On Locality of One-Variable Axioms and Piecewise Combinations

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Abstract. Local theory extensions provide a complete and efficient way for reasoning about satisfiability of certain kinds of universally quantified formulas modulo a background theory. They are therefore useful in automated reasoning, software verification, and synthesis. In this paper, we 1) provide a sufficient condition for locality of axioms with one variable specifying functions and relations, 2) show how to obtain local axiomatizations from non-local ones in some cases, 3) show how to obtain piecewise combination of different local theory extensions.

1 Introduction

Our goal is to develop techniques for efficient automated reasoning about quantified formulas with applications in verification and synthesis. Our focus is on classes of formulas for which we can build *decision procedures* for the satisfiability problem. Decision procedures are guaranteed to terminate and therefore often result in tools that behave better in practice than techniques based on semi-decidable logics. Moreover, the completeness proofs for most decision procedures also lead to the possibility of generating models and counter-models of formulas. Such models are essential for providing diagnostics in verification tools and symbolic execution [TdH08,CDH+09]. Moreover, they can be used as a basis for highly declarative extensions of programming languages [KMPS10].

Efficient decision procedures are often designed for quantifier-free languages, such as quantifier-free linear arithmetic, or quantifier-free fragment of first-order logic. A great expressive power can be obtained by additionally introducing universally quantified formulas (axioms) that typically define properties of fresh uninterpreted function and relation symbols. Reasoning about such axioms reduces to the problem of sufficient instantiation of quantifiers. Our paper focuses on a remarkable class of axioms that define *local theory extensions.* For these axioms it is possible to compute, given a quantifier free formula, a finite set of instantiations of axioms that are sufficient to reason about the satisfiability of this formula in the presence of the axioms. In this way, local theory extensions yield decision procedures for certain quantified formulas.

The contributions of this paper are identifying new infinite classes of axioms that form local theory extensions, as well as identifying new useful methods of obtaining new local theory extensions from existing ones.

- Among the new classes of axioms that we identify as local are those that have the form of implicit function definitions. The key property that ensures locality is that the axiom always uses the fresh function symbol with the same tuple of quantified variables.
- We also show local axioms that introduce relations instead of functions, possibly constraining the relation to be total or functional.
- The conditions under which some of our classes of axioms are local can be stated as $\forall\exists$ formulas. By adding formulas that imply these conditions to the axioms, we can transform non-local axioms into local ones. In some cases such formulas can be computed automatically by applying quantifier elimination to the locality conditions.
- We point to further simple but useful constructions for generating local axioms, such as using different axioms on different subsets of a function domain. This allows us to combine our newly identified local axioms with previously known ones to define or approximate interesting classes of functions.

In this way, we develop a framework where axioms can be used as a flexible form of definitions of functions and relations. The locality conditions guarantee that we do not lose completeness in reasoning about quantifier-free formulas by "unfolding" the axioms only for those terms to which the new function symbol is applied in the quantifier-free formula. Special cases of these observations have been used implicitly before [WKL$^+$06a, MN05a], the present paper explains these results using local theory extensions, identifies a more general class of local axioms. Among the new applications we point out is reasoning about functional programs. Contracts of functions in functional programs can be described using local axioms. If the only information known about the functions are their contracts, then a complete way of checking the satisfiability of any quantifier-free statement involving these functions is to instantiate local axioms.

2 Background

We use the usual notation and terminology of first-order logic. This section introduces additional terminology and gives a short introduction to local theory extensions. For more details, we refer to [SS05, IJSS08, Jac10, ISS10].

Theories and models. Consider a signature $\Pi = (\Sigma, \mathsf{Pred})$, where Σ is a set of function symbols and Pred a set of predicate symbols (both with given arities). Throughout the paper, we allow formulas to contain additional constant symbols not specifically given in the signature. A *Π-structure* $\mathcal{M}$ consists of a non-empty set of elements M, a total function $f^{\mathcal{M}} : M^n \to M$ for every n-ary function symbol $f \in \Sigma$, as well as a set $P^{\mathcal{M}} \subseteq M^n$ for every n-ary predicate symbol $P \in \mathsf{Pred}$. We regard *(Π-)theories* as sets of (Π-)formulas closed under consequences, defined by a set of *axioms*. A given Π-structure $\mathcal{M}$ is a *model* of a theory $\mathcal{T}$ iff every axiom of $\mathcal{T}$ is satisfied by $\mathcal{M}$. If a formula F is satisfied by a Π-structure $\mathcal{M}$, we write $\mathcal{M} \models F$. If F is true in all models of $\mathcal{T}$, we write $\models_{\mathcal{T}} F$. A formula F_2 is a *consequence* of F_1 (modulo $\mathcal{T}$), written $F_1 \models_{\mathcal{T}} F_2$, if

F_2 is true in every model of $\mathcal{T}$ that also satisfies F_1. If no model of $\mathcal{T}$ satisfies F, we write $F \models_{\mathcal{T}} \Box$, where $\Box$ represents the empty clause.

Reasoning in Local Theory Extensions. *Theory extensions* extend a given theory with new function symbols, defined by a set of axioms. *Locality* of the extension ensures that reasoning about these symbols can be efficiently reduced to reasoning in the base theory by finite instantiation of the axioms. The additional symbols are called *extension symbols*, terms starting with extension symbols are called *extension terms*.

Consider a background theory $\mathcal{T}$ with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred}_0)$. In the following, we allow formulas that are not restricted to this signature, but may contain additional function symbols given by a set Σ_1. If F_1 and F_2 are formulas in the signature $\Pi = (\Sigma_0 \cup \Sigma_1, \mathsf{Pred}_0)$, we regard $\mathcal{T}$ as a Π-theory, where extension symbols are not constrained by $\mathcal{T}$.

An *augmented Π-clause* is a Π-formula $\forall \overline{x}.\ \Phi(\overline{x}) \vee C(\overline{x})$, where $\Phi(\overline{x})$ is an arbitrary Π_0-formula and $C(\overline{x})$ is a disjunction of Π-literals. We say that it is *Σ_1-ground* if $C(\overline{x})$ is ground. A *theory extension* of a theory $\mathcal{T}$ with signature Π_0 is given by a set $\mathcal{K}$ of augmented Π-clauses, representing axioms for the extension symbols. We use $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$ to denote the extension of a theory $\mathcal{T}$ with a set of axioms $\mathcal{K}$. Note that $F_1 \models_{\mathcal{T}\cup\mathcal{K}} F_2$ iff $\mathcal{K} \cup F_1 \models_{\mathcal{T}} F_2$.

A substitution σ is a function from variables to terms, by $F\sigma$ we denote the result of simultaneously replacing each free variable x in F with $\sigma(x)$. For a set of formulas $\mathcal{K}$, define $\mathsf{st}(\mathcal{K})$ as the set of ground subterms appearing in $\mathcal{K}$. For a set of Π-formulas $\mathcal{K}$ and a Π-formula G, let

$$\begin{aligned}\mathcal{K}[G] = \{\ F\sigma \mid\ & F \in \mathcal{K} \text{ and } \sigma \text{ is such that}\\ & f(\overline{t})\sigma \in \mathsf{st}(\mathcal{K} \cup G) \text{ for each extension subterm } f(\overline{t}) \text{ of } F,\\ & \text{and } \sigma(x) = x \text{ if } x \text{ does not appear in an extension term } \}.\end{aligned}$$

We consider theory extensions $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$ with the following locality property:

(ELoc) For every set G of Σ_1-ground augmented Π-clauses, we have $\mathcal{K} \cup G \models_{\mathcal{T}} \Box \iff \mathcal{K}[G] \cup G \models_{\mathcal{T}} \Box$

Hierarchical reasoning. Using (ELoc), we can reduce satisfiability problems in the extended theory to the base theory:

Step 1. By (ELoc), $\mathcal{K} \cup G \models_{\mathcal{T}} \Box \iff \mathcal{K}[G] \cup G \models_{\mathcal{T}} \Box$.

Step 2. Since all occurrences of extension symbols are in ground terms, we can effectively remove the extension symbols using Ackermann's reduction, i.e. replace extension terms with fresh constant symbols and add the necessary instances of the congruence axiom (see [SS05]).

If $\mathcal{K}[G] \cup G$ is ground, we can handle the function symbols by a combination of the reasoner for the base theory and a reasoner for free function symbols.

Decidability. Satisfiability of G modulo $\mathcal{T} \cup \mathcal{K}$ is decidable whenever $\mathcal{K}[G] \cup G$ is finite and belongs to a decidable fragment of $\mathcal{T}$ (after removing the additional function symbols). In particular, if G is ground, and all universally quantified variables in $\mathcal{K}$ appear in extension terms, then $\mathcal{K}[G] \cup G$ is ground, and decidability of the ground fragment of $\mathcal{T}$ is sufficient.

Identifying Local Theory Extensions. To formulate a sufficient condition for theory extensions satisfying (ELoc), we need some additional definitions.

A *partial Π-structure* is the same as a Π-structure, except that function symbols may be assigned partial functions. (We assume that constants are always defined.) In a partial structure $\mathcal{M}$, terms are evaluated wrt. a variable assignment β like in total structures, except that the evaluation of $\beta(f(t_1, \ldots, t_n))$ is undefined if either $(\beta(t_1), \ldots, \beta(t_n))$ is not in the domain of $f^{\mathcal{M}}$, or at least one of the $\beta(t_i)$ is undefined. A partial Π-structure $\mathcal{M}$ and a variable assignment β *weakly satisfy* a literal L if either all terms in L are defined and the usual notion of satisfaction applies, or if at least one of the terms in L is undefined. Based on weak satisfaction of literals, weak satisfaction of formulas is defined recursively in the usual way. If $\mathcal{M}$ satisfies F for all variable assignments β, $\mathcal{M}$ is a *weak partial model* of F.

For $\Pi = (\Sigma, \mathsf{Pred})$, a total Π-structure $\mathcal{M}$ is a *completion* of a partial Π-structure $\mathcal{M}'$ if $M = M'$ and

1. for every $f \in \Sigma$: $f^{\mathcal{M}}(\overline{x}) = f^{\mathcal{M}'}(\overline{x})$ whenever $f^{\mathcal{M}'}(\overline{x})$ is defined, and
2. for every $P \in \mathsf{Pred}$: $P^{\mathcal{M}} = P^{\mathcal{M}'}$.

