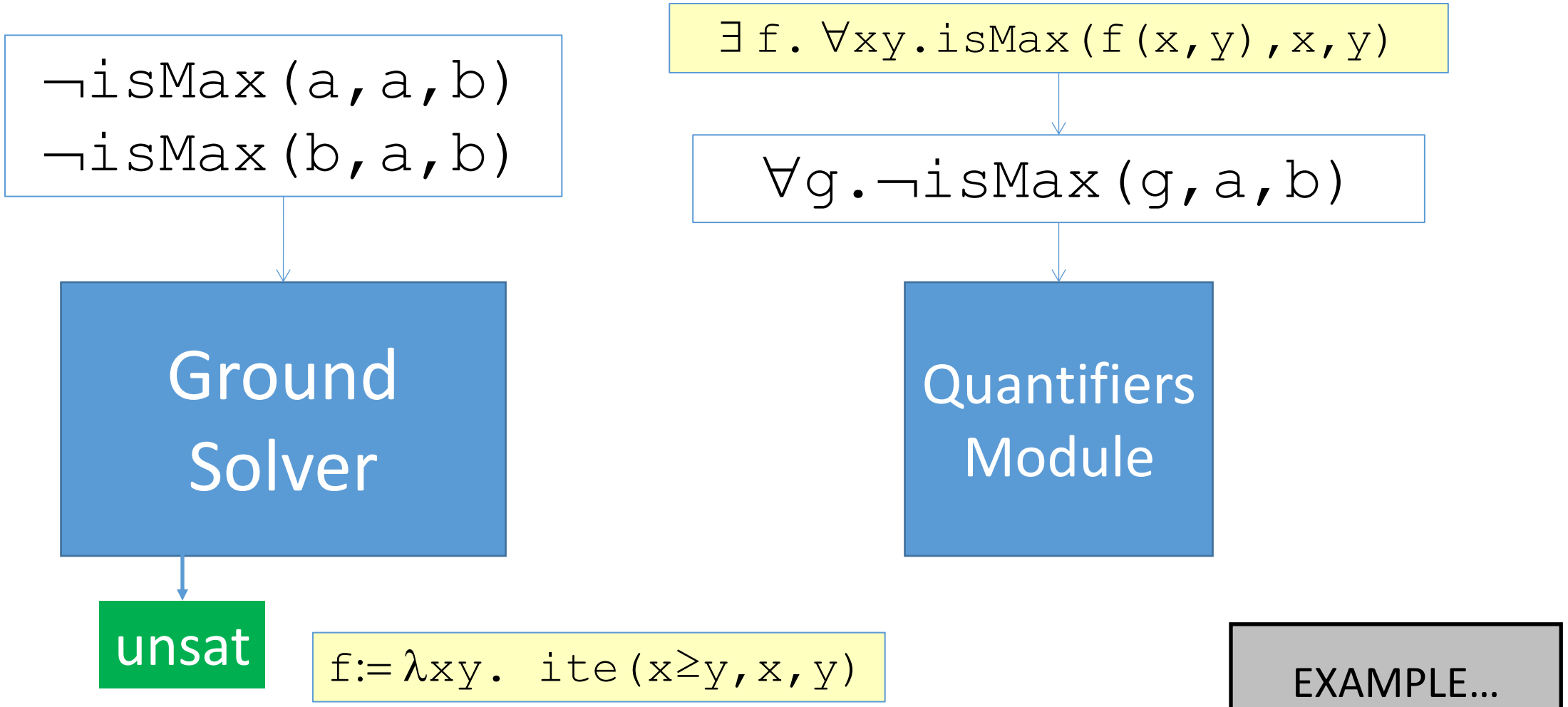


Synthesis: Solutions



⇒Solution can be extracted from unsatisfiable core of instantiations a/g, b/g

