Passively Mobile Communicating Machines that Use Restricted Space[☆,☆☆]

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Abstract

We propose a new theoretical model for passively mobile Wireless Sensor Networks, called PM, standing for *Passively mobile Machines*. The main modification w.r.t. the Population Protocol model [Angluin *et al.* 2006] is that agents now, instead of being automata, are Turing Machines. We provide general definitions for unbounded memories, but we are mainly interested in computations upper-bounded by plausible space limitations. However, we prove that our results hold for more general cases. We focus on *complete communication graphs* and define the complexity classes $\mathbf{PMSPACE}(f(n))$ parametrically, consisting of all predicates that are stably computable by some PM protocol that uses $\mathcal{O}(f(n))$ memory in each agent. We provide a protocol that generates unique identifiers from scratch only by using $\mathcal{O}(\log n)$ memory, and use it to provide an exact characterization of the classes $\mathbf{PMSPACE}(f(n))$ when $f(n) = \Omega(\log n)$: *they are precisely the classes of all symmetric predicates in* $\mathbf{NSPACE}(nf(n))$. In this way, we also provide a space hierarchy of the PM model when the memory bounds are $\Omega(\log n)$. We next explore the computability of the PM model when the protocols use $o(\log\log n)$ space per machine and prove that $\mathbf{SEM} = \mathbf{PMSPACE}(f(n))$ when $f(n) = o(\log\log n)$, where **SEM** denotes the class of the semilinear predicates. Finally, by showing that $\mathbf{SEM} \subsetneq \mathbf{PMSPACE}(f(n))$ when $f(n) = \mathcal{O}(\log\log n)$ the threshold behavior of the $\log\log n$ bound is revealed.

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1. Introduction - Population Protocols

Theoretical models for Wireless Sensor Networks (WSNs) have received great attention over the past few years. Recently, Angluin *et al.* [3] proposed the *Population Protocol* (*PP*) model. Their aim was to model sensor networks consisting of tiny computational devices with sensing capabilities that follow some unpredictable and uncontrollable mobility pattern. Due to the minimalistic nature of their model, the class of computable predicates was proven to be fairly small: it is the class of *semilinear predicates*, thus, not including multiplications, exponentiations, and many other important operations on input variables. Additionally, according to the work of Delporte-Gallet *et al.* [4], we only know how to transform any protocol that computes a function in the failure-free model into a protocol that can tolerate $\mathcal{O}(1)$ crash failures.[1] Moreover, Guerraoui and Ruppert [5] showed that any function computable by a population protocol tolerating one Byzantine agent is trivial. On the other hand, Angluin, Aspnes, and Eisenstat [6] described a population protocol that computes majority tolerating $\mathcal{O}(\sqrt{n})$ Byzantine failures. However, that protocol was designed for a much more restricted setting, where the scheduler chooses the next interaction randomly and uniformly (see the probabilistic population protocols briefly discussed in Subsection 1.1).

The work of Angluin *et al.* shed light and opened the way towards a brand new and very promising direction. The lack of control over the interaction pattern, as well as its inherent nondeterminism, gave rise to a variety of new theoretical models for WSNs. Those models draw most of their beauty precisely from their inability to organize interactions in a convenient and predetermined way. In fact, the Population Protocol model was the minimalistic starting-point of this area of research. Most efforts are now towards strengthening the model of Angluin *et al.* with extra realistic and implementable assumptions, in order to gain more computational power and/or speed-up the time to convergence and/or improve fault-tolerance [7, 5].

In this work, we want to allow agents to use $f(n)$ space for various f, where n is the population size, while preserving the uniformity and anonymity properties of Population Protocols. We think of each agent as being a Turing Machine[2]. In particular, we propose a new theoretical model for passively mobile sensor networks, called the *PM* model. It is a model of Passively mobile Machines (that we keep calling agents) with sensing capabilities, equipped with two-way

[1] Although the letter 'O' is usually used in the Complexity Theory literature for the Big-Oh notation, we have chosen here to use its calligraphic version '$\mathcal{O}$' in order to avoid confusion with the output function of protocols.

[2] As common in the CS literature, we abbreviate a "Turing Machine" by "TM" and by "NTM" when we want to emphasize that the TM is Nondeterministic.

communication. We initially focus on PM protocols that use $\mathcal{O}(\log n)$ memory, which is an interesting space bound, since (as we shall prove) it allows the assignment of unique identifiers[3] to the agents of the population and plays a major role on the computational capabilities of the model. In addition, we explore the computability of the PM model on different space bounds in order to get an insight of the trade-off between computational power and resource (memory) availability. Does more available memory to the agents imply increased computational power? How are the computational capabilities affected under modifications of the available memory? As we shall see, in PM protocols that use $f(n) = \Omega(\log n)$ space, agents can be organized into a distributed NTM that makes use of all the available space. In the case, where $f(n) = o(\log \log n)$ however, we show that the PM protocols are computationally equal to Population Protocols. Thus, we provide exact characterizations for the input symmetric computations performed by communicating TMs using the above space bounds.

1.1. Other Previous Work

In [3], the *Probabilistic Population Protocol* model was proposed, in which the scheduler selects randomly and uniformly the next pair to interact. Some recent work has concentrated on performance, supported by this random scheduling assumption (see e.g. [8]). [9] proposed a generic definition of probabilistic schedulers and a collection of new fair schedulers, and revealed the need for the protocols to adapt when natural modifications of the mobility pattern occur. [10, 11] considered a huge population hypothesis (population going to infinity), and studied the dynamics, stability and computational power of probabilistic population protocols by exploiting the tools of continuous nonlinear dynamics.

In addition, several extensions of the basic model have been proposed in order to more accurately reflect the requirements of practical systems. The *Mediated Population Protocol* (*MPP*) model of [7] was based on the assumption that each edge of the interaction graph can store a state. It has been recently proven [12] that in the case of complete graphs the corresponding class of stably computable predicates is the symmetric subclass of $\mathbf{NSPACE}(n^2)$, rendering the MPP model extremely powerful. Guerraoui and Ruppert [5] made another natural assumption: each agent has its own unique id and can store up to a constant number of other agents' ids. In this model, which they named *Community Protocol* model, the only permitted operation on ids is comparison. It was proven that the corresponding class consists of all symmetric predicates in $\mathbf{NSPACE}(n \log n)$. In [13], Angluin *et al.* studied what properties of restricted interaction graphs are stably computable, gave protocols for some of them, and proposed an extension of the model with *stabilizing inputs* in order to resolve the resistance of population protocols to composability. In [14], MPP's ability to decide graph properties was studied and it was proven that connectivity is undecidable. Some other works incorporated agent failures [4] and

[3] Throughout the text we abbreviate the word "identifier" by "id" and we use "uid" when we want to emphasize the fact that the identifier is "unique".

gave to some agents slightly increased computational power [15] (heterogeneous systems). Recently, Bournez *et al.* [16] investigated the possibility of studying population protocols via game-theoretic approaches. For some introductory texts to the subject of population protocols see [17, 18, 19] and for a survey on mediated population protocols see [20]. Finally, the *Static Synchronous Sensor Field* (SSSF) [21, 22] is a very promising recently proposed model that addresses networks of tiny heterogeneous computational devices and additionally allows processing over constant flows (*streams*) of data originating from the environment. The latter feature is totally absent from the models discussed so far and is required by various sensing problems. See [23] for a joint survey on population-protocol-like models and static synchronous sensor fields.

2. Our Results - Roadmap

In Section 3, we begin with a formal definition of the PM model. The section proceeds with a thorough description of the systems' functionality and then provides definitions of *configurations* and *fair executions*. In Section 4, first *stable computation* and then the class $\mathbf{PMSPACE}(f(n))$ (stably computable predicates by the PM model using $\mathcal{O}(f(n))$ space in each agent) is defined. In section 5, we give two examples of PM protocols where $\mathcal{O}(\log n)$ space is used in each agent; since those compute non-semilinear predicates, it is established that PM protocols using $\mathcal{O}(\log n)$ space are strictly stronger than population protocols. In Section 6, we show that the PM model using $\mathcal{O}(f(n))$ space can simulate a NTM (Theorem 4) of space $\mathcal{O}(nf(n))$ for any $f(n) = \Omega(\log n)$. This along with Theorem 5, where we prove that $\mathbf{PMSPACE}(f(n))$ is a subset of the symmetric subclass of $\mathbf{NSPACE}(nf(n))$, $\mathbf{SNSPACE}(nf(n))$, provide the following exact characterizations: $\mathbf{PMSPACE}(f(n)) = \mathbf{SNSPACE}(nf(n))$ for all $f(n) = \Omega(\log n)$. Based on the results of this section, we observe that the space bound $\Theta(\log n)$ acts as a threshold for the PM's computational power and we establish a space hierarchy theorem for the PM model, when the corresponding protocols use $\Omega(\log n)$ space (Theorem 9). In section 7, we examine the interesting case of the $o(\log logn)$ space bounded protocols, showing that this particular bound acts as another computability threshold. In fact we show that the class of semilinear predicates $\mathbf{SEM} = \mathbf{PMSPACE}(f(n))$ when $f(n) = o(\log\log n)$ and that $\mathbf{SEM} \subsetneq \mathbf{PMSPACE}(f(n))$ when $f(n) = \mathcal{O}(\log\log n)$. Finally, in Section 8 we discuss some future research directions.

3. The Model

In this section, we formally define the PM model and describe its functionality. In what follows, we denote by $G = (V, E)$ the (directed) interaction graph: V is the set of agents, or *population*, and E is the set of permissible ordered pairwise interactions between these agents. We provide definitions for general interaction graphs and unbounded memories, although in this work we deal with complete interaction graphs only and are mainly interested in computations that

are space-bounded by a logarithm of the population size. We generally denote by n the population size (i.e. $n \equiv |V|$).