For theory extensions $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$ with base signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ and extended signature $\Pi = (\Sigma_0 \cup \Sigma_1, \mathsf{Pred})$, we consider the completability property

($\mathsf{Comp_w}$) For every weak partial Π-model $\mathcal{M}$ of $\mathcal{T} \cup \mathcal{K}$ where Σ_0-functions are total, there exists a completion which is a model of $\mathcal{T} \cup \mathcal{K}$

A formula F is Σ_1-*flat* if it does not contain occurrences of function symbols below a Σ_1-symbol. A Σ_1-flat formula F is Σ_1-*linear* if all extension terms in F which contain the same variable are syntactically equal, and no extension term in F contains two or more occurrences of the same variable.

Theorem 1 (Completability implies extended locality [SS05]). *If $\mathcal{K}$ consists of Σ_1-linear augmented clauses and the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$ satisfies* ($\mathsf{Comp_w}$)*, then it also satisfies* (ELoc).

Combinations and chains of extensions. There are two ways of modularly combining local theory extensions. If we have two extensions $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}_1$ and $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}_2$ that introduce disjoint sets of function symbols and individually satisfy ($\mathsf{Comp_w}$), then the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}_1 \cup \mathcal{K}_2$ also satisfies ($\mathsf{Comp_w}$) [SS08]. On the other hand, an extended theory can of course be extended again, so we can have *chains of extensions* with $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}_1$ and $\mathcal{T} \cup \mathcal{K}_1 \subseteq \mathcal{T} \cup \mathcal{K}_1 \cup \mathcal{K}_2$ (and so on), if every extension satisfies ($\mathsf{Comp_w}$).

3 Locality of One-Variable Axioms

In this section, we first introduce a general locality result for axioms with only one variable. When looking at the function as a binary relation which is functional and total, this amounts to putting an upper bound on the relation, since $\Phi(x, f(x))$ is equivalent to $f(x) = y \rightarrow \Phi(x, y)$.

We then generalize this result from functions to general relations, and specifying not only an upper, but also a lower bound. After that, we look at possibilities to obtain relations which are either functional or total, and finally we consider again functions with both lower and upper bound.

3.1 Locality for One-Variable Axioms

Theorem 2. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, f a fresh function symbol and $\Phi(x, f(x))$ a $(\Sigma_0 \cup \{f\}, \mathsf{Pred})$-formula with x as its only free variable and $f(x)$ the only term in which x appears below f.*

The theory extension $\mathcal{T} \subseteq \mathcal{T} \cup \{\forall x.\ \Phi(x, f(x))\}$ satisfies $(\mathsf{Comp_w})$ if and only if $\models_{\mathcal{T}} \forall x \exists y.\ \Phi(x, y)$.

We can obtain extensions with multiple function symbols using the general results for combinations and chains of local extensions explained in Section 2. Since $(\mathsf{Comp_w})$ implies (ELoc), Theorem 2 subsumes the decidability result from [WKL$^+$06b] in a much simpler form by expressing it as a locality result. For all examples given there, reasoning in local theory extensions given by one-variable axioms is sufficient.

Example 1 (Field Constraint Analysis for Skip Lists). One of the examples in [WKL$^+$06b] are skip lists, which are (doubly-linked) lists with an additional $\mathsf{nextSub}$ pointer that nondeterministically points to some element reachable by several applications of the usual next pointer, i.e. $\mathsf{nextSub}$ is specified by

$$\forall x.\ \mathsf{nextSub}(\mathsf{null}) = \mathsf{null} \wedge (x \neq \mathsf{null} \rightarrow \mathsf{reachable}(x, \mathsf{nextSub}(x)))$$

If the ground fragment of the base theory which specifies the underlying list structure is decidable (e.g. MSOL), then for its extension with an additional field $\mathsf{nextSub}$ defined as above the ground fragment remains decidable. Together with update rules which preserve decidability (see [WKL$^+$06b] or [IJSS08]), we can check whether universal formulas are invariants for a given update rule.

3.2 Generalization to Binary Relations

In this section, we generalize our first result to binary relations $r(x, y)$ which are not necessarily functional or total, defined by formulas $L(x, y)$ and $U(x, y)$, giving lower and upper bounds for $r(x, y)$.

Pure binary relations. First, we use only the terminology introduced so far, which does not allow us to introduce new predicate symbols. Thus, we model our relation $r(x, y)$ as a binary function $f(x, y)$ mapping into the booleans, with $f(x, y) = \mathit{true} \iff r(x, y)$. Then, we obtain the following

Theorem 3. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, $L(x, y)$ and $U(x, y)$ Π-formulas with x, y as their only free variables. Consider the extension of $\mathcal{T}$ with a fresh function symbol f defined by*

$$\mathsf{D_f} = \left\{ \begin{array}{l} \forall x, y.\ L(x, y) \rightarrow f(x, y) = \mathit{true} \\ \forall x, y.\ f(x, y) = \mathit{true} \rightarrow U(x, y) \end{array} \right\}.$$

The extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{D_f}$ satisfies $(\mathsf{Comp_w})$ if and only if $\models_{\mathcal{T}} L(x, y) \rightarrow U(x, y)$.

We can optimize the instantiation of axioms by not modeling the relation by a function, but directly encoding it as a relation. To this end, we generalize the definition of $\mathcal{K}[G]$. Assume that an *extension literal* is a literal built from an extension predicate symbol, and let

$$\begin{array}{rl} \mathcal{K}[G] = \{\ F\sigma \mid & F \in \mathcal{K} \text{ and } \sigma \text{ is such that for each extension literal} \\ & L(\overline{x}) \text{ in } F,\ \neg L(\overline{x})\sigma \text{ is a ground literal in } (\mathcal{K} \cup G), \\ & \text{and } \sigma(x) = x \text{ if } x \text{ does not appear in an extension literal}\}. \end{array}$$

That is, we are matching extension literals in $\mathcal{K}$ to ground literals of opposite polarity in $\mathcal{K} \cup G$. For simplicity, we do not consider extension functions and extension predicates simultaneously.

Since Theorem 1 does not apply in this case, we prove extended locality (wrt. the set of instances defined above) directly.

Theorem 4. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, $L(x, y)$ and $U(x, y)$ Π-formulas with x, y as their only free variables. Consider the extension of $\mathcal{T}$ with a fresh binary predicate symbol $r(x, y)$ defined by*

$$\mathsf{D_r} = \left\{ \begin{array}{l} \forall x, y.\ L(x, y) \rightarrow r(x, y) \\ \forall x, y.\ r(x, y) \rightarrow U(x, y) \end{array} \right\}.$$

If $\models_{\mathcal{T}} L(x, y) \rightarrow U(x, y)$ holds, then the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{D_r}$ satisfies (ELoc).

Total binary relations. Now, consider relations that are total, but not necessarily functional. To define these, we use a $\forall\exists$-formula (which is not in the fragment of formulas usually considered for local theory extensions), and a special rule to instantiate it. If $\mathcal{K}$ is a set of formulas of the form $\forall x.\ \exists y.\ F(x, y, r(x, y))$ (i.e. x and y are the only variables and their only appearance below r is in $r(x, y)$), and G a set of Σ_1-ground augmented Π-clauses, let

$$\begin{array}{rl} \mathcal{K}[G] = \{\ \exists y.F(t_1, y, r(t_1, y)) \mid & \forall x.\exists y.\ F(x, y, r(x, y)) \in \mathcal{K} \text{ and there is} \\ & \text{a ground literal } \neg r(t_1, t_2) \text{ in } (\mathcal{K} \cup G) \quad \}. \end{array}$$

Theorem 5. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, $L(x,y)$ and $U(x,y)$ Π-formulas with x, y as their only free variables. Consider the extension of $\mathcal{T}$ with a fresh binary predicate symbol $r(x,y)$ defined by*

$$\mathsf{T_r} = \{\ \forall x.\exists y.\ r(x,y)\ \},\ \text{and}$$

$$\mathsf{D_r} = \left\{ \begin{array}{l} \forall x, y.\ L(x,y) \ \rightarrow\ r(x,y) \\ \forall x, y.\ r(x,y) \ \rightarrow\ U(x,y) \end{array} \right\}.$$

If both $\models_{\mathcal{T}} L(x,y) \rightarrow U(x,y)$ and $\models_{\mathcal{T}} \forall x.\ \exists y.\ U(x,y)$ hold, then the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{T_r} \cup \mathsf{D_r}$ satisfies the modified locality condition

$$\mathsf{T_r} \cup \mathsf{D_r} \cup G \models_{\mathcal{T}} \Box \iff \mathsf{T_r}[G] \cup \mathsf{D_r}[\mathsf{T_r}[G] \cup G] \cup G \models \Box.$$

Functional binary relations. Now, consider relations which are not necessarily total, but are functional. In this case, $L(x,y)$ must be functional (because otherwise no relation with $L(x,y)$ as a lower bound can be functional), and to obtain a decision procedure we also need a predicate $D(x)$ in the base theory which is equivalent to $\exists y.\ L(x,y)$, i.e. $D(x)$ is true whenever x is in the domain of the partial function described by $L(x,y)$. Then we can formulate

Theorem 6. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, $L(x,y)$ and $U(x,y)$ Π-formulas with x, y as their only free variables. Let $D(x)$ be a predicate in the base theory with $\models_{\mathcal{T}} D(x) \leftrightarrow (\exists y.\ L(x,y))$. Consider the extension of $\mathcal{T}$ with a fresh binary predicate symbol $r(x,y)$ defined by*

$$\mathsf{D_r^f} = \left\{ \begin{array}{ll} \forall x, y. & L(x,y) \ \rightarrow\ r(x,y) \\ \forall x, y. & r(x,y) \ \rightarrow\ U(x,y) \\ \forall x_1, x_2, y_1, y_2. & r(x_1,y_1) \wedge r(x_2,y_2) \wedge x_1 = x_2 \ \rightarrow\ y_1 = y_2 \\ \forall x, y. & r(x,y) \wedge D(x) \ \rightarrow\ L(x,y) \end{array} \right\}.$$

If $\models_{\mathcal{T}} L(x,y) \rightarrow U(x,y)$ and $\models_{\mathcal{T}} \forall x_1, x_2, y.\ L(x_1,y) \wedge L(x_2,y) \rightarrow x_1 = x_2$, then the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{D_r^f}$ satisfies (ELoc).