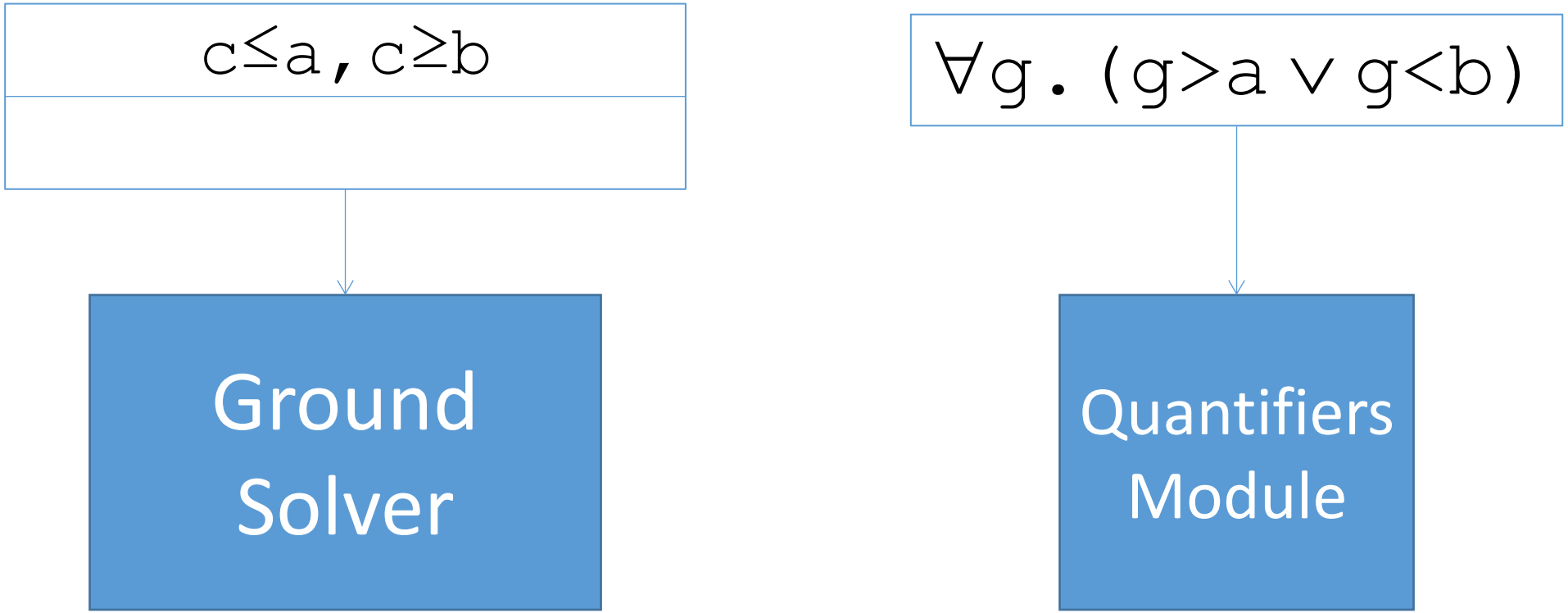
Synthesis: Solutions



EXAMPLE…

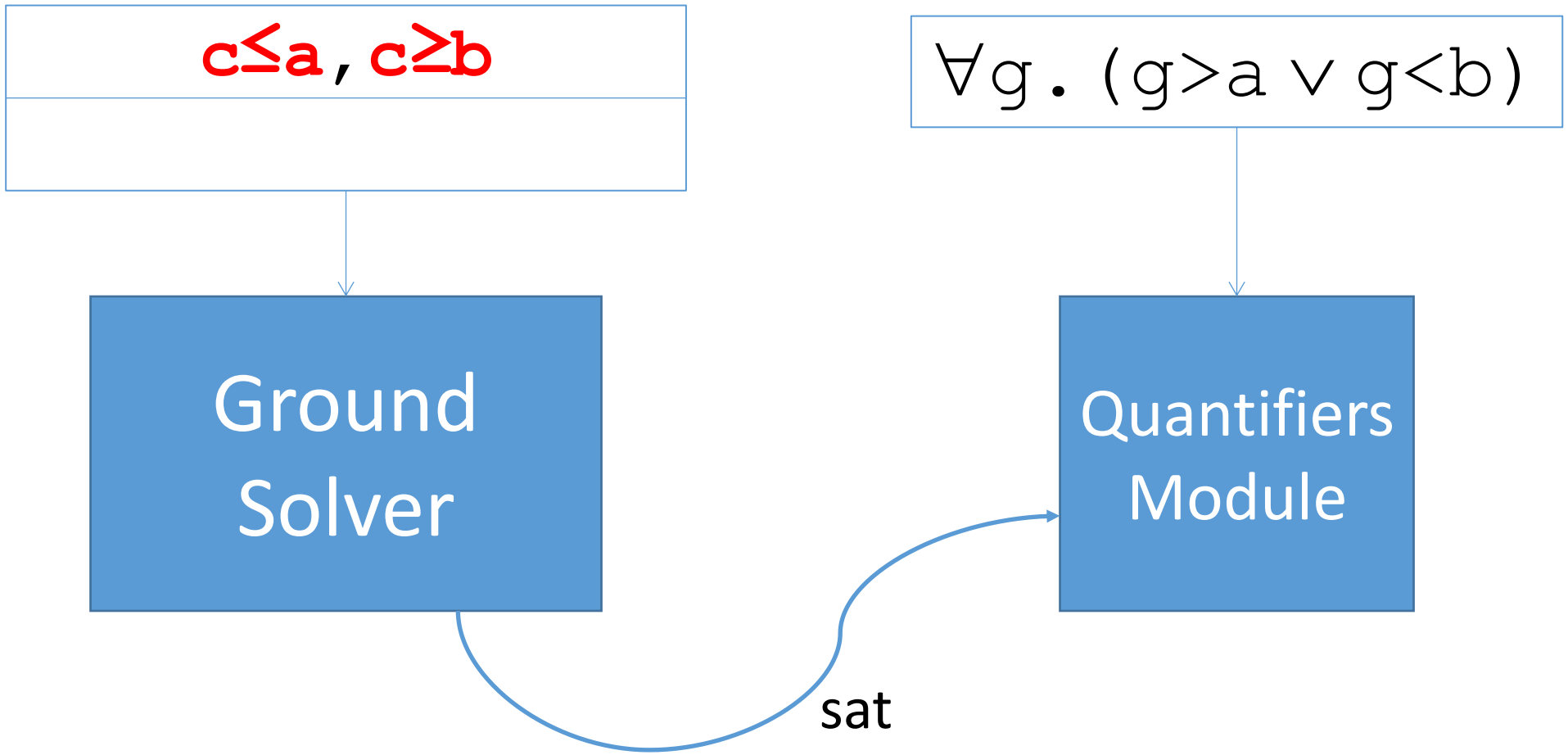
⇒ Desired function, after simplification