Definition 1. *A* PM *protocol is a 6-tuple* $(X, \Gamma, Q, \delta, \gamma, q_0)$ *where* X, Γ *and* Q *are all finite sets and*

1. X *is the* input alphabet*, where* $\sqcup \notin X$*,*
2. Γ *is the* tape alphabet*, where* $\sqcup \in \Gamma$ *and* $X \subset \Gamma$*,*
3. Q *is the set of* states*,*
4. $\delta : Q \times \Gamma^4 \to Q \times \Gamma^4 \times \{L, R\}^4 \times \{0, 1\}$ *is the* internal transition function*,*
5. $\gamma : Q \times Q \to Q \times Q$ *is the* external transition function *(or* interaction transition function*), and*
6. $q_0 \in Q$ *is the* initial state*.*

Each agent is equipped with the following:

- A *sensor* in order to sense its environment and receive a piece of the input.
- Four read/write *tapes*: the *working tape*, the *output tape*, the *incoming message tape* and the *outgoing message tape*. We assume that all tapes are bounded to the left and unbounded to the right.
- A *control unit* that contains the state of the agent and applies the transition functions.
- Four *heads* (one for each tape) that read from and write to the cells of the corresponding tapes and can move one step at a time, either to the left or to the right.
- A binary *working flag* either set to 1 meaning that the agent is *working* internally or to 0 meaning that the agent is *ready* for interaction.

Initially, all agents are in state q_0, their working flag is set to 1, and all their cells contain the *blank symbol* $\sqcup$. We assume that all agents concurrently receive their sensed input (different agents may sense different data) as a response to a global start signal. The input to each agent is a symbol from X and is written on the leftmost cell of its working tape.

When its working flag is set to 1 we can think of an agent working as a usual multitape Turing Machine (but it additionally writes the working flag). In particular, while the working flag is set to 1 the internal transition function δ is applied, the control unit reads the symbols under the heads and its own state, updates all of them, moves each head one step to the left or to the right, and sets the working flag to 0 or 1, according to δ.

As it is common in the population protocol literature, an *adversary* selects ordered pairs of agents (edges from E) to interact. Assume now that two agents u and v are about to interact with u being the *initiator* of the interaction and v being the *responder*. Let $f : V \to \{0, 1\}$ be a function returning the current value of each agent's working flag. If at least one of $f(u)$ and $f(v)$ is equal to 1, then nothing happens, because at least one agent is still working internally. Otherwise

$(f(u) = f(v) = 0)$, both agents are ready and an *interaction* is established. In the latter case, the external transition function γ is applied, the states of the agents are updated accordingly, the outgoing message of the initiator is copied to the leftmost cells of the incoming message tape of the responder (replacing its contents and writting $\sqcup$ to all other previously non-blank cells) and vice versa (we call this the *message swap*), and finally the working flags of both agents are again set to 1. These operations could be handled by the protocols themselves, but then protocol descriptions would become awkward. So, we simply think of them as automatic operations performed by the hardware. These operations are also considered as atomic, that is, the interacting agents cannot take part in another interaction before the completion of these operations and, moreover, either all operations totally succeed or are all totally aborted (in which case, the states of the interacting agents are restored).

Note that the assumption that the internal transition function δ is only applied when the working flag is set to 1 is weak. In fact, an equivalent way to model this is to assume that δ is of the form $\delta : Q \times \Gamma^4 \times \{0,1\} \to Q \times \Gamma^4 \times \{L, R, S\}^4 \times \{0,1\}$, that it is always applied, and that for all $q \in Q$ and $a \in \Gamma^4$, $\delta(q, a, 0) = (q, a, S^4, 0)$ is satisfied, where S means that the corresponding head "stays put". The same holds for the assumptions that γ is not applied if at least one of the interacting agents is working internally and that the working flags are set to 1 when some established interaction comes to an end; it is equivalent to an extended γ of the form $\gamma : Q^2 \times \{0,1\}^2 \to Q^2 \times \{0,1\}^2$, that is applied in every interaction, and for which $\gamma(q_1, q_2, f_1, f_2) = (q_1, q_2, f_1, f_2)$ if $f_1 = 1$ or $f_2 = 1$, and $\gamma(q_1, q_2, f_1, f_2) = (\gamma_1(q_1, q_2), \gamma_2(q_1, q_2), 1, 1)$ if $f_1 = f_2 = 0$, hold for all $q_1, q_2 \in Q$, and we could also further extend δ and γ to handle the exchange of messages, but for sake of simplicity we have decided to leave such details out of the model.

Since each agent is a TM, we use the notion of a configuration to capture its "state". An *agent configuration* is a tuple $(q, l_w, r_w, l_o, r_o, l_{im}, r_{im}, l_{om}, r_{om}, f)$, where $q \in Q$, $l_i, r_i \in \Gamma^*$, and $f \in \{0,1\}$. q is the state of the control unit, l_w (l_o, l_{im}, l_{om}) is the string of the working (output, incoming message, outgoing message) tape to the left of the head (including the symbol scanned), r_w (r_o, r_{im}, r_{om}) is the string of the working (output, incoming message, outgoing message) tape to the right of the head (excluding the blank cells), and f is the working flag indicating whether the agent is ready to interact $(f = 0)$ or carrying out some internal computation $(f = 1)$. Let $\mathcal{B}$ be the set of all agent configurations. Given two agent configurations $A, A' \in \mathcal{B}$, we say that A *yields* A' if A' follows A by a single application of δ.

A *population configuration* is a mapping $C : V \to \mathcal{B}$, specifying the agent configuration of each agent in the population. Let C, C' be population configurations and let $u \in V$. We say that C *yields* C' via *agent transition* u, denoted $C \xrightarrow{u} C'$, if $C(u)$ yields $C'(u)$ and $C'(w) = C(w)$, $\forall w \in V - \{u\}$.

Denote by $q(A)$ the state component of an agent configuration A and similarly for the other components (e.g. $l_w(A)$, $r_{im}(A)$, $f(A)$, and so on). Let $s_{tp}(A) = l_{tp}(A) r_{tp}(A)$, that is, we obtain by concatenation the whole contents of tape $tp \in \{w, o, im, om\}$. Given a string s and $1 \leq i, j \leq |s|$ de-

note by $s[\ldots i]$ its prefix $s_1 s_2 \ldots s_i$ and by $s[j \ldots]$ its suffix $s_j s_{j+1} \ldots s_{|s|}$. If $i, j > |s|$ then $s[\ldots i] = s\sqcup^{i-|s|}$ (i.e. $i - |s|$ blank symbols appended to s) and $s[j \ldots] = \varepsilon$. For any external transition $\gamma(q_1, q_2) = (q_1', q_2')$ define $\gamma_1(q_1, q_2) = q_1'$ and $\gamma_2(q_1, q_2) = q_2'$. Given two population configurations C and C', we say that C *yields* C' via *encounter* $e = (u, v) \in E$, denoted $C \xrightarrow{e} C'$, if one of the following two cases holds:

Case 1 (only for this case, we define $C^u \equiv C(u)$ to avoid excessive number of parentheses):

- $f(C(u)) = f(C(v)) = 0$, which guarantees that both agents u and v are ready for interaction under the population configuration C.
- $C'(u) = (\gamma_1(q(C^u), q(C^v)),\ l_w(C^u),\ r_w(C^u),\ l_o(C^u),\ r_o(C^u),$
  $s_{om}(C^v)[\ldots |l_{im}(C^u)|],\ s_{om}(C^v)[|l_{im}(C^u)| + 1 \ldots],$
  $l_{om}(C^u), r_{om}(C^u), 1),$
- $C'(v) = (\gamma_2(q(C^u), q(C^v)),\ l_w(C^v),\ r_w(C^v),\ l_o(C^v),\ r_o(C^v),$
  $s_{om}(C^u)[\ldots |l_{im}(C^v)|],\ s_{om}(C^u)[|l_{im}(C^v)| + 1 \ldots],$
  $l_{om}(C^v), r_{om}(C^v), 1),$ and
- $C'(w) = C(w),\ \forall w \in V - \{u, v\}$.

Case 2:

- $f(C(u)) = 1$ or $f(C(v)) = 1$, which means that at least one agent between u and v is working internally under the population configuration C, and
- $C'(w) = C(w),\ \forall w \in V$. In this case no effective interaction takes place, thus the population configuration remains the same.

Generally, we say that C *yields* (or *can go in one step to*) C', and write $C \to C'$, if $C \xrightarrow{e} C'$ for some $e \in E$ (via encounter) or $C \xrightarrow{u} C'$ for some $u \in V$ (via agent transition), or both. We say that C' is *reachable* from C, and write $C \xrightarrow{*} C'$ if there is a sequence of population configurations $C = C_0, C_1, \ldots, C_t = C'$ such that $C_i \to C_{i+1}$ holds for all $i \in \{0, 1, \ldots, t-1\}$. An *execution* is a finite or infinite sequence of population configurations $C_0, C_1 \ldots$, so that $C_i \to C_{i+1}$. An infinite execution is *fair* if for all population configurations C, C' such that $C \to C'$, if C appears infinitely often then so does C'. This global *fairness condition* is a restriction imposed on the adversary to ensure that the protocol makes progress. A *computation* is an infinite fair execution.

The *space used by an agent running any protocol* $\mathcal{A}$ is the number of Turing tape cells used to store its configuration, that is the sum of the number of bits for its finite-state control and for the contents of its four tapes. Note that by the definition of the internal (δ) and external (γ) transition functions the space required for computing the next configuration is $\mathcal{O}(1)$ since Γ and Q are finite sets. Thus the above definition also includes the space needed by an agent to compute its updated state after an interaction. In addition, we say that a *PM*

protocol $\mathcal{A}$ uses $\mathcal{O}(f(n))$ space in each agent if the maximum space used by any agent for storing any configuration over all computations is $\mathcal{O}(f(n))$.

We assume that the input alphabet X, the tape alphabet Γ, and the set of states Q are all sets whose cardinality is fixed and independent of the population size (i.e. all of them are of cardinality $\mathcal{O}(1)$). Thus, protocol descriptions have also no dependence on the population size and the PM model *preserves uniformity*. Moreover, PM protocols are *anonymous*, since there is no room in the state of the agents for unique identifiers (though here there is plenty of room on the tapes to create such ids). Uniformity and anonymity are two outstanding properties of the basic population protocol model [3].