Total and functional binary relations. Consider again functional and total relations, i.e. total functions. Previous we considered only an upper bound on the function. We now consider axiomatizations of the form

$$(L(x,y) \ \rightarrow\ f(x) = y) \wedge (f(x) = y \ \rightarrow\ U(x,y)).$$

For the formula above to be consistent, $L(x,y)$ must be functional. Furthermore, the construction which ensures that $f(x)$ remains functional when adding values from $L(x,y)$ requires us to have a predicate $D(x)$ defined by $\models_{\mathcal{T}} D(x) \leftrightarrow \exists y. L(x,y)$ in the base theory. However, if validity of such a predicate is decidable in the base theory, we can show that we have not gained expressivity by adding lower bounds:

by the definition of $D(x)$ and functionality of $L(x, y)$, we have

$$\begin{aligned} & (L(x,y) \rightarrow f(x) = y) \wedge (f(x) = y \rightarrow U(x,y) \\ \iff & (D(x) \wedge f(x) = y \rightarrow L(x,y)) \wedge (f(x) = y \rightarrow U(x,y). \end{aligned}$$

We can rewrite the above condition to a one-variable axiom $\forall x.\phi(x, f(x))$ where $\phi(x, y)$ is the formula $(D(x) \rightarrow L(x, y)) \wedge U(x, y)$.

3.3 Beyond One-Variable Axioms

We next discuss possible extensions of our results for one-variable axioms and analyze whether locality is preserved.

Loss of locality for two variables. In general, the straightforward modification of Theorem 2 for axioms with two or more variables does not hold:

Example 2 (Loss of locality for more than one variable). Consider the background theory $\mathcal{T}_{\mathbb{Z}}$ and its extension with

$$\mathsf{SMon}(f) = \forall x_1, x_2.\ x_1 < x_2 \rightarrow f(x_1) < f(x_2).$$

Even though $\models_{\mathcal{T}_{\mathbb{Z}}} \forall x_1, x_2.\ \exists y_1, y_2.\ x_1 < x_2 \rightarrow y_1 < y_2$, the extension $\mathcal{T}_{\mathbb{Z}} \subseteq \mathcal{T}_{\mathbb{Z}} \cup \mathsf{SMon}(f)$ is not local (and thus cannot satisfy $(\mathsf{Comp_w})$): For $G = \{f(0) = 0, f(2) = 1\}$, $\mathsf{SMon}(f)[G] \cup G$ is $\mathcal{T}_{\mathbb{Z}}$-satisfiable, but $\mathsf{SMon}(f) \cup G$ is not.

Unary functions over tuples of variables. For obtaining the decidability results in this Section, it is not strictly necessary for the axiom to only contain one variable. The important part is that there is only one term $f(\overline{x})$ in Φ, where $\overline{x}$ can consist of several variables. Then, we can consider f as a unary function over tuples, and use parts of the tuple $(\overline{x})$ in the rest of the formula by projection.