Counterexample-Guided Instantiation

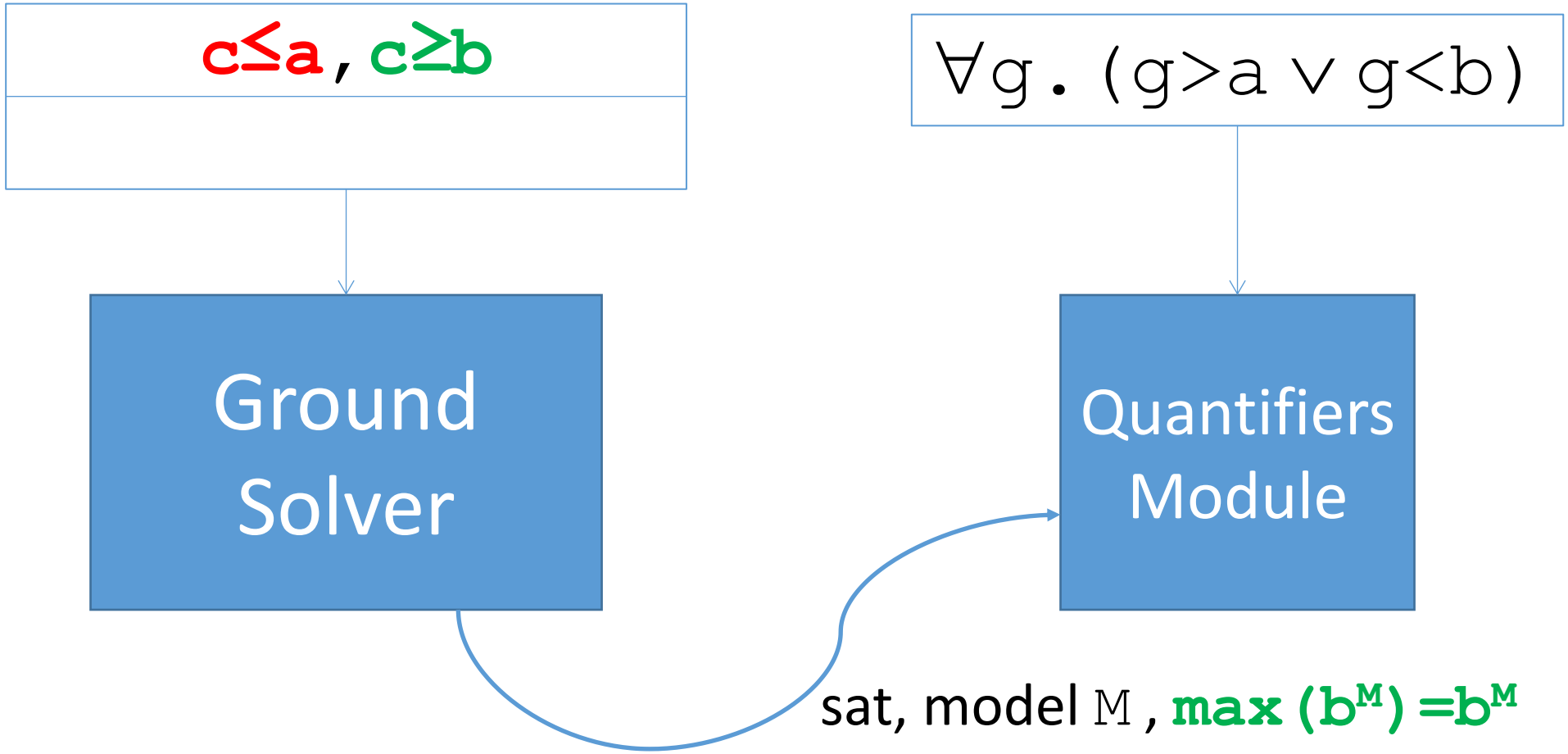


- **Consider example:** ∀g. (g>a ∨ g<b)