4. Stably Computable Predicates

The predicates that we consider here are of the following form. The input (also called an *input assignment*) to the population is any string $x = \sigma_1\sigma_2\ldots\sigma_n \in X^*$, with n being the size of the population. In particular, by assuming an ordering over V, the input to agent i is the symbol σ_i, $1 \leq i \leq n$.[4] Any mapping $p : X^* \to \{0,1\}$ is a *predicate on input assignments*.

Definition 2. *A predicate on input assignments p is called* symmetric *if for every $x \in X^*$ and any x' which is a permutation of x's symbols, it holds that $p(x) = p(x')$.*

In words, permuting the input symbols does not affect the symmetric predicate's outcome. From each predicate p a language L_p is derived that is the set of all strings that make p true or equivalently, $L_p = \{x \in X^* \mid p(x) = 1\}$. If a predicate p is symmetric, L_p is a symmetric language, that is, *for each input string $x \in L_p$ any permutation of x's symbols x' also belongs in L_p.*

A population configuration C is called *output stable* if for every configuration C' that is reachable from C it holds that $O(C') = O(C)$, where $O(C) \in \{0,1\}$ according to the output value that all agents agree to. In other words, the system does not change its overall output in any subsequent step and no matter how the computation proceeds. A predicate on input assignments p is said to be *stably computable* by a PM protocol $\mathcal{A}$ in a graph family $\mathcal{U}$ if, for any input assignment $x \in X^*$, any computation of $\mathcal{A}$, on any interaction graph from $\mathcal{U}$ of order $|x|$, contains an output stable configuration in which all agents have $p(x)$ written on their output tape. In what follows, we always assume that the graph family under consideration contains only complete interaction graphs.

We say that a predicate p over X^* belongs to $\mathbf{SPACE}(f(n))$ ($\mathbf{NSPACE}(f(n))$) if there exists some deterministic (nondeterministic, resp.) TM that decides L_p using $\mathcal{O}(f(n))$ space. Throughout this work, we use $\mathbf{SSPACE}(f(n))$ and $\mathbf{SNSPACE}(f(n))$ to denote the $\mathbf{SPACE}(f(n))$'s and $\mathbf{NSPACE}(f(n))$'s

[4] For simplicity and w.l.o.g. we have made the assumption that the input to each agent is a single symbol but our definitions can with little effort be modified to allow the agents receive whole strings as input.

restrictions to symmetric predicates, respectively. In addition, we denote by **SEM**, the class of the semilinear predicates.

Definition 3. *Let* **PMSPACE**$(f(n))$ *be the class of all predicates that are stably computable by some PM protocol that uses* $\mathcal{O}(f(n))$ *space in each agent (and in all of its tapes).*

All agents are identical and do not initially have unique ids, thus, stably computable predicates by the PM model on complete interaction graphs have to be symmetric. We prove this statement in the following lemma:

Lemma 1. *All predicates in* **PMSPACE**$(f(n))$ *are symmetric for any function* $f(n)$.

Proof. Take any space function $f(n)$ and any $p \in$ **PMSPACE**$(f(n))$ and let $\mathcal{A}$ be the PM that stably computes it. Take also any input assignment $x = \sigma_1\sigma_2 \ldots \sigma_n$ and let $\pi : V \to V$ be any permutation of $V = \{1, 2, \ldots, n\}$. Now consider the input assignment $x' = \sigma_{\pi(1)}\sigma_{\pi(2)} \ldots \sigma_{\pi(n)}$, which is a permutation of x. Take any fair, w.r.t. $\mathcal{A}$, infinite interaction sequence [5] $e_1, e_2, \ldots$, where $e_i \in E$, and replace each $e_i = (j, k)$ with $(\pi(j), \pi(k))$ to obtain a new infinite interaction sequence, which is well defined due to the fact that the interaction graph is complete. Now consider the two infinite executions of $\mathcal{A}$ that correspond to the two interaction sequences on inputs x and x', respectively. Obviously, $x'_w = x_{\pi(w)}$, so that for the initial configurations C'_0 and C_0 we have that $C'_0(w) = C_0(\pi(w))$ for all agents $w \in V$. Assume that the above holds for some interaction step i, that is, $C'_i(w) = C_i(\pi(w))$ for all $w \in V$. It is not hard to see that the same must hold for step $i + 1$, consequently both infinite executions pass in each step through the same multiset of states. This together with the fact that one execution is fair implies that the other must also be fair. So, we have obtained two computations of $\mathcal{A}$ on inputs x and x', respectively, that forever provide the same multisets of output symbols. Now, the fact that p is stably computable implies that $p(x) = p(x')$, which in turn implies that p has to be symmetric. □

5. Two Examples

5.1. Multiplication of Variables

We present now a PM protocol that stably computes the predicate $(N_c = N_a \cdot N_b)$ using $\mathcal{O}(\log n)$ space (on the complete interaction graph of n nodes) that is, all agents eventually decide whether the number of cs in the input assignment is the product of the number of as and the number of bs. We give a high-level description of the protocol.

Initially, all agents have one of a, b and c written on the first cell of their working memory (according to their sensed value). That is, the set of input

[5] By a *fair interaction sequence* we mean one that leads to a computation of $\mathcal{A}$.

symbols is $X = \Sigma = \{a, b, c\}$. Each agent that receives input a goes to state a and becomes ready for interaction (sets its working flag to 0). Agents in state a and b both do nothing when interacting with agents in state a and agents in state b. An agent in c initially creates in its working memory three binary counters, the a-counter that counts the number of as, the b-counter, and the c-counter, initializes the a and b counters to 0, the c-counter to 1, and becomes ready. When an agent in state a interacts with an agent in state c, a becomes $\bar{a}$ to indicate that the agent is now sleeping, and c does the following (in fact, we assume that c goes to a special state c_a in which it knows that it has seen an a, and that all the following are done internally, after the interaction; finally the agent restores its state to c and becomes again ready for interaction): it increases its a-counter by one (in binary), multiplies its a and b counters, which can be done in binary in logarithmic space (binary multiplication is in **LOGSPACE**), compares the result with the c-counter, copies the result of the comparison to its output tape, that is, 1 if they are equal and 0 otherwise, and finally it copies the comparison result and its three counters to the outgoing message tape and becomes ready for interaction. Similar things happen when a b meets a c (interchange the roles of a and b in the above discussion). When a c meets a c, the responder becomes $\bar{c}$ and copies to its output tape the output bit contained in the initiator's message. The initiator remains to c, adds the a-counter contained in the responder's message to its a-counter, the b and c counters of the message to its b and c counters, respectively, multiplies again the updated a and b counters, compares the result to its updated c counter, stores the comparison result to its output and outgoing message tapes, copies its counters to its outgoing message tape and becomes ready again. When a $\bar{a}$, $\bar{b}$ or $\bar{c}$ meets a c they only copy to their output tape the output bit contained in c's message and become ready again (e.g. $\bar{a}$ remains $\bar{a}$), while c does nothing.

Note that the number of cs is at most n which means that the c-counter will become at most $\lceil \log n \rceil$ bits long, and the same holds for the a and b counters, so $\mathcal{O}(\log n)$ memory is required in each tape.

Theorem 1. *The above PM protocol stably computes the predicate* $(N_c = N_a \cdot N_b)$ *using* $\mathcal{O}(\log n)$ *space.*

Proof. Given a fair execution, eventually only one agent in state c will remain, its a-counter will contain the total number of as, its b-counter the total number of bs, and its c-counter the total number of cs. By executing the multiplication of the a and b counters and comparing the result to its c-counter it will correctly determine whether $(N_c = N_a \cdot N_b)$ holds and it will store the correct result (0 or 1) to its output and outgoing message tapes. At that point all other agents will be in one of the states $\bar{a}$, $\bar{b}$, and $\bar{c}$. All these, again due to fairness, will eventually meet the unique agent in state c and copy its correct output bit (which they will find in the message they get from c) to their output tapes. Thus, eventually all agents will output the correct value of the predicate, having used $\mathcal{O}(\log n)$ memory. □

Corollary 1. SEM $\subsetneq$ **PMSPACE**$(\log n)$

Proof. PM protocols using $\mathcal{O}(\log n)$ space can simulate population protocols and $(N_c = N_a \cdot N_b) \in \mathbf{PMSPACE}(\log n)$, which is non-semilinear. □

In the following subsection, we present another PM using $\mathcal{O}(\log n)$ that computes the non-semilinear predicate $(N_1 = 2^t)$, which provides an alternative route to the previous corollary.

5.2. Power of 2

Here, we present a PM protocol that, using $\mathcal{O}(\log n)$ memory, stably computes the predicate $(N_1 = 2^t)$, where $t \in \mathbb{Z}_{\geq 0}$, on the complete interaction graph of n nodes, that is, all agents eventually decide whether the number of 1s in the input assignment is a power of 2.