Example 3 (Specifications in Functional Programming). We can axioms to prove properties of functional programs satisfying a given specification. Consider the specification

```
def f(x) returns r
  requires x > 0
  ensures r > max(0,x)
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We may want to know whether $\forall x.f(f(x)) > 2$. Checking whether this holds amounts to negating and flattening the formula, obtaining $G = \{f(a) = b,\ f(b) = c,\ c \leq 2\}$, and then for every occurrence of f introducing the specification, i.e. $G' = \{a > 0 \rightarrow b > max(0, a),\ b > 0 \rightarrow c > max(0, b),\ c \leq 2\}$. A decision procedure for the ground fragment of $\mathbb{N}$ (with a max function) will be able to prove unsatisfiability of G', i.e. $\forall x.f(f(x)) > 2$ holds for every function which abides to the specification above.

In general, whenever we have functions (e.g. f) specified using contracts (with precondition $P(\overline{x})$ and postcondition $Q(\overline{x}, f(\overline{x}))$ in a decidable theory, we can encode contracts as one-variable axioms over the tuples of arguments (namely $\forall \overline{x}.(P(\overline{x}) \rightarrow Q(\overline{x}, f(\overline{x}))))$. Consequently, our method gives a complete decision procedure for checking the validity of any purely universally quantified formula that contains functions given by the contracts.

Additional quantified variables. The satisfiability problems also remain decidable if Φ contains additional quantified variables, as long as the corresponding fragment of the base theory is decidable. For example, consider a formula of the form $\forall x, z.\Phi(x, z, f(x))$. If $\models_{\mathcal{T}} \forall x.\exists y.\forall z.\Phi(x, z, y)$, then this is a local extension of $\mathcal{T}$. After instantiation, we will have formulas of the form $\forall z.\Phi(t_1, z, f(t_1))$. Because $f(t_1)$ does not contain variables z, we can again remove the function symbol f by stating functionality on the ground instances using Ackerman's encoding. The problem therefore reduces to ground formulas extended with the instances of the form $\forall z.\Phi(t_1, z, r_1)$, which can be handled whenever the base decision procedure supports such universally quantified formulas.

4 Computing Locality-Ensuring Guards

In this section, we will consider the problem of generating axiomatizations which satisfy a locality property. We start with the observation that adding guards to a set of formulas which does not satisfy a locality property can make the resulting axiomatization local wrt. the base theory. We will then show that for one-variable axioms as introduced in Section 3, we can compute such guards efficiently if the background theory allows elimination of a pair of $\forall\exists$-quantifiers.

4.1 Introducing Locality-Ensuring Guards

Locality of a theory extension is a property which depends on the base theory and the extension axioms. By examining the results on local theory extensions, we found that there are often additional requirements on constants, functions or relations appearing in the extension axioms. For example, the following axiom imposes a boundedness condition on the slope of a real-valued function:

$$\forall x, y.\ x \neq y \rightarrow c_1 \leq \frac{f(x) - f(y)}{x - y} \leq c_2. \qquad (\mathsf{BS}_f^{c_1,c_2})$$

The extension $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_{\mathbb{R}} \cup (\mathsf{BS}_f^{c_1,c_2})$ is local if and only if we know that $c_1 \leq c_2$ (otherwise the axiom is even inconsistent). The idea of locality-ensuring guards is to have such requirements not outside of the axiomatization, but put them inside. For the example above, we can extend $\mathcal{T}_{\mathbb{R}}$ either with $c_1 \leq c_2 \cup (\mathsf{BS}_f^{c_1,c_2})$, or with $c_1 \leq c_2 \rightarrow (\mathsf{BS}_f^{c_1,c_2})$. In both cases, the theory extension is local without additional requirements to the axiom.

These two forms define different extensions: by adding the constraint conjunctively, we state that it must be satisfied in any model of the extension. By putting it as the guard of an implication, we state that if the constraint does not hold, then this axiom does not need to be considered. In the example above, the later case means that the function would simply be uninterpreted.

Definition 1. *If $\mathcal{T}$ is a Π_0-theory and $\mathcal{K}$ a set of augmented Π-clauses, then we say that a Π_0-formula F is a* locality-ensuring guard *for $\mathcal{K}$ wrt. $\mathcal{T}$ if for every weak partial Π-model of $\mathcal{T} \cup \mathcal{K} \cup F$ with total Σ_0-functions there is a completion which is a model of $\mathcal{T} \cup \mathcal{K} \cup F$.*

Theorem 7. *If F is a locality-ensuring guard for $\mathcal{K}$ wrt. $\mathcal{T}$, then both $\mathcal{T} \subseteq \mathcal{T} \cup F \cup \mathcal{K}$ and $\mathcal{T} \subseteq \mathcal{T} \cup (F \rightarrow \mathcal{K})$ satisfy* (ELoc).

4.2 Computing Locality-Ensuring Guards for One-Variable Axioms

Since we have expressed the condition for locality of one-variable axioms by a formula in the base theory, we can directly formulate a theorem for locality-ensuring guards of one-variable axioms.

Theorem 8. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, f a fresh function symbol and $\Phi(x, f(x))$ a $(\Sigma_0 \cup \{f\}, \mathsf{Pred})$-formula, where x is the only variable in $\Phi(x, f(x))$ and $f(x)$ is the only term in which x appears below f.*

If F is a Π_0-formula such that $\models_{\mathcal{T}} F \rightarrow \forall x \exists y.\ \Phi(x, y)$, then F is a locality-ensuring guard for $\forall x.\ \Phi(x, f(x))$ wrt. $\mathcal{T}$.

Thus, if our base theory admits elimination of a pair of $\forall\exists$-quantifiers, then we can compute from $\forall x \exists y.\ \Phi(x, y)$ a Π-formula F with $\models_{\mathcal{T}} F \leftrightarrow \forall x \exists y.\ \Phi(x, y)$, i.e. in particular $\models_{\mathcal{T}} F \rightarrow \forall x \exists y.\ \Phi(x, y)$. Compared to adding $\forall x \exists y.\ \Phi(x, y)$ itself and using quantifier elimination every time, this has the benefit that we only need to invoke the quantifier elimination algorithm once, and then can rely on instantiation of $\Phi(x, f(x))$ for checking satisfiability of Σ_1-ground inputs.

Example 4 (Computing Constraints in the Real Numbers). Let t_1 and t_2 be terms in the signature of $\mathcal{T}_{\mathbb{R}}$, with x as their only variable. It has been shown [SS05] that the extension of $\mathcal{T}_{\mathbb{R}}$ with a unary function symbol f satisfying

$$\mathsf{Bound}_{t_1,t_2}(f) = \left\{ \forall\ x.\ t_1[x] \leq f(x) \leq t_2[x] \right\}$$

satisfies (Comp) if $\models_{\mathcal{T}_{\mathbb{R}}} t_1[x] \leq t_2[x]$.

If we assume that $t_1[x]$ and $t_2[x]$ are linear terms, they can be rewritten to slope-intercept form $t_i[x] = m_i \cdot x + b_i$. Then, we can obtain a locality-ensuring guard by eliminating the quantified variables from $\forall x. \exists y.\ m_1 \cdot x + b_1 \leq y \leq m_2 \cdot x + b_2$. A quantifier elimination procedure will reduce this to $m_1 = m_2 \wedge b_1 \leq b_2$ (or an equivalent formula). Thus, (Comp) holds for the extension of $\mathcal{T}_{\mathbb{R}}$ with

$$\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f) = \left\{ \begin{array}{c} m_1 = m_2, \\ b_1 \leq b_2, \\ \forall\ x.\ \ m_1 \cdot x + b_1 \leq f(x) \leq m_2 \cdot x + b_2 \end{array} \right\}.$$

Example 5 (Guards for Axioms with Additional Quantified Variables). Consider the axiom A enforcing that f is growing according to the lower bounds given by function l, where l function is the part of the base theory:

$$\forall x.\ \forall i.\ l(i) \leq x \rightarrow i \leq f(x)$$

We view this formula as $\forall x. \phi(x, f(x))$ where $\phi(x, y)$ is $\forall i.\ l(i) \leq x \rightarrow i \leq f(x)$. The guard for this axiom is then $\forall x. \exists y. \phi(x, y)$. Formula $\phi(x, y)$ means that x is a lower bound for all values $l(i)$ where $y < i$. Therefore, the guard formula G for this example is $\lim_{i \to \infty} l(i) = \infty$. Our results ensure that both $G \wedge A$ and $G \rightarrow A$ are local axioms.

5 Piecewise Combination of Local Extensions

In this section, we consider sets of axioms that separate the domain of an extension function into subsets and define different properties on these subsets. We are not restricted to axioms with one occurrence of an extension term $f(\overline{x})$, but consider sets of axioms $\mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$, where in each axiom there may be up to n extension terms.[1] We first show that we can have several different local axiomatizations in disjoint subsets of the domain, and the resulting "piecewise" axiomatization will again be local. Furthermore, we show that in some cases a piecewise local axiomatization can even be obtained if properties with local axiomatizations are given for non-disjoint subsets of the domain.

In this section, we will use the restriction of formulas $\mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ to a subset of the domain of f, specified by a formula $\Phi(\overline{x})$. We denote by $\bigwedge_{i=1}^{n} \Phi(x_i) \rightarrow \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ the set of augmented Π-clauses $\{\forall \overline{x}_1, \ldots, \overline{x}_n. \bigwedge_{i=1}^{n} \Phi(x_i) \rightarrow F \vee C \mid F \vee C \in \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))\}$. We state that such restrictions do not destroy locality properties:

Lemma 1. *Let $\mathcal{T}$ be a Π_0-theory and $\mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ a set of augmented Π-clauses such that $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ satisfies* ($\mathsf{Comp_w}$). *For any Π_0-formula $\Phi(\overline{x})$, the extension $\mathcal{T} \subseteq \mathcal{T} \cup (\Phi(\overline{x}) \rightarrow \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ also satisfies* ($\mathsf{Comp_w}$).

5.1 Specifications over disjoint subsets

Theorem 9. *Let $\mathcal{T}$ be a Π_0-theory and consider Π_0-formulas $\Phi_1(x), \Phi_2(x)$ such that $\models_{\mathcal{T}} \neg(\Phi_1(x) \wedge \Phi_2(x))$. If $\mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ and $\mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m))$ are Π-formulas such that both $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^{n} \Phi_1(x_i) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ and $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^{m} \Phi_2(x_i) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m))$ satisfy condition* ($\mathsf{Comp_w}$)*, then for*