Counterexample-Guided Instantiation

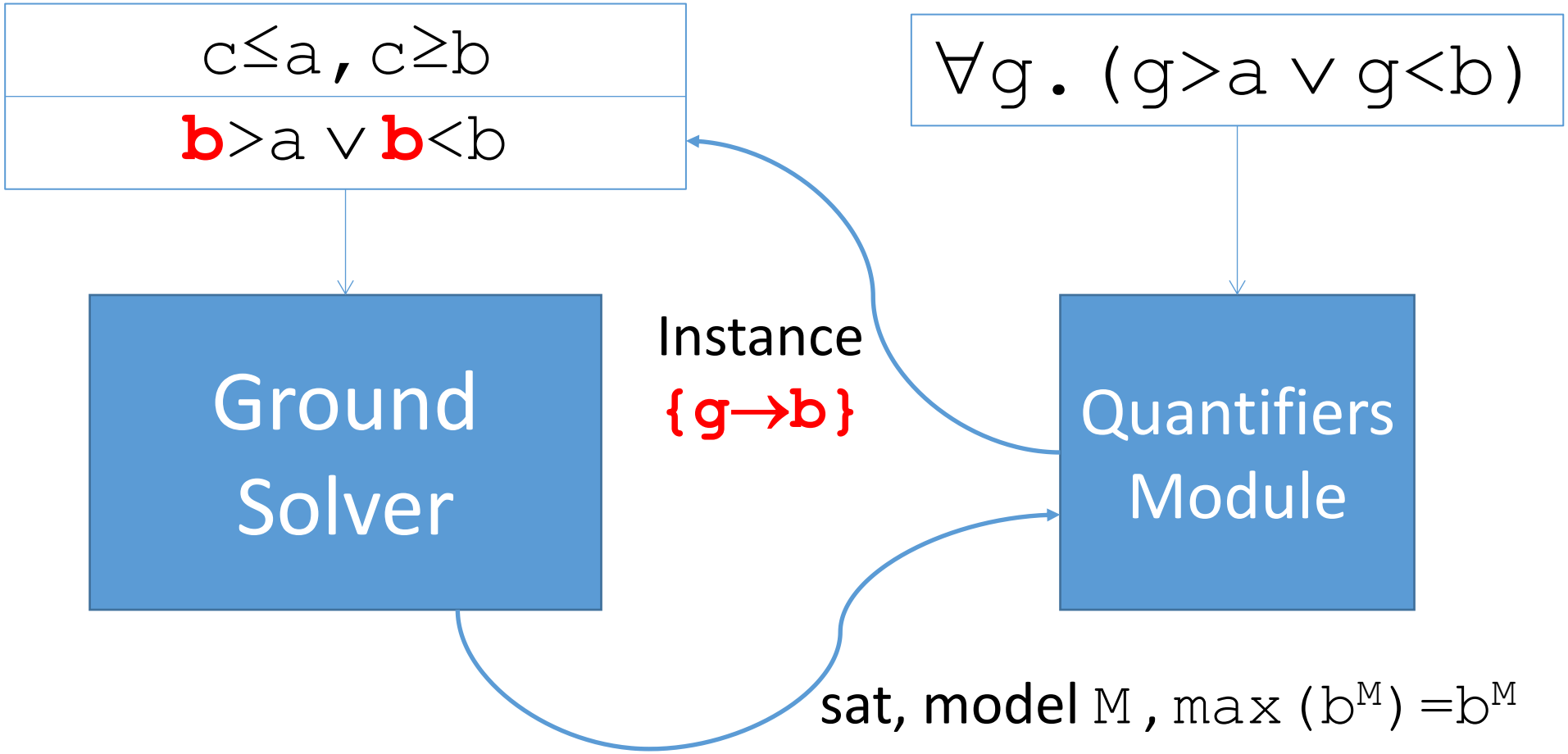


Counterexample-Guided Instantiation

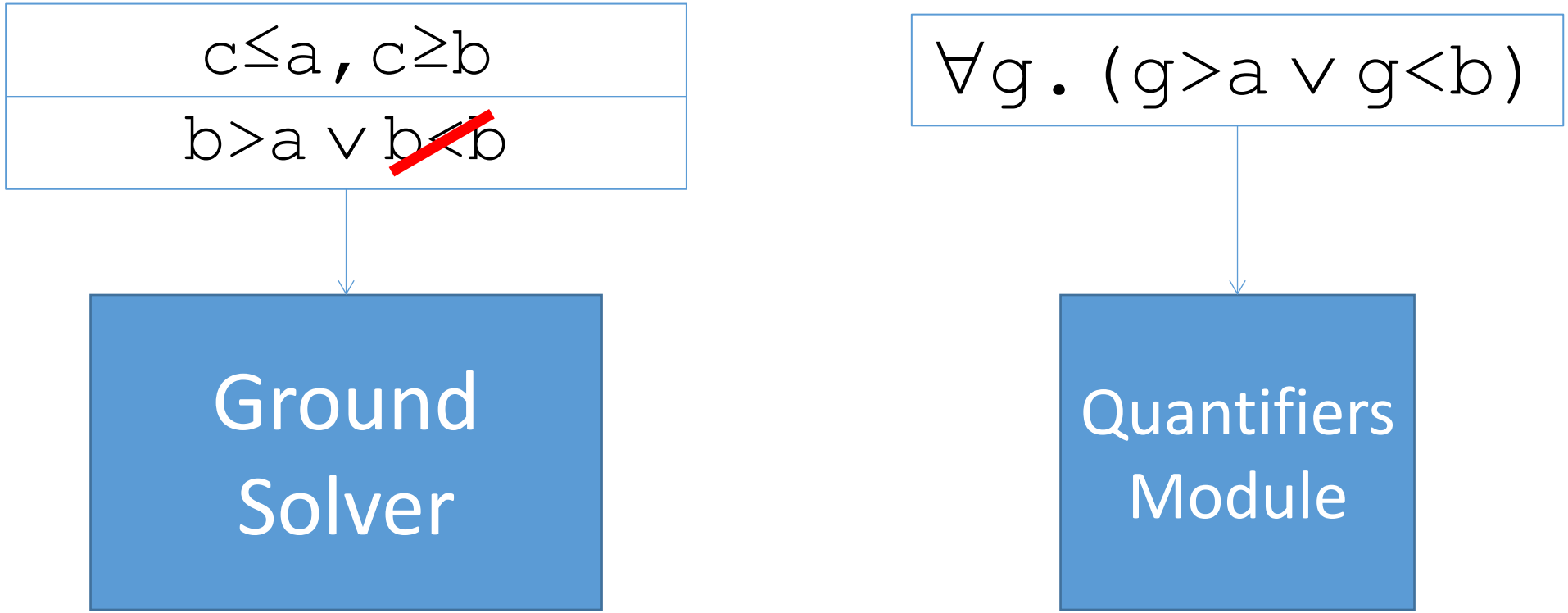


- Take maximal lower bound for c in model M

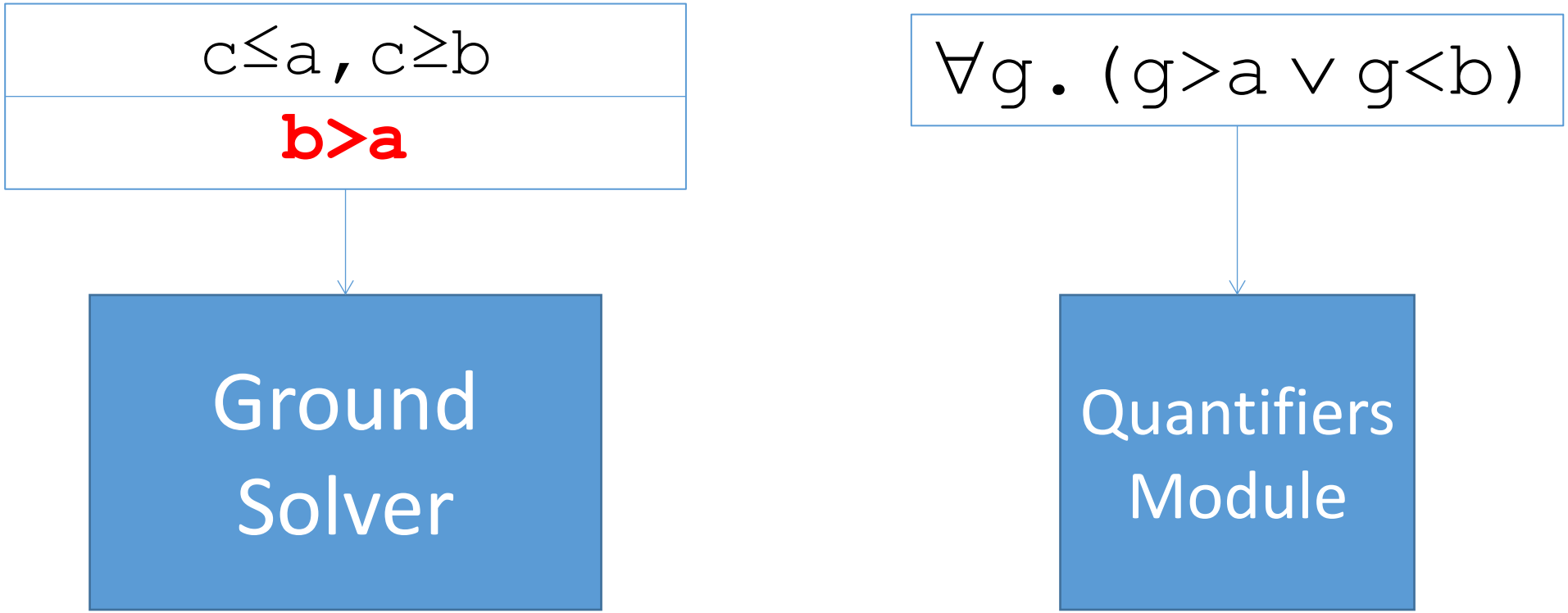
Counterexample-Guided Instantiation



Counterexample-Guided Instantiation

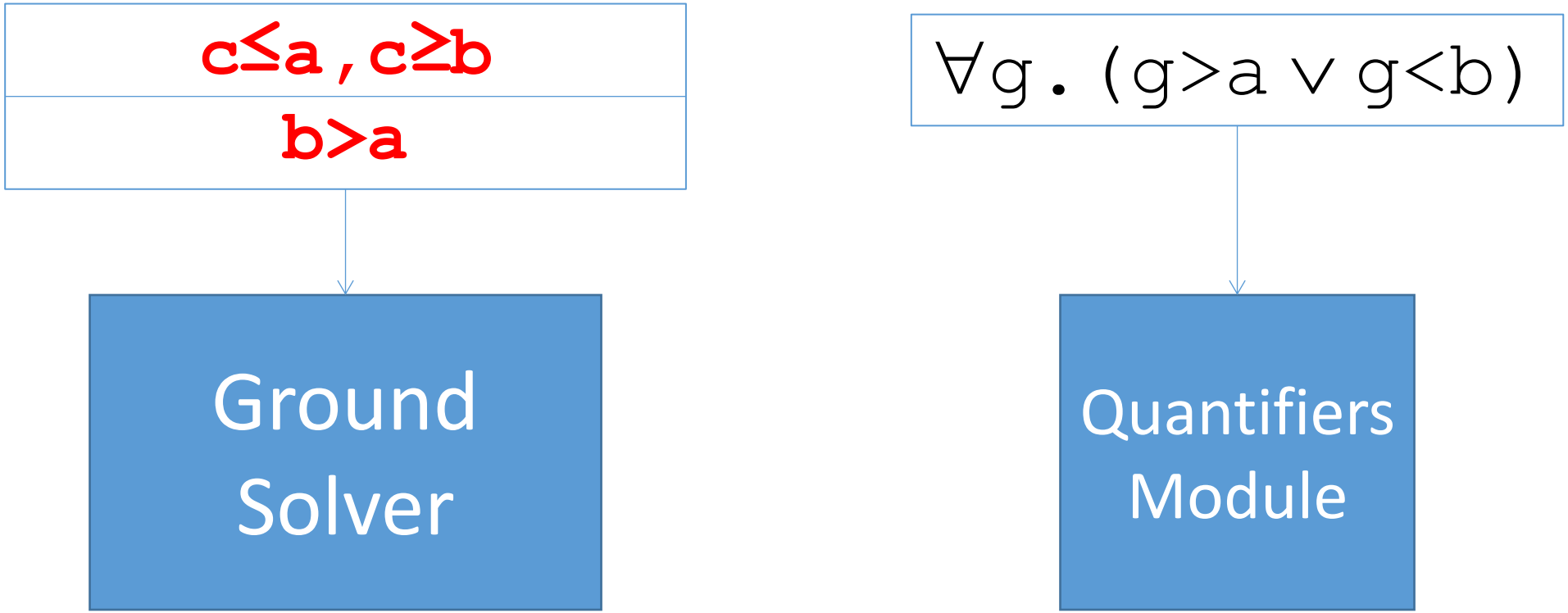


Counterexample-Guided Instantiation



- {b>a} is sat