The set of input symbols is $\Sigma = \{0, 1\}$. All agents that receive 1 create a binary 1-counter to their working tape and initialize it to 1. Moreover, they create a binary $next\_pow\_of2$ block and set it to 2. Finally, they write 1 (which is interpreted as "true") to their output tape, and copy the 1-counter and the output bit to their outgoing message tape before going to state 1 and becoming ready. Agents that receive input 0 write 0 (which is interpreted as "false") to their output tape, go to state 0, and become ready. Agents in state 0 do nothing when interacting with each other. When an agent in state 0 interacts with an agent in state 1, then 0 simply copies the output bit of 1. When two agents in state 1 interact, then the following happen: the initiator sets its 1-counter to the sum of the responder's 1-counter and its own 1-counter and compares its updated value to $next\_pow\_of2$. If it is less than $next\_pow\_of2$ then it writes 0 to the output tape. If it is equal to $next\_pow\_of2$ it sets $next\_pow\_of2$ to $2 \cdot next\_pow\_of2$ and sets its output bit (in the output tape) to 1. If it is greater than $next\_pow\_of2$, then it starts doubling $next\_pow\_of2$ until 1-counter $\geq next\_pow\_of2$ is satisfied. If it is satisfied by equality, then it doubles $next\_pow\_of2$ one more time and writes 1 to the output tape. Otherwise, it simply writes 0 to the output tape. Another implementation would be to additionally send $next\_pow\_of2$ blocks via messages and make the initiator set $next\_pow\_of2$ to the maximum of its own and the responder's $next\_pow\_of2$ blocks. In this case at most one doubling would be required. Finally, in both implementations, the initiator copies the output bit and the 1-counter to its outgoing message tape (in the second implementation it would also copy $next\_pow\_of2$ to the outgoing message tape), remains in state 1, and becomes ready. The responder simply goes to state $\bar{1}$ and becomes ready. An agent in state $\bar{1}$ does nothing when interacting with an agent in state 0 and vice versa. When an agent in state $\bar{1}$ interacts with an agent in state 1, then $\bar{1}$ simply copies the output bit of 1.

Note that $next\_pow\_of2$ can become at most 2 times the number of 1s in the input assignment, and the latter is at most n. Thus, it requires at most $\lceil \log 2n \rceil$ bits of memory. Either way, we can delay the multiplication until another 1 appears, in which case we need at most $\lceil \log n \rceil$ bits of memory for storing $next\_pow\_of2$ (the last unnecessary multiplication will never be done). In addition, all multiplications and comparisons described above can be obviously computed in $\mathcal{O}(\log n)$ space.

6. Space Hierarchy of the PM Model

In this section, we study the behavior of the PM model for various space bounds. Such a study is of particular interest since it is always important to know what computations is a model capable of dispatching according to the capabilities of the available hardware.

6.1. A lower bound

Here we show that for space functions $f(n) = \Omega(\log n)$, the PM model can simulate a NTM of space $\mathcal{O}(nf(n))$ using $\mathcal{O}(f(n))$ space in each agent.

The intuition behind the proof is that with at least $\log n$ memory per agent we can assign unique ids and propagate the size of the population to all agents. The assignment process is presented in Subsection 6.1.1. Since the agents do not know when the process terminates, the simulation is reinitialized in a fashion similar to the one described in [12]. The agents line up according to their ids and the simulation accesses their tapes in a modular way. The nondeterministic choices are made by exploiting the inherent nondeterminism of the interaction pattern. The full proof is presented in Subsection 6.1.2.

6.1.1. Assigning Unique IDs by Reinitiating Computation

In this subsection, we prove that PM protocols can assume the existence of unique consecutive ids and knowledge of the population size at the space cost of $\mathcal{O}(\log n)$ (Theorem 2). In particular, we present a PM protocol that correctly assigns unique consecutive ids to the agents and informs them of the correct population size using only $\mathcal{O}(\log n)$ memory, without assuming any initial knowledge of none of them. We show that this protocol can simulate any PM protocol that assumes the existence of these ids and knows the population size.

Definition 4. *Let* $\mathbf{PLM} \equiv \mathbf{PMSPACE}(\log n)$*. In words it is the class of all predicates that are stably computable by some PM protocol that uses* $\mathcal{O}(\log n)$ *space in each agent (and in all of its tapes, excluding the space used for the read-only tape).*

Definition 5. *Let* IPM *('I' standing for "Ids") be the extension of the PM model in which the agents have additionally the unique ids* $\{0, 1, \ldots, n-1\}$ *and in which each agent knows the population size (these are read-only information stored in a separate read-only tape).*

Definition 6. *Let* $\mathbf{IPMSPACE}(f(n))$ *be the class of all predicates that are stably computable by some IPM protocol that uses* $\mathcal{O}(f(n))$ *space in every agent (and in all of its tapes, excluding the space used for the read-only tape) and denote by* $\mathbf{SIPMSPACE}(f(n))$ *its symmetric subclass. Similarly to* **PLM** *define* $\mathbf{IPLM} \equiv \mathbf{IPMSPACE}(\log n)$ *and* $\mathbf{SIPLM} \equiv \mathbf{SIPMSPACE}(\log n)$.

Pick any $p \in \mathbf{SIPLM}$. Let $\mathcal{A}$ be the IPM protocol that stably computes it in $\mathcal{O}(\log n)$ space. We now present a PM protocol $\mathcal{I}$, containing protocol $\mathcal{A}$ as

a subroutine (see Protocol 1), that stably computes p, by also using $\mathcal{O}(\log n)$ space. $\mathcal{I}$ is always executed and its job is to assign unique ids to the agents, to inform them of the correct population size and to control $\mathcal{A}$'s execution (e.g. restarts its execution if needed). $\mathcal{A}$, when $\mathcal{I}$ allows its execution, simply reads the unique ids and the population size provided by $\mathcal{I}$ and executes itself normally. We first present $\mathcal{I}$ and then prove that it eventually correctly assigns unique ids and correctly informs the agents of the population size, and that when this process comes to a successful end, it restarts $\mathcal{A}$'s execution in all agents without allowing non-reinitialized agents to communicate with the reinitialized ones. Therefore, at some point, $\mathcal{A}$ will begin its execution reading the correct unique ids and the correct population size (provided by $\mathcal{I}$), thus, it will get correctly executed and will stably compute p.

Protocol 1 $\mathcal{I}$

```
1: if rid == id then // two agents with the same ids interact
2:     if initiator == 1 then // the initiator
3:         id ← id + 1, sid ← id // increases its id by one and writes it in
           the outgoing message tape
4:         ps ← id + 1, sps ← ps // sets the population size equal to its
           updated id plus 1
5:     else // the responder
6:         ps ← id + 2, sps ← ps
7:     end if
8:     // both clear their working block and copy their input symbol into it
9:     // they also clear their output tape
10:    working ← binput, output ← ∅
11: else // two agents whose ids differ interact
12:    if rps > ps then // the one who knows an outdated population size
13:        working ← binput, output ← ∅ // is reinitialized
14:        ps ← rps, sps ← ps // and updates its population size to the
           greater value
15:    else if rps == ps then // they know the same population size
16:        // so they are both already reinitialized and can proceed executing
           A
17:        execute A for 1 step
18:    end if
19: end if
```

We begin by describing $\mathcal{I}$'s variables. id is the variable storing the id of the agent (from which $\mathcal{A}$ reads the agents' ids), sid the variable storing the id that an agent writes in its outgoing message tape in order to send it, and rid the variable storing the id that an agent receives via interaction. The model's definition implies that all variables used for sending information, like sid, preserve their value in future interactions unless altered by the agent. Initially, $id = sid = 0$ for all agents. All agents have an input backup variable $binput$ which they initially

set to their input symbol and make it read-only. Thus, each agent has always available its input via *binput* even if the computation has proceeded. *working* represents the block of the working tape that $\mathcal{A}$ uses for its computation and *output* represents the contents of the output tape. *initiator* is a binary flag that after every interaction becomes true if the agent was the initiator of the interaction and false otherwise (this is easily implemented by exploiting the external transition function). *ps* is the variable storing the population size, *sps* the one used to put it in a outgoing message, and *rps* the received one. Initially, $ps = sps = 1$.

We now describe $\mathcal{I}$'s functionality. Whenever a pair of agents with the same id interact, the initiator increases its id by one and both update their population size value to the greater id plus one. Whenever two agents with different ids and population size values interact, they update their population size variables to the greater size. Thus the correct size (greatest id plus one) is propagated to all agents. Both interactions described above reinitialize the participating agents (restore their input and erase all data produced by the subroutine $\mathcal{A}$, without altering their ids and population sizes). $\mathcal{A}$, runs as a subroutine whenever two agents of different ids and same population sizes interact, using those data provided by $\mathcal{I}$.

The following lemmas provide some important properties of Protocol 1. Lemma 2 shows that $\mathcal{I}$ correctly assigns unique consecutive ids and propagates the correct population size to the agents of the population in a finite number of steps, whereas Lemma 3 guarantees the fairness of subroutine's $\mathcal{A}$ execution.

Lemma 2. *(i) No agent id will get greater than $n-1$, and no ps variable will get greater than n. (ii) $\mathcal{I}$ assigns the ids $\{0, 1, \ldots, n-1\}$ in a finite number of interactions. (iii) $\mathcal{I}$ sets the ps variable of each agent to the correct population size in a finite number of interactions.*

Proof. (i) By an easy induction, in order for an *id* to reach the value v, there have to be at least $v+1$ agents present in the population. Thus, whenever an *id* gets greater than $n-1$, there have to be more than n agents present, which creates a contradiction. Similar arguments hold for the *ps* variables
(ii) Assume on the contrary that it does not. Because of (i), at each point of the computation there will exist at least two agents, u, v such that $id_u = id_v$. Due to fairness, an interaction between such agents shall take place infinitely many times, creating an arbitrarily large *id* which contradicts (i).
(iii) The correctness of the id assignment ((i),(ii)) guarantees that after a finite number of steps two agents, u, v will set their *ps* variables to the correct population size (upon interaction in which $id_u = id_v = n-2$). It follows from (i) that no agent will have its *ps* variable greater than n. Fairness guarantees that each other agent will interact with u or v, updating its *ps* to n. □

Lemma 3. *Given that $\mathcal{I}$'s execution is fair, $\mathcal{A}$'s execution is fair as well.*