$$\begin{aligned} \mathcal{K} = \quad & (\textstyle\bigwedge_{i=1}^{n} \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n)) \\ \cup\, & (\textstyle\bigwedge_{i=1}^{m} \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m)), \end{aligned}$$

the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$ satisfies ($\mathsf{Comp_w}$) *and is a local extension.*

Example 6 (Locality on Disjoint Subsets). Theorem 9 explains some older locality results on piecewise boundedness [SSI07], based on the local extension specifying boundedness of a function [SS05].

The theorem can also be used to define functions which satisfy different monotonicity properties on different subsets of the background theory, based on locality of extensions specifying several variants of strict [Jac10] and non-strict [SS05] monotonicity.

5.2 Specifications over non-disjoint subsets

If the subsets defined by the Φ_i are not disjoint, we can still obtain a decision procedure in some cases:

[1] The case for axioms with only one extension term is a consequence of Theorem 2.

Non-disjoint subsets with a local property in the intersection. If Φ_1 and Φ_2 do not describe disjoint subsets of the domain of f, we can obtain a disjoint case distinction by considering the extension

$$\begin{aligned}\mathcal{K} = \quad & (\bigwedge_{i=1}^{n} \Phi_1(x_i) \wedge \bigwedge_{i=1}^{m} \Phi_2(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n)) \wedge \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m)) \\ \cup\ & (\bigwedge_{i=1}^{n} \Phi_1(x_i) \wedge \neg \bigwedge_{i=1}^{m} \Phi_2(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n)), \\ \cup\ & (\neg \bigwedge_{i=1}^{n} \Phi_1(x_i) \wedge \bigwedge_{i=1}^{m} \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m)).\end{aligned}$$

However, this will only be a local extension if $(\bigwedge_{i=1}^{n} \Phi_1(x_i) \wedge \bigwedge_{i=1}^{m} \Phi_2(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n)) \wedge \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m))$ is a local extension.

Example 7 (Non-disjoint combination of monotonicity and boundedness). Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ and a partial order $\leq$. Let $\mathsf{Mon}(f) = \{\forall x, y.\ x \leq y \rightarrow f(x) \leq f(y)\}$, and $\mathsf{Bound}_{t_1,t_2}(f) = \{\forall x.\ t_1[x] \leq f(x) \leq t_2[x]\}$, for Σ_0-terms t_1 and t_2 with $\models_{\mathcal{T}} t_1[x] \leq t_2[x]$. Then both $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{Mon}(f)$ and $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{Bound}_{t_1,t_2}(f)$ satisfy $(\mathsf{Comp_w})$, as well as $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{Mon}(f) \cup \mathsf{Bound}_{t_1,t_2}(f)$. Suppose $\Phi_1(x)$ and $\Phi_2(x)$ are not disjoint. Then, for

$$\begin{aligned}\mathcal{K} = \quad & (\Phi_1(x) \wedge \Phi_2(x)) \rightarrow \mathsf{Mon}(f) \wedge \mathsf{Bound}_{t_1,t_2}(f) \\ \cup\ & (\Phi_1(x) \wedge \neg\Phi_2(x)) \rightarrow \mathsf{Mon}(f) \\ \cup\ & (\neg\Phi_1(x) \wedge \Phi_2(x)) \rightarrow \mathsf{Bound}_{t_1,t_2}(f),\end{aligned}$$

the extension $\mathcal{T} \cup \mathcal{K}$ also satisfies $(\mathsf{Comp_w})$.

Non-disjoint subsets with a finite intersection. If Φ_1 and Φ_2 are not disjoint and the resulting set of axioms for the intersection does not satisfy $(\mathsf{Comp_w})$, we in general cannot obtain a piecewise combination satisfying $(\mathsf{Comp_w})$. However, if the intersection between Φ_1 and Φ_2 is finite, we can use the more general notion of *Ψ-locality.* To this end, consider a closure operator Ψ on ground terms and define the more general notion of Ψ-completability

$(\mathsf{Comp_w^\Psi})$ For every weak partial Π-model $\mathcal{M}$ of $\mathcal{T} \cup \mathcal{K}$ where Σ_0-functions are total and the definition domain of Σ_1-functions is closed under Ψ, there exists a completion which is a model of $\mathcal{T} \cup \mathcal{K}$,

which implies the Ψ-locality condition

$(\mathsf{ELoc^\Psi})$ For every set G of Σ_1-ground augmented Π-clauses, we have $\mathcal{K} \cup G \models_{\mathcal{T}} \Box \iff \mathcal{K}^{\Psi}[G] \cup G \models_{\mathcal{T}} \Box$

with $\mathcal{K}^{\Psi}[G]$ defined like $\mathcal{K}[G]$, except extension terms may be in $\Psi(\mathsf{st}(\mathcal{K} \cup G))$.

Then, with a suitable Ψ we can prove Ψ-locality for piecewise combinations with finite overlaps:

Theorem 10. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ and consider Π_0-formulas $\Phi_1(x), \Phi_2(x)$ such that in every $\mathcal{T}$-model $\mathcal{M}$, the set $O = \{x \in M | \Phi_1(x) \wedge \Phi_2(x)\}$ is finite. Let furthermore T_0 be a set of Σ_0-terms such that in every such model $\mathcal{M}$, $O \subseteq \{t^{\mathcal{M}} \mid t \in T_0\}$.*

If $\mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ and $\mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m))$ are sets of augmented Π-clauses such that both $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^{n} \Phi_1(x_i) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ and $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^{m} \Phi_2(x_i) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m))$ satisfy $(\mathsf{Comp_w})$, then for

$$\begin{aligned}\mathcal{K} = \quad & (\textstyle\bigwedge_{i=1}^{n} \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n)) \\ \cup\, & (\textstyle\bigwedge_{i=1}^{m} \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m))\end{aligned}$$

and $\Psi(T) = T \cup \{f(t) | t \in T_0\}$, the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$ satisfies $(\mathsf{Comp_w}^{\Psi})$.

Theorem 10 can easily be extended to a combination of arbitrarily many pieces, where $O = \{x \in M | \Phi_i(x) \wedge \Phi_j(x)$, for some $i \neq j\}$ (and T_0 and Ψ change accordingly). We have an example application of the non-disjoint piecewise combination in Section 5.3.

5.3 Piecewise Approximation of Numerical Functions

In this section we introduce local axiomatizations which allow us to approximate numerical functions by piecewise constraints on their slopes and function values.[2]

Bounded values. In Example 4, we have seen local extensions $\mathsf{Bound}_{t_1,t_2}(f)$ and $\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)$ that allow us to bound the values of an extension function by terms in the base theory. If we only want to have either a lower or an upper bound, we can drop the additional conditions. With strict bounds in the condition (i.e. $\forall x.\ t_1[x] < t_2[x]$ or $b_1 < b_2$, respectively), we can also have strict bounds on f.

Bounded slope. (Comp) holds for the extension of $\mathcal{T}_{\mathbb{R}}$ with a unary function symbol f and constants l, u satisfying

$$\mathsf{BS}_{l,u}(f) = \left\{ \begin{array}{l} l \leq u \\ \forall\, x, y.\ x < y \rightarrow (y - x) \cdot l \leq f(y) - f(x) \leq (y - x) \cdot u \end{array} \right\}.$$

We can drop condition $l \leq u$ if we only want to have either a lower or an upper bound, and we can also have strict bounds on the slopes (i.e. $<$ instead of $\leq$).

Combination of bounded slope and bounded values. We can also combine bounded slope and bounded values for the same function, if the bounding terms are linear. $(\mathsf{Comp_w})$ is satisfied by the extension of $\mathcal{T}_{\mathbb{R}}$ with a unary function f satisfying

$$\begin{aligned}\mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f) = \quad & \{l \leq m_1,\ m_1 = m_2,\ m_2 \leq u,\ b_1 \leq b_2\} \\ \cup\, & \mathsf{BS}_{l,u}(f) \\ \cup\, & \mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f).\end{aligned}$$

[2] Some of the local extensions presented in subsection 5.3 have been identified by Sofronie-Stokkermans before or independently of our research. We present them here as a part of the illustration of results from this and the previous sections.

Restriction to an interval. In $\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)$, we have the constraint $m_1 = m_2$, which is necessary to ensure $\forall x.\ m_1 \cdot x + b_1 \leq m_2 \cdot x + b_2$. If we restrict $\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)$ to an interval $[c, d]$, the weaker constraints $m_1 \cdot c + b_1 \leq m_2 \cdot c + b_2$ and $m_1 \cdot d + b_1 \leq m_2 \cdot d + b_2$ already imply $\forall x.\ m_1 \cdot x + b_1 \leq m_2 \cdot x + b_2$. Thus,

$$\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)[c,d] = \left\{ \begin{array}{r} m_1 \cdot c + b_1 \leq m_2 \cdot c + b_2, \\ m_1 \cdot d + b_1 \leq m_2 \cdot d + b_2, \\ \forall\ x.\ \ c \leq x \leq d\ \rightarrow\ m_1 \cdot x + b_1 \leq f(x) \leq m_2 \cdot x + b_2 \end{array} \right\}$$

satisfies (Comp). The same holds if the interval is left-open and the first constraint is replaced by $m_1 \leq m_2$, and if it is right-open and the second constraint is replaced by $m_2 \leq m_1$.

Similarly, for the restriction of $\mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f)$ to an interval $[c, d]$,

$$\begin{aligned} \mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f)[c,d] = \quad & \{\quad m_1 \cdot c + b_1 + (d - c) \cdot l \leq m_2 \cdot d + b_2 \\ & \quad \vee\ m_2 \cdot c + b_2 + (d - c) \cdot u \geq m_1 \cdot d + b_1\}, \\ & \cup\ \mathsf{BS}_{l,u}(f)[c,d] \\ & \cup\ \mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)[c,d] \end{aligned}$$

satisfies ($\mathsf{Comp_w}$). The same holds if the interval is left- or right-open and the first constraint is replaced by $m_1 \leq l \leq m_2\ \vee\ m_1 \leq u \leq m_2$.

Extensions of $\mathcal{T}_{\mathbb{Q}}$ and $\mathcal{T}_{\mathbb{Z}}$. For all extensions mentioned above, the corresponding extensions of $\mathcal{T}_{\mathbb{Q}}$ and $\mathcal{T}_{\mathbb{Z}}$ satisfy (Comp) if constants which are multiplied to variables are fixed rational or integer values, respectively, and only for non-strict bounds in case of $\mathcal{T}_{\mathbb{Z}}$.

Example 8. We can use the local extensions given above, together with Theorem 10, to linearly approximate arbitrary real-valued functions. As a simple example, we can approximate $f(x) = x^2$ by the set of axioms.

$$\left\{ \begin{array}{l} x \leq -1\ \rightarrow\ 1 \leq f(x), \\ -1 \leq x \leq 1\ \rightarrow\ f(x) \leq 1, \\ 1 \leq x\ \rightarrow\ 1 \leq f(x), \\ x < y \leq -1\ \rightarrow\ f(y) - f(x) \leq -2(y - x), \\ -1 \leq x < y \leq 1\ \rightarrow\ -2(y - x) \leq f(y) - f(x) \leq 2(y - x), \\ 1 \leq x < y\ \rightarrow\ 2(y - x) \leq f(y) - f(x) \end{array} \right\}.$$

If this approximation is not fine-grained enough for our needs, we can increase the number of intervals arbitrarily in order to come up with a better approximation.

6 Related Work

The notion of *local theory extensions* was introduced by Sofronie-Stokkermans [SS05]. The approach is based on earlier results by Givan and