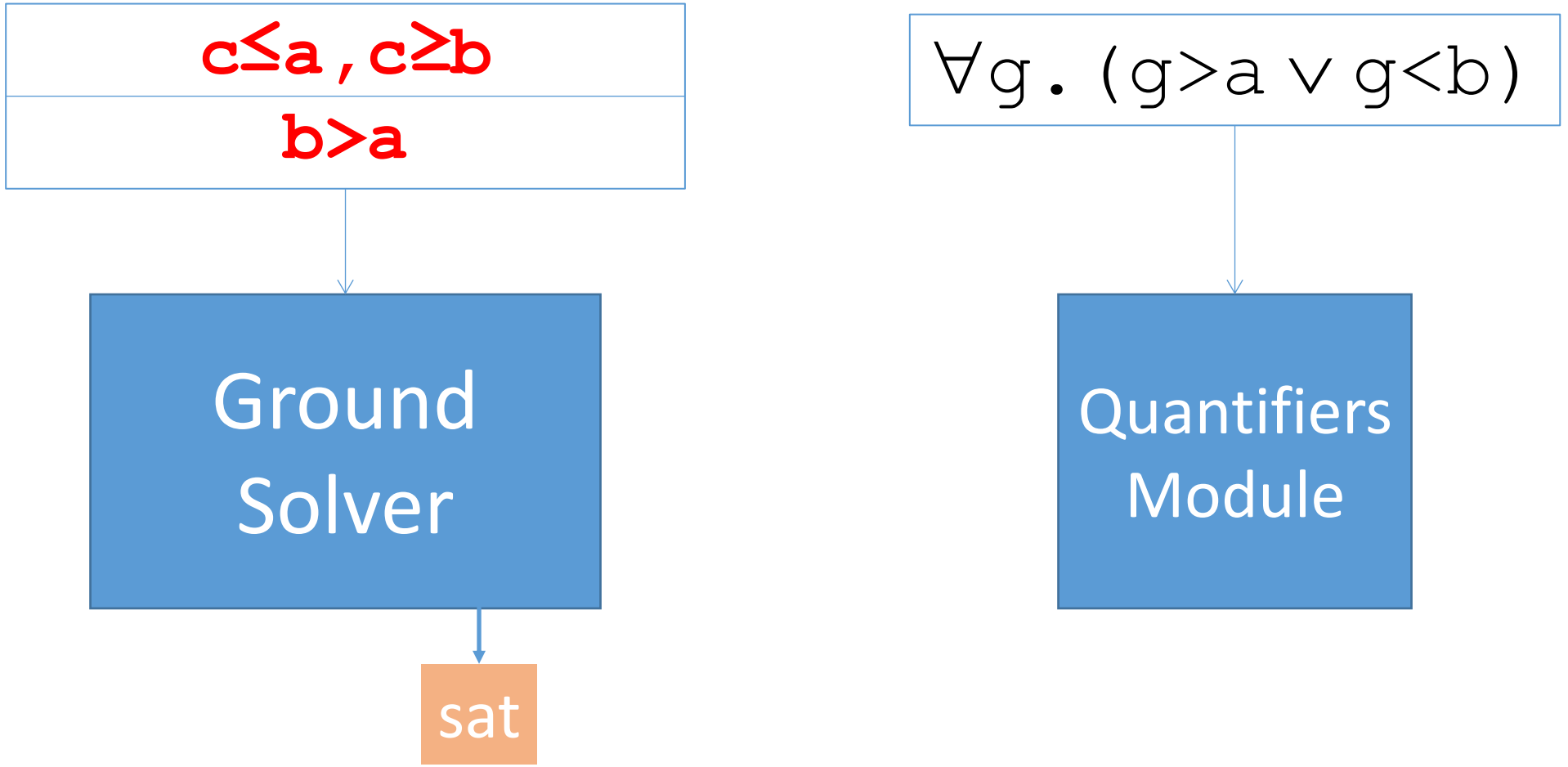
Counterexample-Guided Instantiation



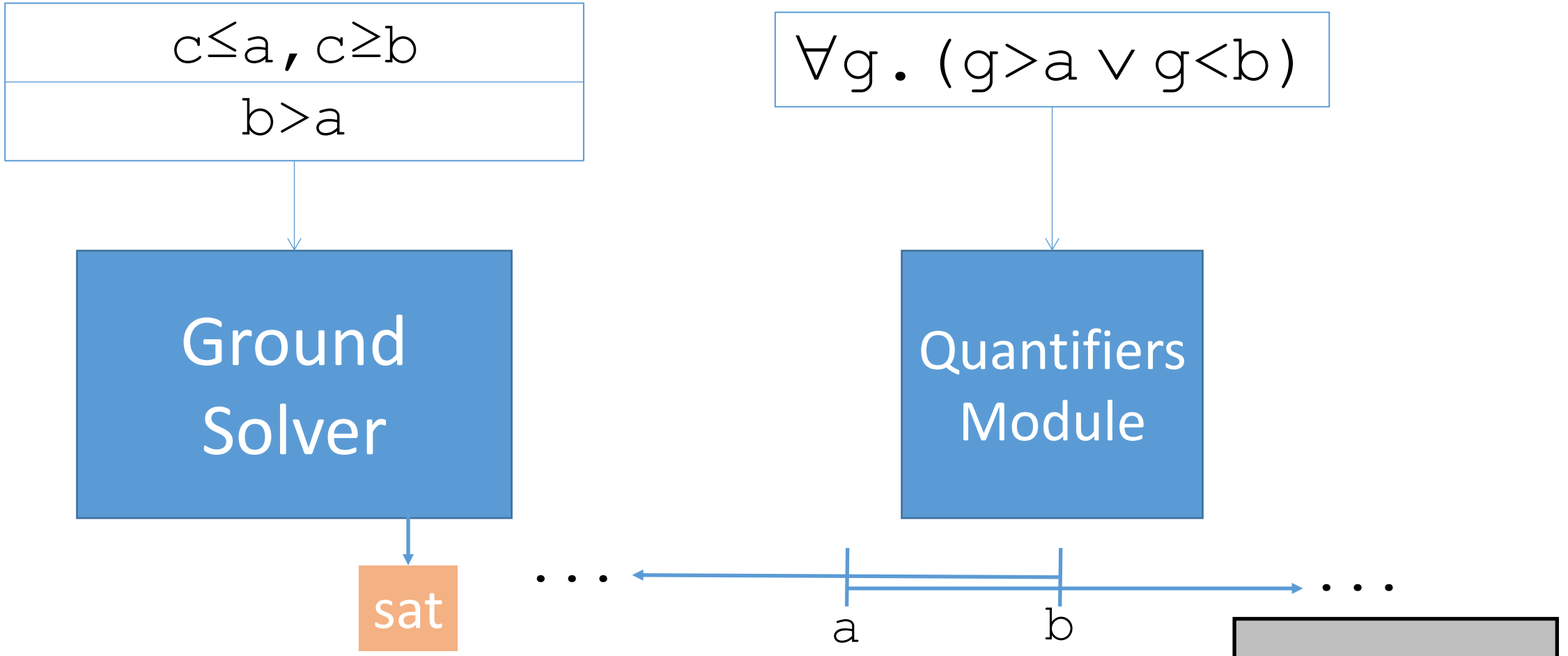
- {b>a} is sat
- ...but {c≤a, c≥b, b>a} is unsat

  ⇒ In other words, there is no model for counterexample c

Counterexample-Guided Instantiation



Counterexample-Guided Instantiation



⇒ All models satisfying b>a also satisfy ∀g. (g>a ∨ g<b)

Counterexample-Guided Instantiation