Proof. Due to the fact that the id-assignment process and the population size propagation are completed in a finite number of steps, it suffices to study fairness

of $\mathcal{A}$'s execution after their completion. The state of each agent may be thought of as containing an $\mathcal{I}$-subcomponent and an $\mathcal{A}$-subcomponent, with obvious contents. Denote by $C_{\mathcal{A}}$ the unique subconfiguration of C consisting only of the $\mathcal{A}$-subcomponents of all agents and note that some $C_{\mathcal{A}}$ may correspond to many superconfigurations C. Assume that $C_{\mathcal{A}} \to C'_{\mathcal{A}}$ and that $C_{\mathcal{A}}$ appears infinitely often (since here we consider $\mathcal{A}$'s configurations, this '$\to$' refers to a step of $\mathcal{A}$'s execution). $C_{\mathcal{A}} \to C'_{\mathcal{A}}$ implies that there exist superconfigurations C, C' of $C_{\mathcal{A}}$, $C'_{\mathcal{A}}$, respectively, such that $C \to C'$ (via some step of $\mathcal{A}$ in the case that $C_{\mathcal{A}} \neq C'_{\mathcal{A}}$). Due to $\mathcal{I}$'s fairness, if C appears infinitely often, then so does C' and so does $C'_{\mathcal{A}}$ since it is a subconfiguration of C'. Thus, it remains to show that C appears infinitely often. Since $C_{\mathcal{A}}$ appears infinitely often, then the same must hold for all of its superconfigurations. The reasoning is as follows. All those superconfigurations differ only in the $\mathcal{I}$-subcomponents, that is, they only differ in some variable checks performed by $\mathcal{I}$ (after the id-assignment process and the population size propagation have come to an end, nothing else is performed by $\mathcal{I}$). But all of them are reachable from and can reach a common superconfiguration of $C_{\mathcal{A}}$ in which no variable checking is performed by $\mathcal{I}$, thus, they only depend on which pair of agents is selected for interaction and they are all reachable from one another. Since at least one of them appears infinitely often then, due to the fairness of $\mathcal{I}$'s execution, all of them must also appear infinitely often and this completes the proof. □

By combining the above lemmata we can prove the following:

Theorem 2. **PLM** = **SIPLM**.

Proof. **PLM** $\subseteq$ **SIPLM** holds trivially, so it suffices to show that **SIPLM** $\subseteq$ **PLM**. We have presented a **PLM** protocol (protocol 1) that assigns the agents unique consecutive ids after a finite number of interactions and informs them of the population size (Lemma 2). It follows directly from the protocol that after that point, further fair execution of $\mathcal{I}$ will result in execution of protocol $\mathcal{A}$ which can take into account the existence of unique ids. Moreover, execution of $\mathcal{A}$ is guaranteed to be fair (Lemma 3). □

6.1.2. **SNSPACE**$(nf(n)) \subseteq$ **PMSPACE**$(f(n))$ *for any* $f(n) = \Omega(\log n)$

We now show that for space functions $f(n) = \Omega(\log n)$, the PM model can simulate a deterministic TM of space $\mathcal{O}(nf(n))$ using $\mathcal{O}(f(n))$ in each agent. This is formally stated by the following theorem:

Theorem 3. **SSPACE**$(nf(n)) \subseteq$ **PMSPACE**$(f(n))$ *for any* $f(n) = \Omega(\log n)$.

Proof. Let $p : X^* \to \{0, 1\}$ be any predicate in **SSPACE**$(nf(n))$ and $\mathcal{M}$ be the deterministic TM that decides p by using $\mathcal{O}(nf(n))$ space. We can construct a PM protocol $\mathcal{A}$ that uses $f(n) = \Omega(\log n)$ space on each agent and that stably computes p by exploiting its knowledge of unique ids and the population size. Such knowledge can be obtained by the protocol $\mathcal{I}$ of Theorem 2 (see Subsection 6.1.1). Note that *protocol* $\mathcal{I}$ *can be executed by any PM protocol whose agents use* $\Omega(\log n)$ *space*. Let x be any input assignment in X^*. Each agent receives

its input symbol according to x (e.g. u receives symbol $x(u)$). We assume for the sake of simplicity that the agents are equipped with an extra tape, the *simulation tape* that is used during the simulation. The agent that has obtained the unique id 0 starts simulating $\mathcal{M}$.

In the general case, assume that currently the simulation is carried out by an agent u having the id i_u. Agent u uses its simulation tape to write symbols according to the transition function of $\mathcal{M}$. Any time the head of $\mathcal{M}$ moves to the right, u moves the head of the simulation tape to the right, pauses the simulation, writes the current state of $\mathcal{M}$ to its outgoing message tape, and passes the simulation to the agent v having id $i_v = (i_u + 1) \mod n$. Any time the head of $\mathcal{M}$ moves to the left, u pauses the simulation, writes the current state of $\mathcal{M}$ to its outgoing message tape, and passes the simulation to the agent v having id $i_v = (i_u - 1) \mod n$. From agent v's perspective, in the first case it just receives the state of $\mathcal{M}$, copies it to its working tape and starts the simulation, while in the second case it additionally moves the head of the simulation tape one cell to the left before it starts the simulation.

It remains to cover the boundary case in which the head of the simulation tape is over the special symbol that indicates the beginning of the tape. In that case, the agent moves the head to the right and continues the simulation himself (notice that this can only happen to the agent that begins the simulation, that is, the one having the id 0).

Whenever, during the simulation, $\mathcal{M}$ accepts, then $\mathcal{A}$ also accepts; that is, the agent that detects $\mathcal{M}$'s acceptance, writes 1 to its output tape and informs all agents to accept. If $\mathcal{M}$ rejects, it also rejects. Finally, note that $\mathcal{A}$ simulates $\mathcal{M}$ not necessarily on input $x = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ but on some x' which is a permutation of x. The reason is that agent with id i does not necessarily obtain σ_i as its input. The crucial remark that completes the proof is that $\mathcal{M}$ accepts x if and only if it accepts x', because p is symmetric.

Because of the above process, it is easy to verify that the $k-$th cell of the simulation tape of any agent u having the id i_u corresponds to the $(n(k-1) + i_u + 1)-$th cell of $\mathcal{M}$. Thus, whenever $\mathcal{M}$ alters $l = \mathcal{O}(nf(n))$ tape cells, any agent u will alter $l' = \frac{l - i_u - 1}{n} + 1 = \mathcal{O}(f(n))$ cells of its simulation tape. □

The next theorem shows how the above approach can be generalized to include NTMs.

Theorem 4. *For any* $f(n) = \Omega(\log n)$ *it holds that* $\mathbf{SNSPACE}(nf(n)) \subseteq \mathbf{PMSPACE}(f(n))$.

Proof. We have already shown that the PM model can simulate a deterministic TM $\mathcal{M}$ of $\mathcal{O}(nf(n))$ space, where $f(n) = \Omega(\log n)$, by using $\mathcal{O}(f(n))$ space (Theorem 3). We now present some modifications that will allow us to simulate a NTM $\mathcal{N}$ of the same memory size. Keep in mind that $\mathcal{N}$ is a decider for some predicate in $\mathbf{SNSPACE}(nf(n))$, thus, it always halts. Upon initialization, each agent enters a reject state (writes 0 to its output tape) and the simulation is carried out as in the case of $\mathcal{M}$.

Whenever a nondeterministic choice has to be made, the corresponding agent gets ready and waits for participating in an interaction. The id of the other participant will provide the nondeterministic choice to be made. One possible implementation of this idea is the following. Since there is a fixed upper bound on the number of nondeterministic choices (independent of the population size), the agents can store them in their memories. Any time a nondeterministic choice has to be made between k candidates the agent assigns the numbers $0, 1, \ldots, k-1$ to those candidates and becomes ready for interaction. Assume that the next interaction is with an agent whose id is i. Then the nondeterministic choice selected by the agent is the one that has been assigned the number $i \bmod k$. Fairness guarantees that, in this manner, all possible paths in the tree representing $\mathcal{N}$'s nondeterministic computation will eventually be followed.

Any time the simulation reaches an accept state, all agents change their output to 1 and the simulation halts. Moreover, any time the simulation reaches a reject state, it is being reinitiated. The correctness of the above procedure is captured by the following two cases.

1. *If $\mathcal{N}$ rejects then every agent's output stabilizes to* 0. Upon initialization, each agent's output is 0 and can only change if $\mathcal{N}$ reaches an accept state. But all branches of $\mathcal{N}$'s computation reject, thus, no accept state is ever reached, and every agent's output forever remains to 0.
2. *If $\mathcal{N}$ accepts then every agent's output stabilizes to* 1. Since $\mathcal{N}$ accepts, there is a sequence of configurations S, starting from the initial configuration C that leads to a configuration C' in which each agent's output is set to 1 (by simulating directly the branch of $\mathcal{N}$ that accepts). Notice that when an agent sets its output to 1 it never alters its output tape again, so it suffices to show that the simulation will eventually reach C'. Assume on the contrary that it does not. Since $\mathcal{N}$ always halts the simulation will be at the initial configuration C infinitely many times. Due to fairness, by an easy induction on the configurations of S, C' will also appear infinitely many times, which leads to a contradiction. Thus the simulation will eventually reach C' and the output will stabilize to 1.

□

6.1.3. Simulating Nondeterministic Recognizers

Here, we generalize the preceding ideas to nondeterministic *recognizers* of $\mathcal{O}(nf(n))$ space, where $f(n) = \Omega(\log n)$. There is a way to stably compute predicates in $\mathbf{SNSPACE}(nf(n))$ even when the corresponding TM $\mathcal{N}$ might loop, by carrying out an approach similar to the one given above. However, since neither an accept nor a reject state may be reached, the simulation is nondeterministically reinitiated at any point that is not in such a state. This choice is also obtained by the nondeterministic interactions. For example, whenever the agent that carries out the simulation interacts with an agent that has an id that is even, the simulation remains unchanged, otherwise it is reinitiated. Notice however that during the simulation, any agent having id i may need to

interact with those having neighboring ids, so those must not be able to cause a reinitiation in the simulation.

Correctness of the above procedure is captured by similar arguments to those in the proof of Theorem 4. If $\mathcal{N}$ never accepts, then no output tape will ever contain a 1, so the simulation stabilizes to 0. If $\mathcal{N}$ accepts there is a sequence of configurations S, starting from the initial configuration C that leads to a configuration C' in which each agent's output is set to 1. Observe that this is a "good" sequence, meaning that no reinitiations take place, and, due to fairness, it will eventually occur.