McAllester [GM02] and Ganzinger [Gan01]. Local theory extensions are one of the few approaches that allow us to obtain decision procedures for quantified satisfiability problems modulo a background theory. They have been shown to be useful in the verification of parameterized systems [JSS07, FJSS07, SS10] and properties of data structures [IJSS08, SS09], in reasoning about certain properties of numerical functions [SS08] and about certain properties of functions in ordered domains [SSI07]. An overview of a large part of the results on local theory extensions can be found in [Jac10].

In addition, there has been other research on handling quantified formulas in a complete way, by identifying decidable fragments of the theory of arrays [BMS06, GNRZ07] or of pointer data structures [MN05b], or certain forms of formulas which ensure decidability with the right instantiation strategy [GdM09].

Other methods for handling quantifiers arise in theories admitting quantifier elimination [Pre29, FV59, Mal71, KR03]. These results typically work for interpreted functions that satisfy particular theory-specific properties; they do not support introducing new function symbols into a theory.

7 Conclusions

We have identified a sufficient and necessary condition for extended locality of one-variable axioms. We have also introduced the notion of locality-ensuring guards, which allows to obtain a local axiomatization from a non-local one. For the case of one-variable axioms, we have shown that locality-ensuring guards can be computed if the base theory supports elimination of a pair of $\forall\exists$-quantifiers.

Additionally, we have presented results for the piecewise combination of different axiomatizations for extension functions, considering different properties in subsets of the domain of the function. If the subsets are disjoint or only have a finite overlap, we can always obtain a local axiomatization. As an example application we have shown how numerical functions can be approximated with local axiomatizations.

To achieve these results, we have in some parts extended the usual framework of local theory extensions. E.g., we have considered extension relations instead of functions and have shown that in this case polarity of literals allows us to further restrict instantiation of axioms. Also, we have used $\forall\exists$-formulas as axioms, where usually the framework of local theory extensions is restricted to universal formulas.

The results we have presented allow us to obtain a number of new decision procedures that reason in a sound and complete way about the satisfiability of quantified formulas modulo a background theory.

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A Locality of One-Variable Axioms

A.1 Locality for One-Variable Axioms

Theorem 11. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, f a fresh function symbol and $\Phi(x, f(x))$ a $(\Sigma_0 \cup \{f\}, \mathsf{Pred})$-formula with x as its only free variable and $f(x)$ the only term in which x appears below f.*

The theory extension $\mathcal{T} \subseteq \mathcal{T} \cup \forall x.\ \Phi(x, f(x))$ satisfies $(\mathsf{Comp_w})$ if and only if $\models_{\mathcal{T}} \forall x \exists y.\ \Phi(x, y)$.

Proof. First, assume $\models_{\mathcal{T}} \forall x \exists y.\ \Phi(x, y)$. We prove that then $\mathcal{T} \subseteq \mathcal{T} \cup \forall x.\ \Phi(x, f(x))$ satisfies $(\mathsf{Comp_w})$: Let $\mathcal{M}$ be a partial model of $\mathcal{T} \cup \forall x.\ \Phi(x, f(x))$, with $f^{\mathcal{M}}(a_i)$ defined for finitely many values $a_1, \dots, a_n$. Consider the structure $\mathcal{M}'$ obtained from $\mathcal{M}$ by letting

$$f^{\mathcal{M}'}(x) = \begin{cases} f^{\mathcal{M}}(x), & \text{if } x = a_i \\ y, \text{ for some } y \text{ with } \mathcal{M} \models \Phi(x, y), & \text{else} \end{cases}$$

Since $\models_{\mathcal{T}} \forall x \exists y.\ \Phi(x, y)$, such a value y always exists (in every model of $\mathcal{T}$). Clearly, the resulting structure is a model of $\mathcal{T} \cup \forall x.\ \Phi(x, f(x))$, i.e. $(\mathsf{Comp_w})$ is satisfied.

Now, assume that $\mathcal{T} \subseteq \mathcal{T} \cup \forall x.\ \Phi(x, f(x))$ satisfies $(\mathsf{Comp_w})$. Since every weak partial model of $\mathcal{T} \cup \forall x.\ \Phi(x, f(x))$ can be completed to a total model of $\mathcal{T} \cup \forall x.\ \Phi(x, f(x))$, we know that for every x there is a value $f(x)$ which satisfies $\Phi(x, f(x))$. In particular, we have $\models_{\mathcal{T}} \forall x \exists y.\ \Phi(x, y)$. $\square$

A.2 Extensions with multiple function symbols

A.3 Generalization to Binary Relations

Pure binary relations.

Theorem 12. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, $L(x, y)$ and $U(x, y)$ Π-formulas with x, y as their only free variables. Consider the extension of $\mathcal{T}$ with a fresh function symbol f defined by*

$$\mathsf{D_f} = \left\{ \begin{array}{l} \forall x, y.\ L(x, y) \rightarrow f(x, y) = true \\ \forall x, y.\ f(x, y) = true \rightarrow U(x, y) \end{array} \right\}.$$

The extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{D_f}$ satisfies $(\mathsf{Comp_w})$ if and only if $\models_{\mathcal{T}} L(x, y) \rightarrow U(x, y)$.

Proof. First, assume that $\models_{\mathcal{T}} L(x, y) \rightarrow U(x, y)$. Let $\mathcal{M}$ be a weak partial model of $\mathcal{T} \cup \mathcal{K}$, where all functions except $f^{\mathcal{M}}$ are total. Consider the completion $\mathcal{M}'$ of $\mathcal{M}$ defined by

$$f^{\mathcal{M}'}(x, y) = \begin{cases} f^{\mathcal{M}}(x, y) & \text{if defined} \\ L^{\mathcal{M}}(x, y) & \text{otherwise} \end{cases}$$

Since whenever $f^{\mathcal{M}}(a, b)$ is defined, we have $L^{\mathcal{M}}(a, b) \rightarrow f^{\mathcal{M}}(a, b) = true$, we conclude $\mathcal{M}' \models L(x, y) \rightarrow f(x, y) = true$. Furthermore, since whenever $f^{\mathcal{M}}(a, b)$ is defined we have $f^{\mathcal{M}}(a, b) = true \rightarrow U^{\mathcal{M}}(a, b)$, and since

$\models_{\mathcal{T}} L(x, y) \rightarrow U(x, y)$ (and $\mathcal{M}$ is in particular a model of $\mathcal{T}$), we conclude $\mathcal{M}' \models f(x, y) = \mathit{true} \rightarrow U(x, y)$. Thus, $\mathcal{M}'$ is a total model of $\mathcal{T} \cup \mathsf{D_f}$.

Now, assume that $\not\models_{\mathcal{T}} L(x, y) \rightarrow U(x, y)$. Let $\mathcal{M}$ be a model of $\mathcal{T}$ such that we have $L(a, b) \wedge \neg U(a, b)$ for some a, b. Then there are partial models of $\mathcal{T} \cup \mathsf{D_f}$ (e.g. the partial model $\mathcal{M}$ in which $f^{\mathcal{M}}$ is completely undefined), but no value of $f^{\mathcal{M}}(a, b)$ can simultaneously satisfy both axioms of $\mathsf{D_r}$. Thus, $(\mathsf{Comp_w})$ does not hold. $\square$

Assume that an *extension literal* is a literal built from an extension predicate symbol, and let

$$\mathcal{K}[G] = \{\ F\sigma \mid \begin{array}{l} F \in \mathcal{K} \text{ and } \sigma \text{ is such that for each extension literal} \\ L(\overline{x}) \text{ in } C,\ \neg L(\overline{x})\sigma \text{ is a ground literal in } (\mathcal{K} \cup G), \\ \text{and } \sigma(x) = x \text{ if } x \text{ does not appear in an extension literal}\}. \end{array}$$

That is, we are matching extension literals in $\mathcal{K}$ to ground literals of opposite polarity in $\mathcal{K} \cup G$.

Theorem 13. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, $L(x, y)$ and $U(x, y)$ Π-formulas with x, y as their only free variables. Consider the extension of $\mathcal{T}$ with a fresh binary predicate symbol $r(x, y)$ defined by*

$$\mathsf{D_r} = \left\{ \begin{array}{l} \forall x, y.\ L(x, y) \rightarrow r(x, y) \\ \forall x, y.\ r(x, y) \rightarrow U(x, y) \end{array} \right\}.$$

If $\models_{\mathcal{T}} L(x, y) \rightarrow U(x, y)$, then the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{D_r}$ satisfies condition (ELoc).

Proof. Let G be a set of Σ_1-ground augmented Π-clauses. "$\Leftarrow$": immediate, since $\mathsf{D_r}[G]$ consists of instances of $\mathsf{D_r}$.

"$\Rightarrow$": Suppose $\mathsf{D_r} \cup G \models_{\mathcal{T}} \square$, but there is a $\mathcal{T}$-model $\mathcal{M}$ for $\mathsf{D_r}[G] \cup G$. Let

$$r^- = r^{\mathcal{M}} \cap \{(t_1^{\mathcal{M}}, t_2^{\mathcal{M}}) \mid r(t_1, t_2) \text{ appears positively in } G\}.$$

The resulting relation satisfies G since we have only removed values which appear only negatively or not at all.

Then, let

$$r^+ = r^- \cup L^{\mathcal{M}}.$$

The resulting relation r^+ still satisfies G, since it includes r^- and for every negative occurrence $\neg r(t_1, t_2)$ in G, we have $L(t_1, t_2) \rightarrow r(t_1, t_2)$ in $\mathsf{D_r}[G]$, i.e. adding $L^{\mathcal{M}}$ cannot make G inconsistent. Moreover, since r^+ includes $L^{\mathcal{M}}$, it clearly satisfies $\forall x, y.\ L(x, y) \rightarrow r^+(x, y)$.

Finally, it also satisfies $\forall x, y.\ r^+(x, y) \rightarrow U(x, y)$: on the one hand, r^- is based on positive literal occurrences $r(t_1, t_2)$ in G, for which we have $r(t_1, t_2) \rightarrow U(t_1, t_2)$ in $\mathsf{D_r}[G]$, and on the other hand we have $\models_{\mathcal{T}} L(x, y) \rightarrow U(x, y)$. $\square$

Total binary relations.

Theorem 14. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, $L(x,y)$ and $U(x,y)$ Π-formulas with x,y as their only free variables. Consider the extension of $\mathcal{T}$ with a fresh binary predicate symbol $r(x,y)$ defined by*

$$\mathsf{T_r} = \{\ \forall x. \exists y.\ r(x,y)\ \}$$

and

$$\mathsf{D_r} = \left\{ \begin{array}{l} \forall x,y.\ L(x,y) \ \rightarrow\ r(x,y) \\ \forall x,y.\ r(x,y) \ \rightarrow\ U(x,y) \end{array} \right\}.$$

If both $\models_{\mathcal{T}} L(x,y) \rightarrow U(x,y)$ and $\models_{\mathcal{T}} \forall x.\ \exists y.\ U(x,y)$ hold, then the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{T_r} \cup \mathsf{D_r}$ satisfies the modified locality condition

$$\mathsf{T_r} \cup \mathsf{D_r} \cup G \models_{\mathcal{T}} \Box \iff \mathsf{T_r}[G] \cup \mathsf{D_r}[\mathsf{T_r}[G] \cup G] \cup G \models \Box.$$

Proof. "$\Leftarrow$": immediate, since $\mathsf{T_r}[G] \cup \mathsf{D_r}[\mathsf{T_r}[G] \cup G]$ consists of instances of $\mathsf{T_r}$ and $\mathsf{D_r}$.

"$\Rightarrow$": Let $G^* = \mathsf{T_r}[G] \cup \mathsf{D_r}[\mathsf{T_r}[G] \cup G] \cup G$ and suppose $\mathsf{T_r} \cup \mathsf{D_r} \cup G \models_{\mathcal{T}} \Box$, but there is a $\mathcal{T}$-model $\mathcal{M}$ for G^*. Similar to the proof of Theorem 4, let

$$r^- = r^{\mathcal{M}} \cap \{(t_1^{\mathcal{M}}, t_2^{\mathcal{M}}) \mid r(t_1,t_2) \text{ appears positively in } \mathsf{T_r}[G] \cup G\},$$

and

$$r^+ = r^- \cup L^{\mathcal{M}}.$$

By the same reasoning as above, the resulting relation satisfies $\mathsf{T_r}[G] \cup G$.

Then, let

$$U^- = U^{\mathcal{M}} \setminus \{(t_1^{\mathcal{M}}, t_2^{\mathcal{M}}) \mid r(t_1,t_2) \text{ appears in } \mathsf{T_r}[G] \cup G\}$$