- For linear real and integer arithmetic:
  - With one quantifier alternation:
    - **Sound** and **complete** (terminating) **[Reynolds/King/Kuncak, draft 2015]**
  - With arbitrary quantifier alternations:
    - Effective in practice, for both “sat” and “unsat”

Counterexample-Guided Instantiation in CVC4

- Highly competitive for synthesis applications
  - **Won**, GENERAL/LIA divisions of SygusComp 2015
- Applicable to arbitrary quantified formulas as well
  - **Won**, LIA/LRA divisions of SMT COMP 2015
  - **Won**, first-order theorems division of CASC J7
  - 2nd place, first-order theorems division of CASC 25
  - **Won**, first-order non-theorems division of CASC 25

Counterexample-Guided Instantiation in CVC4

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Conclusion

- CVC4 + quantified formulas can be used for:
  - Theorem proving and verification
  - Finite model finding (`--finite-model-find`)
  - Function synthesis (`--cegqi`, on *.sl)
  - …and more:
    - Inductive Theorem Proving (`--quant-ind`)
    - Model finding for recursive functions (`--fmf-fun`)
    - …

⇒ *All techniques work in combination with the wide array of ground theories CVC4 supports*

Thanks!

- CVC4 is publicly available at:

  http://cvc4.cs.nyu.edu/web/