6.2. An upper bound

We first prove that $\mathbf{PMSPACE}(f(n)) \subseteq \mathbf{SNSPACE}(nf(n))$.

Theorem 5. *For any function $f(n)$ it holds that* $\mathbf{PMSPACE}(f(n)) \subseteq \mathbf{SNSPACE}(nf(n))$.

Proof. We will now present a NTM $\mathcal{M}$ of space $\mathcal{O}(nf(n))$ that can decide a language L_p corresponding to any predicate $p \in \mathbf{PMSPACE}(f(n))$. To accept the input (assignment) x, $\mathcal{M}_\mathcal{A}$ must verify two conditions: That there exists a configuration C reachable from the initial configuration corresponding to x in which the output tape of each agent indicates that p holds, and that there is no configuration C' reachable from C under which p is violated for some agent.

The first condition is verified by guessing and checking a sequence of configurations. Starting from the initial configuration, each time $\mathcal{M}_\mathcal{A}$ guesses configuration C_{i+1} and verifies that C_i yields C_{i+1}. This can be caused either by an agent transition u, or an encounter (u, v). In the first case, the verification can be carried out as follows: $\mathcal{M}_\mathcal{A}$ guesses an agent u so that C_i and C_{i+1} differ in the configuration of u, and that $C_i(u)$ yields $C_{i+1}(u)$. It then verifies that C_i and C_{i+1} differ in no other agent configurations. Similarly, in the second case $\mathcal{M}_\mathcal{A}$ nondeterministically chooses agents u, v and verifies that encounter (u, v) leads to C' by ensuring that: (a) both agents have their working flags cleared in C, (b) the tape exchange takes place in C', (c) both agents update their states according to γ and set their working flags to 1 in C' and (d) that C_i and C_{i+1} differ in no other agent configurations. In each case, the space needed is $\mathcal{O}(nf(n))$ for storing C_i, C_{i+1}, plus $\mathcal{O}(f(n))$ extra capacity for ensuring the validity of each agent configuration in C_{i+1}.

If the above hold, $\mathcal{M}_\mathcal{A}$ replaces C_i with C_{i+1} and repeats this step. Otherwise, $\mathcal{M}_\mathcal{A}$ drops C_{i+1}. Any time a configuration C is reached in which p holds, $\mathcal{M}_\mathcal{A}$ computes the complement of a similar reachability problem: it verifies that there exists no configuration reachable from C in which p is violated. Since **NSPACE** is closed under complement for all space functions $g(n) = \Omega(\log n)$ (see Immerman-Szelepcsényi theorem), this condition can also be verified in $\mathcal{O}(n \log n)$ space. Note that for any reasonable function $f(n)$, $g(n) = nf(n) \geq \log n$, as required by the Immerman-Szelepcsényi theorem. Thus, L_p can be decided in $\mathcal{O}(nf(n))$ space by some NTM, so $L_p \in \mathbf{SNSPACE}(nf(n))$. □

Using a different representation of population configurations we can prove that $\mathbf{PMSPACE}(f(n)) \subseteq \mathbf{SNSPACE}(2^{f(n)}(f(n) + \log n))$.

Theorem 6. *For any function $f : \mathbb{N} \to \mathbb{N}$, any predicate in* $\mathbf{PMSPACE}(f(n))$ *is also in* $\mathbf{SNSPACE}(2^{f(n)}(f(n) + \log n))$.

Proof. Take any $p \in \mathbf{PMSPACE}(f(n))$. Let $\mathcal{A}$ be the PM protocol that stably computes predicate p in space $\mathcal{O}(f(n))$. $L_p = \{(\sigma_1\sigma_2 \dots \sigma_n) \mid \sigma_i \in X \text{ for all } i \in \{1, \dots, n\} \text{ and } p(\sigma_1\sigma_2 \dots \sigma_n) = 1\}$ is the language corresponding to p ($X \subset \Sigma^*$ is the set of input strings). We describe a NTM $\mathcal{N}$ that decides L_p in $g(n) = \mathcal{O}(2^{f(n)}(f(n) + \log n))$ space.

Note that each agent uses memory of size $\mathcal{O}(f(n))$. So, by assuming a binary tape alphabet $\Gamma = \{0, 1\}$ (the alphabet of the agents' tapes), an assumption which is w.l.o.g., there are $2^{\mathcal{O}(f(n))}$ different agent configurations (internal configurations) each of size $\mathcal{O}(f(n))$. $\mathcal{N}$ stores a population configuration by storing all these agent configurations, consuming for this purpose $\mathcal{O}(f(n)2^{f(n)})$ space, together with a number per agent configuration representing the number of agents in that agent configuration under the current population configuration. These numbers sum up to n and each one of them requires $\mathcal{O}(\log n)$ bits, thus, $\mathcal{O}(2^{f(n)} \log n)$ extra space is needed, giving a total of $\mathcal{O}(2^{f(n)}(f(n) + \log n))$ space needed to store a population configuration. The reason that such a representation of population configurations suffices is that when k agents are in the same internal configuration there is no reason to store it k times. The completeness of the interaction graph allows us to store it once and simply indicate the number of agents that are in this common internal configuration, that is, k.

Now $\mathcal{N}$ does the same as the NTM of Theorem 5 does. The main difference is that it now store the population configurations according to the new representation we discussed above. □

The upper bounds shown in Theorem 5 are obviously better for functions $f(n) = \Omega(\log n)$ than those presented by Theorem 6. Note however, that Theorem 5 also holds for $f(n) = o(\log n)$ and for those space functions the upper bounds are worse than those of Theorem 6. In order to realize this, consider the function $f(n) = c$ (the memory of each agent is independent of the population size, thus this is the basic PP model). According to Theorem 5 the upper bound is the trivial $\mathbf{SNSPACE}(n)$, whereas the Theorem 6 decreases the upper bound to $\mathbf{SNSPACE}(\log n)$. This behavior is expected due to the configuration representation of the population used by those theorems. When the configuration is stored as n-vector where each element of the vector holds the internal configuration of an agent (representation used in Theorem 5) then as the memory size grows the additional space needed is a factor n of that growth. On the other hand, when a configuration is represented as a vector of size equal to the number of all possible internal configurations where each element is the number of agents that are in the corresponding internal configuration (as in Theorem 6) then the size of the vector grows exponentially to the memory growth. Therefore tighter upper bounds are obtained by Theorem 5 for functions $f(n) = \Omega(\log n)$

and by Theorem 6 for $f(n) = o(\log n)$. Note that for $f(n) = \log n$ the bounds by both theorems are the same.

Using the following theorem we can prove that $\mathbf{PMSPACE}(f(n)) \subsetneq \mathbf{SNSPACE}(nf(n))$ for any $f(n) = o(\log n)$.

Theorem 7 (Symmetric Space Hierarchy Theorem)**.** *For any function $f : \mathbb{N} \to \mathbb{N}$, a symmetric language L exists that is decidable in $\mathcal{O}(f(n))$ (non)deterministic space but not in $o(f(n))$ (non)deterministic space.*

Proof. Follows immediately from the unary (tally) separation language presented in [24] and the fact that any unary language is symmetric. □

Corollary 2. *For any function $f(n) = o(\log n)$ it holds that* $\mathbf{PMSPACE}(f(n)) \subsetneq \mathbf{SNSPACE}(nf(n))$.

Proof. By considering Theorem 6 and the symmetric space hierarchy theorem (Theorem 7) it suffices to show that $2^{f(n)}(f(n) + \log n) = o(nf(n))$ for $f(n) = o(\log n)$. We have that

$$2^{f(n)}(f(n) + \log n) = 2^{o(\log n)}\mathcal{O}(\log n) = o(n)\mathcal{O}(\log n),$$

which obviously grows slower than $nf(n) = n \cdot o(\log n)$. □

So, for example, if $f(n) = \log \log n$, then $\mathbf{PMSPACE}(\log \log n) \subseteq \mathbf{SNSPACE}(\log^2 n)$ which is, by the symmetric space hierarchy theorem, strictly smaller than $\mathbf{SNSPACE}(nf(n)) = \mathbf{SNSPACE}(n \log \log n)$. Another interesting example is obtained by setting $f(n) = c$. In this case, we obtain the $\mathbf{SNSPACE}(\log n) = \mathbf{SNL}$ (for *Symmetric* **NL**) upper bound for population protocols of Angluin *et al.* [3] (in [25] they obtained a better bound which is the semilinear predicates and constitutes an exact characterization of population protocols).

6.3. $\mathbf{PMSPACE}(f(n)) = \mathbf{SNSPACE}(nf(n))$ *for any* $f(n) = \Omega(\log n)$

In this section we use the previously shown lower and upper bounds to provide exact characterizations for $\mathbf{PMSPACE}(f(n))$, when $f(n) = \Omega(\log n)$ and to formally state the *Space Hierarchy Theorem of the PM model*.

From Theorems 5 and 4 we get exact characterizations for all $\mathbf{PMSPACE}(f(n))$, when $f(n) = \Omega(\log n)$. It is formally stated as:

Theorem 8. *For any $f(n) = \Omega(\log n)$ it holds that* $\mathbf{PMSPACE}(f(n)) = \mathbf{SNSPACE}(nf(n))$.

Thus, by considering together Corollary 2 and Theorem 8 we obtain the following corollary.

Corollary 3. *$f(n) = \Theta(\log n)$ acts as a threshold for the computational power of the PM model.*

Theorem 9 (Space Hierarchy Theorem of the PM model)**.** *For any two functions $f, g : \mathbb{N} \to \mathbb{N}$, where $f(n) = \Omega(\log n)$ and $g(n) = o(f(n))$ there is a predicate p in* $\mathbf{PMSPACE}(f(n))$ *but not in* $\mathbf{PMSPACE}(g(n))$.