be the restriction of $U^{\mathcal{M}}$ to values not represented by literals $r(t_1,t_2)$ in $\mathsf{T_r}[G] \cup G$, and finally

$$r^* = r^+ \cup U^-.$$

The resulting relation r^* satisfies $\mathsf{T_r}[G] \cup G$ because r^+ satisfies $\mathsf{T_r}[G] \cup G$ and U^- only adds literals that are not represented in $\mathsf{T_r}[G] \cup G$.

Moreover, it satisfies $\mathsf{T_r}$ because $\mathsf{T_r}[G]$ ensures that for every t_1 appearing in some literal $r(t_1,t_2)$ in $\mathsf{T_r}[G] \cup G$, there is some a with $r^-(t_1,a)$, and for values (x,y) not represented by literals in $\mathsf{T_r}[G] \cup G$, $r^+(x,y)$ is the same as $U^{\mathcal{M}}(x,y)$, with $\models_{\mathcal{T}} \forall x. \exists y. U(x,y)$ by assumption.

Since r^* includes $L^{\mathcal{M}}$, in particular it satisfies $\forall x,y.\ L(x,y) \rightarrow r^*(x,y)$.

Finally, r^* satisfies $\forall x,y.\ r^*(x,y) \rightarrow U(x,y)$: for positive literals $r(t_1,t_2)$ in $\mathsf{T_r}[G] \cup G$, we have $r(t_1,t_2) \rightarrow U(t_1,t_2)$ in G^*, for values (x,y) with $L^{\mathcal{M}}(x,y)$ it follows from $\models_{\mathcal{T}} L(x,y) \rightarrow U(x,y)$, and for values with $U^{\mathcal{M}}(x,y)$ it is immediate. $\Box$

Functional binary relations.

Theorem 15. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, $L(x,y)$ and $U(x,y)$ Π-formulas with x, y as their only free variables. Let $D(x)$ be a predicate in the base theory with $\models_{\mathcal{T}} D(x) \leftrightarrow (\exists y.\ L(x,y))$. Consider the extension of $\mathcal{T}$ with a fresh binary predicate symbol $r(x,y)$ defined by*

$$\mathsf{D}^{\mathsf{f}}_{\mathsf{r}} = \left\{ \begin{array}{ll} \forall x, y. & L(x,y) \rightarrow r(x,y) \\ \forall x, y. & r(x,y) \rightarrow U(x,y) \\ \forall x_1, x_2, y_1, y_2. & r(x_1,y_1) \wedge r(x_2,y_2) \wedge x_1 = x_2 \rightarrow y_1 = y_2 \\ \forall x, y. & r(x,y) \wedge D(x) \rightarrow L(x,y) \end{array} \right\}.$$

If $\models_{\mathcal{T}} L(x,y) \rightarrow U(x,y)$ and $\models_{\mathcal{T}} \forall x_1, x_2, y.\ L(x_1,y) \wedge L(x_2,y) \rightarrow x_1 = x_2$, then the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{D}^{\mathsf{f}}_{\mathsf{r}}$ satisfies (ELoc).

Proof. "$\Leftarrow$": immediate, since $\mathsf{D}_{\mathsf{r}}[G]$ consists of instances of D_{r}.

"$\Rightarrow$": Suppose $\mathsf{D}^{\mathsf{f}}_{\mathsf{r}} \cup G \models_{\mathcal{T}} \Box$, but there is a $\mathcal{T}$-model $\mathcal{M}$ for $\mathsf{D}^{\mathsf{f}}_{\mathsf{r}}[G] \cup G$. We use the same construction as in the proof of Theorem 4: Let

$$r^- = r^{\mathcal{M}} \cap \{(t_1^{\mathcal{M}}, t_2^{\mathcal{M}}) \mid r(t_1, t_2) \text{ appears positively in } G\}.$$

The resulting relation satisfies G since we have only removed values which appear only negatively or not at all.

Then, let

$$r^+ = r^- \cup L^{\mathcal{M}}.$$

By the same arguments as in the proof of Theorem 4, the resulting relation r^+ satisfies G, $\forall x, y.\ L(x,y) \rightarrow r^+(x,y)$ and $\forall x, y.\ r^+(x,y) \rightarrow U(x,y)$.

Additionally, it satisfies $\forall x_1, x_2, y_1, y_2.\ r^+(x_1,y_1) \wedge r^+(x_2,y_2) \wedge x_1 = x_2 \rightarrow y_1 = y_2$: 1) for every pair of positive occurrences $r(t_1,t_2), r(t_3,t_4)$ in G, we have $r(t_1,t_2) \wedge r(t_3,t_4) \wedge t_1 = t_3 \rightarrow t_2 = t_4$ in $\mathsf{D}^{\mathsf{f}}_{\mathsf{r}}[G]$, so r^+ is functional on the set of values defined by positive literal occurrences in G, 2) $L^{\mathcal{M}}$ is functional itself, so r^+ is functional on the set of values which are not defined by positive literal occurrences in G, and finally 3) if $r(t_1,t_2)$ is a postive literal occurrence in G, then $\mathsf{D}^{\mathsf{f}}_{\mathsf{r}}[G]$ also contains $r(t_1,t_2) \wedge D(t_1) \rightarrow L(t_1,t_2)$, which together with functionality of $L^{\mathcal{M}}$ ensures that adding $L^{\mathcal{M}}$ to r^- cannot destroy functionality. $\Box$

B Computing Locality-Ensuring Guards

Theorem 16. *If F is a locality-ensuring guard for $\mathcal{K}$ wrt. $\mathcal{T}$, then both*

$$\mathcal{T} \subseteq \mathcal{T} \cup F \cup \mathcal{K} \quad \text{and} \quad \mathcal{T} \subseteq \mathcal{T} \cup (F \rightarrow \mathcal{K})$$

satisfy (ELoc).

Proof. For $\mathcal{T} \subseteq \mathcal{T} \cup F \cup \mathcal{K}$, this is immediately clear from the definition. For $\mathcal{T} \subseteq \mathcal{T} \cup (F \rightarrow \mathcal{K})$, we can distinguish partial models which satisfy F and models which do not satisfy F: if $\mathcal{M} \models F$, by definition it can be completed to a total model of $\mathcal{T} \subseteq \mathcal{T} \cup F \cup \mathcal{K}$, which in particular satisfies $\mathcal{T} \cup (F \rightarrow \mathcal{K})$; if $\mathcal{M} \not\models F$, then every completion of $\mathcal{M}$ satisfies $(F \rightarrow \mathcal{K})$. □

Theorem 17. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, f a fresh function symbol and $\Phi(x, f(x))$ a $(\Sigma_0 \cup \{f\}, \mathsf{Pred})$-formula, where x is the only variable in $\Phi(x, f(x))$ and $f(x)$ is the only term in which x appears below f.*

If F is a Π_0-formula such that $\models_{\mathcal{T}} F \rightarrow \forall x \exists y.\ \Phi(x, y)$, then F is a locality-ensuring guard for $\forall x.\ \Phi(x, f(x))$ wrt. $\mathcal{T}$.

Proof. If $\models_{\mathcal{T}} F \rightarrow \forall x \exists y.\ \Phi(x, y)$, then every partial Π-model with completely defined Σ_0-functions which satisfies F must also satisfy $\forall x \exists y.\ \Phi(x, y)$. Since we know that all partial models satifying $\forall x \exists y.\ \Phi(x, y)$ can be completed to total models by Theorem 2, this in particular holds for all models satisfying F. □

C Piecewise Combination of Local Extensions

Lemma 2. *Let $\mathcal{T}$ be a Π_0-theory and $\mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ a set of augmented Π-clauses such that $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ satisfies $(\mathsf{Comp_w})$. For any Π_0-formula $\Phi(\overline{x})$, the extension $\mathcal{T} \subseteq \mathcal{T} \cup (\Phi(\overline{x}) \rightarrow \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ also satisfies $(\mathsf{Comp_w})$.*

Proof. Since $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ satisfies $(\mathsf{Comp_w})$, every partial model of $\mathcal{T} \cup \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ can be completed. For every partial model $\mathcal{M}$ of $\mathcal{T} \cup (\Phi(\overline{x}) \rightarrow \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$, there is a partial model $\mathcal{M}'$ of $\mathcal{T} \cup (\Phi(\overline{x}) \rightarrow \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ which is also a partial model of $\mathcal{T} \cup \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ (it can be obtained from $\mathcal{M}$ by letting $f^{\mathcal{M}'}(x)$ be defined only if $\mathcal{M} \models \Phi(x)$). By assumption, we can complete $\mathcal{M}'$ to a total model $\mathcal{M}''$ of $\mathcal{T} \cup \mathcal{K}(f(\overline{x}_1), \ldots, f(\overline{x}_n))$. Now, let

$$\overline{f}(x) = \begin{cases} f^{\mathcal{M}}(x), & \text{if defined} \\ f^{\mathcal{M}''}(x), & \text{if } f^{\mathcal{M}}(x) \text{ undefined and } \mathcal{M} \models \Phi(x) \\ \textit{arbitrary}, \text{ else} \end{cases}$$

□

C.1 Specifications over disjoint subsets

Theorem 18. *Let $\mathcal{T}$ be a Π_0-theory and consider Π_0-formulas $\Phi_1(x), \Phi_2(x)$ such that $\models_{\mathcal{T}} \neg(\Phi_1(x) \wedge \Phi_2(x))$. If $\mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ and $\mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m))$ are Π-formulas such that both $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^n \Phi_1(x_i) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n))$ and $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^m \Phi_2(x_i) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m))$ satisfy condition $(\mathsf{Comp_w})$, then for*

$$\mathcal{K} = \begin{array}{l} (\bigwedge_{i=1}^n \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \ldots, f(\overline{x}_n)) \\ \cup\ (\bigwedge_{i=1}^m \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \ldots, f(\overline{x}_m)), \end{array}$$

the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$ satisfies $(\mathsf{Comp_w})$ and is a local extension.

Proof. Consider a partial model $\mathcal{M}$ of $\mathcal{T} \cup \mathcal{K}$. Since $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n))$ and $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m))$ satisfy $(\mathsf{Comp_w})$, by Lemma 1 also $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^n \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n))$ and $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^m \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m))$ satisfy $(\mathsf{Comp_w})$.

Thus, we can complete $\mathcal{M}$ to a total model $\mathcal{M}_1$ of $\mathcal{T} \cup (\bigwedge_{i=1}^n \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n))$ and to a total model $\mathcal{M}_2$ of $\mathcal{T} \cup (\bigwedge_{i=1}^m \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m))$. Then, define

$$\overline{f}(x) = \begin{cases} f^{\mathcal{M}_1}(x), & \text{if } \mathcal{M} \models \Phi_1(x) \\ f^{\mathcal{M}_2}(x), & \text{if } \mathcal{M} \models \Phi_2(x) \\ \mathit{arbitrary}, \text{ else} \end{cases}$$

It is easy to see that the resulting structure satisfies $\mathcal{T} \cup \mathcal{K}$. □

C.2 Specifications over non-disjoint subsets

Theorem 19. *Let $\mathcal{T}$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ and consider Π_0-formulas $\Phi_1(x), \Phi_2(x)$ such that in every $\mathcal{T}$-model $\mathcal{M}$, the set $O = \{x \in M | \Phi_1(x) \wedge \Phi_2(x)\}$ is finite. Let furthermore T_0 be a set of Σ_0-terms such that in every such model $\mathcal{M}$, $O \subseteq \{t^{\mathcal{M}} \mid t \in T_0\}$.*

If $\mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n))$ and $\mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m))$ are sets of augmented Π-clauses such that both $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^n \Phi_1(x_i) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n))$ and $\mathcal{T} \subseteq \mathcal{T} \cup (\bigwedge_{i=1}^m \Phi_2(x_i) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m))$ satisfy $(\mathsf{Comp_w})$, then for

$$\mathcal{K} = \begin{array}{l} (\bigwedge_{i=1}^n \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n)) \\ \cup (\bigwedge_{i=1}^m \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m)) \end{array}$$

and $\Psi(T) = T \cup \{f(t) | t \in T_0\}$, the extension $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$ satisfies $(\mathsf{Comp_w}^{\Psi})$.

Proof. We prove that $(\mathsf{Comp_w}^{\Psi})$ is satisfied by $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{K}$: Let $\mathcal{M}$ be a partial model of $\mathcal{T} \cup \mathcal{K}$ in which $f^{\mathcal{M}}(a_i)$ is defined for a finite set of values $a_1, \dots, a_k$, where for every $t \in T_0$ we have $t^{\mathcal{M}} = a_i$ for some i. Since both $(\bigwedge_{i=1}^n \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n))$ and $(\bigwedge_{i=1}^m \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m))$ satisfy $(\mathsf{Comp_w})$ individually, there are completions f_1, f_2 of $f^{\mathcal{M}}$, satisfying these sets of axioms. We define another completion $\overline{f}$ of $f^{\mathcal{M}}$ by

$$\overline{f}(x) = \begin{cases} f^{\mathcal{M}}(x), & \text{if defined} \\ f_1(x), & \text{if } f^{\mathcal{M}}(x) \text{ undefined and } \mathcal{M} \models \Phi_1(x) \\ f_2(x), & \text{if } f^{\mathcal{M}}(x) \text{ undefined and } \mathcal{M} \models \Phi_2(x) \\ \mathit{arbitrary}, \text{ else} \end{cases}$$

We need to show that this completion satisfies both $(\bigwedge_{i=1}^n \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n))$ and $(\bigwedge_{i=1}^m \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m))$. If $a_1, \dots, a_n$ are values with $\Phi_1^{\mathcal{M}}(a_i)$ for each i, then $\overline{f}(a_i) = f_1(a_i)$, and since $\mathcal{M}_1 \models (\bigwedge_{i=1}^n \Phi_1(x_i)) \rightarrow \mathcal{K}_1(f(\overline{x}_1), \dots, f(\overline{x}_n))$, we know that this axiom will also be satisfied by $\overline{f}$. A similar argument holds for values $a_1, \dots, a_m$ with $\Phi_2^{\mathcal{M}}(a_i)$ and axiom $(\bigwedge_{i=1}^m \Phi_2(x_i)) \rightarrow \mathcal{K}_2(f(\overline{x}_1), \dots, f(\overline{x}_m))$. Finally, we know that for all values a_i with $\Phi_1^{\mathcal{M}}(a_i) \wedge \Phi_2^{\mathcal{M}}(a_i)$ we know that $f^{\mathcal{M}}(a_i)$ is defined, so $f_1(a_i) = f_2(a_1)$.

Thus, we know that for points in the intersection, both of the axioms hold. Clearly, for values a_i where both $\Phi_i^{\mathcal{M}}(a_i)$ are false, the axioms are satisfied trivially. $\square$

D Piecewise Approximation of Numerical Functions

Bounded slope.

Theorem 20. *Consider an extension of the theory of real arithmetic $\mathcal{T}_{\mathbb{R}}$ with a unary function symbol f and constants l, u satisfying*

$$\mathsf{BS}_{l,u}(f) = \left\{ \begin{array}{l} l \leq u \\ \forall\, x, y.\ x < y \rightarrow (y - x) \cdot l \leq f(y) - f(x) \leq (y - x) \cdot u \end{array} \right\}.$$

The extension $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_{\mathbb{R}} \cup \mathsf{BS}_{l,u}(f)$ satisfies (Comp).

Proof. Let $\mathcal{M}$ be a finite partial model of $\mathcal{T}_{\mathbb{R}} \cup \mathsf{BS}_{l,u}(f)$, where $f^{\mathcal{M}}(a_i)$ is defined for finitely many values $a_1, \dots, a_n \in \mathbb{R}$. Assume wlog. $a_i < a_{i+1}$ for $1 \leq i < n$. Then we can extend $\mathcal{M}$ to a total model $\mathcal{M}'$ of $\mathcal{T}_{\mathbb{R}} \cup \mathsf{BS}_{l,u}(f)$ by linear interpolation: the a_i separate $\mathbb{R}$ into $n+1$ intervals $(-\infty, a_1), (a_1, a_2), \dots, (a_{n-1}, a_n), (a_n, \infty)$ where $f^{\mathcal{M}}$ is undefined. We define the slopes c_i for each of these intervals by $c_0 = c_n = \frac{u^{\mathcal{M}} + l^{\mathcal{M}}}{2}$ and $c_i = \frac{f^{\mathcal{M}}(a_{i+1}) - f^{\mathcal{M}}(a_i)}{a_{i+1} - a_i}$ for $1 \leq i < n$. Then, define

$$f^{\mathcal{M}'}(x) = \begin{cases} f^{\mathcal{M}}(a_i) & \text{if } x = a_i \\ f^{\mathcal{M}}(a_1) - (a_1 - x) \cdot c_0 & \text{if } x \in (-\infty, a_1) \\ f^{\mathcal{M}}(a_i) + (x - a_i) \cdot c_i & \text{if } x \in (a_i, a_{i+1}), 1 \leq i < n \\ f^{\mathcal{M}}(a_n) + (x - a_n) \cdot c_n & \text{if } x \in (a_n, \infty) \end{cases}$$

Since $(a_{i+1} - a_i) \cdot l^{\mathcal{M}} \leq f^{\mathcal{M}}(a_{i+1}) - f^{\mathcal{M}}(a_i) \leq (a_{i+1} - a_i) \cdot u^{\mathcal{M}}$ holds for $1 \leq i \leq n$ (and $l^{\mathcal{M}} \leq u^{\mathcal{M}}$), we have $l^{\mathcal{M}} \leq c_i \leq u^{\mathcal{M}}$ for $0 \leq i \leq n$, i.e. the slope of the resulting function is bounded by l and u on every interval. Since the function $f^{\mathcal{M}'}$ is piecewise linear, the axiom of bounded slope will hold for any $x, y \in \mathbb{R}$. Thus, the model $\mathcal{M}'$ resulting from $\mathcal{M}$ by using the completion $f^{\mathcal{M}'}$ of $f^{\mathcal{M}}$ is a total model of $\mathcal{T}_{\mathbb{R}} \cup \mathsf{BS}_{l,u}(f)$. $\square$

We can obtain an easy corollary of this theorem by considering a strict bounded slope, and constraining l to be strictly less than u. Similarly, the proof from above still works if we allow both strict and non-strict bounds (as long as we have $l < u$ whenever one of the bounds is strict). Finally, we can also allow only lower or only upper bound.

Combination of bounded slope and bounded values.

Theorem 21. *Consider linear terms $m_i \cdot x + b_i$ and an extension of $\mathcal{T}_{\mathbb{R}}$ with a unary function symbol f satisfying*

$$\begin{aligned}\mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f) = \quad & \{l \le m_1,\ m_1 = m_2,\ m_2 \le u,\ b_1 \le b_2\} \\ & \cup \mathsf{BS}_{l,u}(f) \\ & \cup \mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)\end{aligned}$$

The extension $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_{\mathbb{R}} \cup \mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f)$ satisfies $(\mathsf{Comp_w})$.

Proof. We can prove locality by showing completability of finite partial models of $\mathcal{T}_{\mathbb{R}} \cup \mathcal{K}$ in the same way as in the proof of Theorem 20, except that we let $c_0 = c_n = m_1^{\mathcal{M}}$. Clearly, the resulting structure satisfies the ground constraints and $\mathsf{BS}_{l,u}(f)$. It also satisfies $\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)$ between a_1 and a_n because it is satisfied at all a_i and the points defined by linear interpolation satisfy it because the bounding terms are linear. For $x < a_0$, $\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)$ is satisfied because it is satisfied in a_0 and the slope on $(-\infty, a_0)$ is $m_1^{\mathcal{M}}$, i.e. the same as for the bounding terms. The same holds for $x > a_n$. Thus, the model $\mathcal{M}'$ resulting from $\mathcal{M}$ by using the completion $f^{\mathcal{M}'}$ of $f^{\mathcal{M}}$ is a total model of $\mathcal{T}_{\mathbb{R}} \cup \mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f)$. □

Restriction to an interval.

Corollary 1. *The extension of $\mathcal{T}_{\mathbb{R}}$ with*

$$\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)[c,d] = \left\{\begin{array}{ll} & m_1 \cdot c + b_1 \le m_2 \cdot c + b_2, \\ & m_1 \cdot d + b_1 \le m_2 \cdot d + b_2, \\ \forall\ x. & c \le x \le d\ \rightarrow\ m_1 \cdot x + b_1 \le f(x) \le m_2 \cdot x + b_2 \end{array}\right\}$$

satisfies $(\mathsf{Comp_w})$.

Proof. This is a direct consequence of Theorems 2 and 8 and the fact that $m_1 \cdot c + b_1 \le m_2 \cdot c + b_2 \wedge m_1 \cdot d + b_1 \le m_2 \cdot d + b_2$ is equivalent to $\forall x. \exists y.\ c \le x \le d\ \rightarrow\ m_1 \cdot x + b_1 \le y \le m_2 \cdot x + b_2$.

Corollary 2. *The extension of $\mathcal{T}_{\mathbb{R}}$ with*

$$\begin{aligned}\mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f)[c,d] = \quad & \{\quad m_1 \cdot c + b_1 + (d-c) \cdot l \le m_2 \cdot d + b_2 \\ & \quad \vee\ m_2 \cdot c + b_2 + (d-c) \cdot u \ge m_1 \cdot d + b_1\}, \\ & \cup \mathsf{BS}_{l,u}(f)[c,d] \\ & \cup \mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)[c,d]\end{aligned}$$

satisfies $(\mathsf{Comp_w})$.

Proof. For I_1, the ground constraint ensures that $BV^{\sim_4}_{t_1,t_2}(f)$ is satisfiable in b_1, there are no intersections of t_1 and t_2 in the interval, and there are values between l and u which are also between p and q. Thus, a function $\overline{f}$ which satisfies $BV^{\sim_4}_{t_1,t_2}(\overline{f})$ in b_1 and has a slope c with $l \sim_1 c \sim_1 u$ and $q \le c \le p$ will satisfy $\mathsf{BSV}^{\sim}_{l,u,t_1,t_2}(f(I_1))$. To complete a partial model of $\mathsf{BSV}^{\sim}_{l,u,t_1,t_2}(f(I_1))$, we use linear interpolation from b_1 to the left, and use a slope c as above for points smaller than the smallest defined point.

The ground constraint ensures that either the linear function f_1 with $f_1(c) = m_1 \cdot c + b_1$ and slope l will have $f_1(d) \leq m_2 \cdot d + b_2$, and thus satisfy $\mathsf{BS}_{l,u}(f_1)[c,d]$ and $\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f_1)[c,d]$, or the linear function f_2 with $f_2(c) = m_2 \cdot c + b_2$ and slope u will have $f_2(d) \geq m_1 \cdot d + b_1$, and thus satisfy $\mathsf{BS}l, u(f_2)[c,d]$ and $\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f_2)[c,d]$. That is, the constraint ensures that $\mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f)[c,d]$ is consistent. We can complete partial models of $\mathsf{BSV}_{l,u,m_1,b_1,m_2,b_2}(f)[c,d]$ by linear interpolation between neighbouring defined points, and using slopes c_1, c_2 at the ends of the interval such that $\mathsf{BS}_{l,u}(f)[c,d]$ and $\mathsf{LinBound}_{m_1,b_1,m_2,b_2}(f)[c,d]$ are satisfied. These slopes exist because of the argument above. $\square$

Extensions of Rationals and Integers Another corollary is obtained when considering the theory of linear rational arithmetic $\mathcal{T}_{\mathbb{Q}}$ instead of the reals. Since this theory does allow multiplication only with fixed values, we need l and u to be fixed rational values.

Corollary 3. *For fixed values of $l, u \in \mathbb{Q}$, the extension $\mathcal{T}_{\mathbb{Q}} \subseteq \mathcal{T}_{\mathbb{Q}} \cup \mathsf{BS}_{l,u}(f)$, satisfies* $(\mathsf{Comp_w})$.

Proof. The completion from Theorem 20 works also for the rational numbers, as long as l and u have rational values. One can easily verify that all the c_i will also be rational, and the resulting structure will be a total model of $\mathcal{T}_{\mathbb{Q}} \cup \mathsf{BS}_{l,u}(f)$. $\square$

Finally, we can consider the theory of integers $\mathcal{T}_{\mathbb{Z}}$. In this case, locality only holds for non-strict boundedness of the slope:

Corollary 4. *For fixed values of $l, u \in \mathbb{Z}$, the extension $\mathcal{T}_{\mathbb{Z}} \subseteq \mathcal{T}_{\mathbb{Z}} \cup \mathsf{BS}_{l,u}(f)$ satisfies* $(\mathsf{Comp_w})$.

Proof. For the integers, we can use a similar completion as above, except that we need to round $f^{\mathcal{M}'}(x)$ to the nearest integer value for every integer x. One can easily verify that the resulting structure will be a total model of $\mathcal{T}_{\mathbb{Z}} \cup \mathcal{K}$.

This does not work for slopes which are strictly between l and u because the rounding may destroy the strict ordering. $\square$