Proof. From Theorem 7 we have that for any such functions f, g there is a language $L \in \mathbf{SNSPACE}(nf(n))$ so that $L \notin \mathbf{SNSPACE}(ng(n))$. From Theorem 8 we have that $\mathbf{PMSPACE}(f(n)) = \mathbf{SNSPACE}(nf(n))$, therefore $p_L \in \mathbf{PMSPACE}(f(n))$ (where p_L is the symmetric predicate that corresponds to the symmetric language L). We distinguish two cases. If $g(n) = \Omega(\log n)$ then from Theorem 8 we have that $\mathbf{PMSPACE}(g(n)) = \mathbf{SNSPACE}(ng(n))$ and so $L \notin \mathbf{PMSPACE}(g(n))$ or equivalently $p_L \notin \mathbf{PMSPACE}(g(n))$. If $g(n) = o(\log n)$ then from Corollary 2 we have that $\mathbf{PMSPACE}(g(n)) \subsetneq \mathbf{SNSPACE}(ng(n)) \subsetneq \mathbf{SNSPACE}(nf(n)) = \mathbf{PMSPACE}(f(n))$. □

In simple words, Theorem 9 says that for the space bounds discussed in this section, protocols using more memory can compute more things.

7. A threshold in the Computability of the PM Model

In this section we explore the computability of the PM model when the protocols use $o(\log\log n)$ space. We show that $\log\log n$ acts as a threshold under which the PM protocols become computationally equal to Population Protocols.

We will prove that $\mathbf{SEM} = \mathbf{PMSPACE}(f(n))$ when $f(n) = o(\log\log n)$. Moreover, we will prove that this bound is a threshold, so that $\mathbf{SEM} \subsetneq \mathbf{PMSPACE}(f(n))$ when $f(n) = \mathcal{O}(\log\log n)$.

7.1. $o(\log\log n)$ Threshold

Here, we prove an interesting limitation in the computability of the PM model when the memory bounds are too restrictive.

Theorem 10. *Any PM protocol $\mathcal{A}$ that uses $f(n) = o(\log\log n)$ space can only compute semilinear predicates.*

Proof Idea. For each execution and each agent configuration C_u of an agent u in that execution we define a binary tree of "ancestor agent configurations" which has C_u as its root and each node is either a leaf, corresponding to an initial agent configuration, or a parent node C_v which has as children the agent configurations whose interaction produced C_v.

Define the *depth* $d(C)$ of an agent configuration C as the height of the shortest tree with C at its root.

If we only consider trees that have *minimum size* (w.r.t. the number of nodes) among all trees of the same depth, then no agent configuration appears twice on the same path, because else we could replace the subtree rooted at the later occurrence with the smaller subtree rooted at the earlier one. But then any such tree yields an execution that has the following property: *(Property P) "It has at least $d(C)$ distinct agent configurations and must have started from a population of at most $2^{(d(C)-1)}$ initial agents."*

Now consider the complementary cases:

(a) For any C, the depth $d(C)$ (of *distinct* configurations) is bounded by a constant. But there is only a constant number of binary trees of a depth bounded by a constant. So the protocol has only finitely reachable agent configurations. Then this passively mobile protocol can be converted to a *finite-states* protocol (i.e., a standard population protocol).

(b) $d(C)$ grows with the population size, n, in some executions. The longest path in the tree of C may (in an execution) have all nodes representing (agent) configurations of the same agent, u. This means that some other agents produced agent configurations that caused the existence of the $d(C)$ *distinct* configurations of this agent u. At the worst, in some execution, all n agents may have been involved in the tree of C, i.e., the tree has n nodes (by fairness, all agents will meet with u). But then u has to have local memory enough to keep *at least* $\log n$ *distinct configurations* by Property P, i.e., $\Omega(\log \log n)$ local memory (in any reasonable, at least binary, tape alphabet).

In the following, we present a rigorous proof of Theorem 10.

Definition 7. *Let $\mathcal{A}$ be a PM protocol executed in a population V of size n. Define* agent configuration graph*, $R_{\mathcal{A},V} = \{U, W, F\}$, a graph such that:*

- *U is the set of the agent configurations that can occur in any execution of $\mathcal{A}$ such that the working flag is set to 0.*
- *W is the set of edges (u, v), $u, v \in U$ so that there exists an edge (u, v) when there exists an agent configuration w so that an interaction between two agents with configurations u, w will lead the first one to configuration v.*
- *$F : W \to \{u_1, u2, \dots\}, u_i \in U \times \{i, r\}$ is an edge labeling function so that when an agent k being in configuration u enters configuration v via a single interaction with an agent being in configuration w, and k acts as $x \in \{i, r\}$ (initiator-responder) in the interaction, then $\{w, x\} \in F((u, v))$, while $F((u, v)) = \emptyset$ in any other case. Notice that F is a function.*

In other words, U contains the configurations that an agent may enter in any possible execution, when we do not take into consideration the ones that correspond to internal computation, while W defines the transitions between those configurations through interactions defined by F. Note that the model's description excludes infinite sequences of blank cells from the agent configurations. Also, notice that in general, $R_{A,V}$ depends not only on the protocol $\mathcal{A}$, but also on the population V. We call a $u \in U$ *initial node* iff it corresponds to an initial agent configuration.

Definition 8. *Two agent transition graphs $R_{\mathcal{A},V} = \{U, W, F\}$ and $R_{\mathcal{A}',V'} = \{U', W', F'\}$ are called* equal*, denoted by $R_{\mathcal{A},V} = R_{\mathcal{A}',V'}$, iff $U = U'$, $W = W'$ and $F(e) = F'(e), \forall e \in W$.*

Because of the uniformity property, we can deduce the following theorem:

Theorem 11. *Let $R_{\mathcal{A},V}$, $R_{\mathcal{A},V'}$ be two agent configuration graphs corresponding to a protocol $\mathcal{A}$ for any two different populations V, V' of size n and n' respectively, where $n < n'$. Then, there exists a subgraph R^* of $R_{\mathcal{A},V'}$ such that $R^* = R_{\mathcal{A},V}$, and whose initial nodes contains all the initial nodes of $R_{\mathcal{A},V'}$.*

Proof. Indeed, let V_1', V_2' be a partitioning of V' such that $V_1' = V$, and observe the agent configuration graph that is yielded by the execution of $\mathcal{A}$ in V_1'. Since both populations execute the same protocol A the transitions are the same, thus all edges in $R_{\mathcal{A},V}$ will be present in $R_{\mathcal{A},V_1'}$ between the common pairs of nodes and their F labels will be equal as well since $V_1' = V$. Therefore $R_{\mathcal{A},V} = R_{\mathcal{A},V_1'}$. Moreover, since the initial nodes are the same for both populations, they must be in $R_{\mathcal{A},V_1'}$. Finally, $R_{\mathcal{A},V_1'}$ is a subgraph of $R_{\mathcal{A},V'}$, as $V_1' \subset V'$, and the proof is complete. □

The above theorem states that while we explore populations of greater size, the corresponding agent configuration graphs are only enhanced with new nodes and edges, while the old ones are preserved.

Given an agent configuration graph, we associate each node a with a value $r(a)$ inductively, as follows:

Base Case For any initial node a, $r(a) = r_{init} = 1$.

Inductive Step For any other node a, $r(a) = min(r(b) + r(c))$ such that a is reachable from b through an edge that contains c in its label, and b, c have already been assigned an r value.

Lemma 4. *Let $R_{\mathcal{A},V} = \{U, W, F\}$ be an agent configuration graph. Every node in $R_{\mathcal{A},V}$ get associated with an r value.*

Proof. Assume for the shake of the contradiction that there is a maximum, non empty set of nodes $U' \subset U$ such that $\forall v \in U'$, v does not get associated with an r value. Then $B = U - U'$, and $C = (B, U')$ defines a cut, with all the initial nodes being in B. We examine any edge (u, v) with label L that crosses the cut, having an arbitrary $(w, x) \in L$. Obviously $u \in B$ and $v \in U'$, and u is associated with a value $r(u)$. Since v is not associated with any r value, the same must hold for node w (otherwise $r(v) = r(u) + r(w)$). We now examine the first agent c that enters a configuration corresponding to some $v \in U'$. Because of the above observation, this could only happen through an interaction with an agent being in a configuration that is also in U' which creates the contradiction. □

Note that for any given protocol and population size, the r values are *unique* since the agent configuration graph is unique. The following lemma captures a bound in the r values when the corresponding protocol uses $f(n) = o(\log \log n)$ space.

Lemma 5. *Let r_{max-i} be the $i-th$ greatest r value associated with any node in an agent configuration graph. For any protocol $\mathcal{A}$ that uses $f(n) = o(\log \log n)$, there exists a n_0 such that for any population of size $n > n_0$, $r_{max} < \frac{n}{2}$.*

Proof. Since $f(n) = o(\log\log n)$, $\lim_{n\to\infty} \frac{f(n)}{\log\log n} = 0$, so $\lim_{n\to\infty} \frac{\log\log n}{f(n)} = \infty$ and $\lim_{n\to\infty} \frac{\log n}{2^{f(n)}} = \infty$. It follows from the last equation that for any positive integer M, there exists a fixed n_0 such that $\frac{\log n}{2^{f(n)}} > M$ for any $n > n_0$.

Fix any such n and let $k = |U| \leq 2^{f(n)}$ in the corresponding agent configuration graph. Since any node is associated with an r value, there can be at most k different such values. Now observe that $r_{max} \leq 2 \cdot r_{max-1} \leq \cdots \leq 2^k \cdot r_{init} \leq 2^{2^{f(n)}} < 2^{\frac{\log n}{M}} \leq \sqrt[M]{n} \leq \frac{n}{2}$ for $n > max(n_0, 2)$ and $M \geq 2$ □

Note that these r-values are a part of our theoretical analysis and are not stored on the population (there is not enough space on the agents to store them). In the following analysis, we consider for any practical purposes that $M = 2$.

Given a protocol $\mathcal{A}$ and a population V we define the following property:

Definition 9. Property Q(a)*: Given a node a in the agent configuration graph $R_{\mathcal{A},V}$, there exists a subpopulation of V of size $r(a)$ and a fair execution of the corresponding protocol $\mathcal{A}$ that will lead an agent to the configuration a.*

The following lemma proves that the r values correspond to the above reachability property.

Lemma 6. *$Q(a)$ holds for any node of $R_{\mathcal{A},V}$.*

Proof. We prove the above lemma by generalized induction in the r values.

Base Case $Q(a)$ obviously holds for any initial node u, since $r_{init} = 1$.

Inductive Step We examine any non initial node u that has been associated with a value $r(u) = r(a) + r(b)$, for some a, b. The inductive hypothesis guarantees that $Q(a)$ and $Q(b)$ hold. Then, a population of size $r(a)+r(b)$ can lead two agents to configurations a and b independently. Then an interaction between those agents will take one of them to configuration u, so $Q(u)$ holds too.

□

Lemmata 5 and 6 lead to the following:

Lemma 7. *For any protocol $\mathcal{A}$ that uses $f(n) = o(\log\log n)$ there exists a fixed n_0 such that for any population of size $n > n_0$ and any pair of agent configurations u, v, the interaction (u, v) can occur.*

Proof. Indeed, because of the Lemma 5, there exists a n_0 such that for any $n > n_0$, $r(a) < \frac{n}{2}$ for any a. With that in mind, Lemma 6 guarantees that in any such population, any interaction (u, v) can occur since any of the agent configurations u, v can occur independently, by partitioning the population in two subpopulations of size $\frac{n}{2}$ each. □

We can now complete our proof of Theorem 10:

Proof. Because of the uniformity constraint, A can be executed in any population of arbitrary size. We choose a fixed n_0 as defined in Lemma 5 and examine the population L of size $n = n_0$. Let $R_{\mathcal{A},L}$ be the corresponding agent configuration graph. Let L' be any population of size $n' > n$ and $R_{\mathcal{A},L'}$ the corresponding agent configuration graph. Because of the theorem 11, $R_{\mathcal{A},L'}$ contains a subgraph K, such that $K = R_{\mathcal{A},L}$, and the initial nodes of $R_{\mathcal{A},L'}$ are in K. Let $U^* = U' - U$, and k the first agent configuration that appears in L' such that $k \in U^*$ through an interaction (u, v)(k can't be an initial configuration, thus it occurs through some interaction). Then $u, v \in U$, and the interaction (u, v) can occur in the population L too (Lemma 7), so that $k \in U$, which refutes our choice of k creating a contradiction. So, $U^* = \emptyset$, and the set of agent configurations does not change as we examine populations of greater size. Since the set of agent configurations remains described by the *fixed* $R_{\mathcal{A},L}$, the corresponding predicate can be computed by the Population Protocol model, thus it is semilinear. □

Theorem 10 guarantees that for any protocol that uses only $f(n) = o(\log\log n)$ space in each agent, there exists a population of size n_0 in which it stops using extra space. Since n_0 is fixed, we can construct a protocol based on the agent configuration graph which uses constant space[6], and thus can be executed in the Population Protocol Model.

So far, we have established that $\mathbf{PMSPACE}(f(n)) \subseteq \mathbf{SEM}$ when $f(n) = o(\log\log n)$. Since the inverse direction holds trivially, we can conclude that $\mathbf{PMSPACE}(f(n)) = \mathbf{SEM}$.

Theorem 10 practically states that when the memories available to the protocols are strictly smaller than $\log\log n$ (asymptotically) then these PM protocols are nothing more than population protocols, and although their memory is still dependent on the population size, they cannot exploit it as such; instead they have to use it as a constant memory much like population protocols do.

7.2. The Logarithmic Predicate

We will now present the non-semilinear logarithmic predicate, and devise a PM protocol that computes it using $\mathcal{O}(\log\log n)$ space in each agent.

We define the logarithmic predicate as follows: During the initialization, each agent receives an input symbol from $X = \{a, 0\}$, and let N_a denote the number of agents that have received the symbol a. We want to compute whether $\log N_a = t$ for some arbitrary t. We give a high level protocol that computes this predicate, and prove that it can be correctly executed using $\mathcal{O}(\log\log n)$ space.

Each agent u maintains a variable x_u, and let out_u be the variable that uses to write its output. Initially, any agent u that receives a as his input symbol sets $x_u = 1$ and $out_u = 1$, while any other agent v sets $x_v = 0$ and $out_v = 1$.

[6]Notice that this fixed agent configuration graph can be viewed as a deterministic finite automaton.

Our main protocol consists of two subprotocols, $\mathcal{A}$ and $\mathcal{B}$, that are executed concurrently. Protocol $\mathcal{A}$ does the following: whenever an interaction occurs between two agents, u, v, with u being the initiator, if $x_u = x_v > 0$, then $x_u = x_u + 1$ and $x_v = 0$. Otherwise, nothing happens. Protocol $\mathcal{B}$ runs in parallel, and computes the semilinear predicate of determining whether there exist 0, two or more agents having $x > 0$. If so, it outputs 0, otherwise it outputs 1. Observe that $\mathcal{B}$ is executed on *stabilizing inputs*, as the $x-$variables fluctuate before they stabilize to their final value. However, it is well known that the semilinear predicates are also computable under this constraint [25].

Lemma 8. *The main protocol uses $\mathcal{O}(\log\log n)$ space.*

Proof. As protocol $\mathcal{B}$ computes a semilinear predicate, it only uses $\mathcal{O}(c)$ space, with c being a constant. To examine the space bounds of $\mathcal{A}$, pick any agent u. We examine the greatest value that can be assigned to the variable x_u. Observe that in order for x_u to reach value k, there have to be at least 2 preexisting $x-$ variables with values $k-1$. Through an easy induction, it follows that there have to be at least 2^k pre-existing variables with the value 1. Since $2^k \leq N_a$, $k \leq \log N_a \leq \log n$, so x_u is upper bounded by $\log n$, thus consuming $\mathcal{O}(\log\log n)$ space. □

Lemma 9. *For every agent u, eventually $out_u = 1$ if $\log N_a = t$ for some arbitrary t, and $out_u = 0$ otherwise.*

Proof. Indeed, the execution of protocol $\mathcal{B}$ guarantees that all agents will set $out = 1$ iff eventually there exists only one agent u that has a non-zero value assigned in x_u. Assume that $x_u = k$ for some k. Then, because of the analysis of lemma 8 during the initialization of the population will exist 2^k $x-$variables set to 1. Since each of those variables corresponds to one a assignment, $N_a = 2^k \Rightarrow \log N_a = k$. On the other hand, if the answer of the protocol is 0 then there are $t > 1$ agents in the population with $x-$variables set to different values $x_1, x_2, \cdots, x_t$ otherwise they could have effective interactions with each other. Therefore, there should have initially existed $2^{x_1-1} + 2^{x_2-1} + \cdots + 2^{x_t-1}$ agents with input a. This, however, means that $N_a \neq 2^k$ for any k since each number can be uniquely expressed as a sum of distinct powers of 2. Thus the protocol correctly outputs 0. □

Thus, we have presented a non-semilinear predicate that can be computed by a PM protocol using $\mathcal{O}(\log\log n)$ space. Combining this result with the one presented in the previous subsection, we obtain the following theorem:

Theorem 12 (*Threshold Theorem*)**.** $\mathbf{SEM} = \mathbf{PMSPACE}(f(n))$ *when* $f(n) = o(\log\log n)$. *Moreover,* $\mathbf{SEM} \subsetneq \mathbf{PMSPACE}(f(n))$ *when* $f(n) = \mathcal{O}(\log\log n)$.

Theorem 12 exactly reflects the situation for the class of all regular languages **REG**, where $\mathbf{REG} = \mathbf{SPACE}(o(\log\log n)) \subsetneq \mathbf{SPACE}(\mathcal{O}(\log\log n))$ [26, 27] and [28] (Theorem 5.1.3, pages 29-30). The proof presented in this work however is quite different.

8. Conclusions - Future Research Directions

We proposed the PM model, an extension of the PP model [3], in which the agents are communicating TMs. Throughout our work, we studied the computational power of the new model when the space used by each agent is bounded by a function $f(n)$ of the population size. To do so, we presented protocols in which the number of states used by any execution on n agents is bounded by $\mathcal{O}(c^{f(n)})$ (so that each state can be represented by $\mathcal{O}(f(n))$ Turing tape cells), where c constant, and the new states of the interacting agents are computable in $f(n)$-space by a Turing Machine. Although the model preserves uniformity and anonymity, interestingly, we have been able to prove that the agents can *organize themselves into a NTM* that makes full use of the agents' total memory (i.e. of $\mathcal{O}(nf(n))$ space) when $f(n) = \Omega(\log n)$. The agents are initially identical and have no global knowledge of the system, but by executing an *iterative reinitiation process* they are able to assign *unique consecutive ids* to themselves and get informed of the population size. In this manner, we showed that $\mathbf{PMSPACE}(f(n))$, which is the class of predicates stably computable by the PM model using $\mathcal{O}(f(n))$ memory, contains all symmetric predicates in $\mathbf{NSPACE}(nf(n))$. Moreover, by proving that $\mathbf{PMSPACE}(f(n)) \subseteq \mathbf{SNSPACE}(nf(n))$, we concluded that for $f(n) = \Omega(\log n)$, it is precisely equal to the class consisting of all symmetric predicates in $\mathbf{NSPACE}(nf(n))$. We also explored the behavior of the PM model for space bounds $f(n) = o(\log n)$ and proved that $\mathbf{SEM} = \mathbf{PMSPACE}(f(n))$ when $f(n) = o(\log \log n)$. Finally, we showed that this bound acts as a threshold, that is, $\mathbf{SEM} \subsetneq \mathbf{PMSPACE}(f(n))$ when $f(n) = \mathcal{O}(\log \log n)$.

Many interesting questions remain open. Is the PM model *fault-tolerant*? What preconditions are needed in order to achieve satisfactory fault-tolerance? What is the behavior of the model when the agents use $\mathcal{O}(f(n))$ memory, where $f(n) = o(\log n)$ and $f(n) = \Omega(\log \log n)$?

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