

Lecture 7

Loops and Recursion

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Loops

Loops: Example

Consider the set of variables $V = \{x, y\}$ and this program L :

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while ( $x > 0$ ) {  
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- ▶ $k = 1$: $x > 0 \wedge x' = x - y \wedge y' = y \wedge x' \leq 0$
- ▶ $k > 0$: $x > 0 \wedge x' = x - ky \wedge x' \leq 0 \wedge y' = y$

Solution:

$$(x \leq 0 \wedge x' = x \wedge y' = y) \vee$$
$$(\exists k. k > 0 \wedge x > 0 \wedge x' = x - ky \wedge x' \leq 0 \wedge y' = y)$$

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$$\exists k. k > 0 \wedge y > 0 \wedge x > 0 \wedge y \mid (x - x') \wedge k = (x - x')/y \wedge x' \leq 0 \wedge y' = y$$

Apply one-point rule to eliminate k

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$$\exists k. k > 0 \wedge y > 0 \wedge x > 0 \wedge y | (x - x') \wedge k = (x - x') / y \wedge x' \leq 0 \wedge y' = y$$

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$$((x - x') / y) > 0 \wedge y > 0 \wedge x > 0 \wedge y | (x - x') \wedge x' \leq 0 \wedge y' = y$$

which is also equivalent to simply

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Formula for Loop

Meaning of

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while (x > 0) {  
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is given by formula

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What happens if initially $x > 0 \wedge y \leq 0$?

- ▶ in the formula
- ▶ in the program

Integer Programs with Loops

Even if loop body is in Presburger arithmetic, the semantics of a loop need not be.

Integer programs with loops are Turing complete and can compute all computable functions.

Even if we cannot find Presburger arithmetic formula, we may be able to find

- ▶ a formula in a richer logic
- ▶ a property of the meaning of the loop
(e.g. formula for the superset)

To help with these tasks, we give mathematical semantics of loops

Useful concept for this is transitive closure: $r^* = \bigcup_{n \geq 0} r^n$
(We may or may not have a general formula for r^n or r^*)

A few more facts about relations

Let $r \subseteq S \times S$ and $\Delta = \{(x, x) \mid x \in S\}$. Then

$$\Delta \circ r = r = r \circ \Delta$$

We say that r is **reflexive** iff $\forall x \in S. (x, x) \in r$.

► equivalently, reflexivity means $\Delta \subseteq r$

Relation r is **transitive** iff

$$\forall x, y, z. ((x, y) \in r \wedge (y, z) \in r \rightarrow (x, z) \in r)$$

which is the same as saying $r \circ r \subseteq r$

Transitive Closure of a Relation

$r \subseteq S \times S$. Define $r^0 = \Delta$ and $r^{n+1} = r \circ r^n$. Then $(x_0, x_n) \in r^n$ iff $\exists x_1, \dots, x_{n-1}$ such that $(x_i, x_{i+1}) \in r$ for $0 \leq i \leq n-1$. Define reflexive transitive closure of r by

$$r^* = \bigcup_{n \geq 0} r^n$$

Properties that follow from the definition:

- ▶ $(x_0, x_n) \in r^*$ iff there exists $n \geq 0$ and $\exists x_1, \dots, x_{n-1}$ such that $(x_i, x_{i+1}) \in r$ for $0 \leq i \leq n-1$ (a path in the graph r)
- ▶ r^* is a reflexive and transitive relation
- ▶ If s is a reflexive transitive relation and $r \subseteq s$, then $r^* \subseteq s$
 - ▶ r^* is the smallest reflexive transitive relation containing r
- ▶ $(r^{-1})^* = (r^*)^{-1}$
- ▶ $r_1 \subseteq r_2$ implies $r_1^* \subseteq r_2^*$
- ▶ $r^* = \Delta \cup (r \circ r^*)$ and, likewise, $r^* = \Delta \cup (r^* \circ r)$

Towards meaning of loops: unfolding

Loops can describe an infinite number of basic paths
(for a larger input, program takes a longer path)

Consider loop

$$L \equiv \mathbf{while}(F)c$$

We would like to have

$$\begin{aligned} L &\equiv \mathbf{if}(F)(c; L) \\ &\equiv \mathbf{if}(F)(c; \mathbf{if}(F)(c; L)) \end{aligned}$$

For $r_L = \rho(L)$, $r_c = \rho(c)$, $\Delta_1 = \Delta_{\tilde{F}}$, $\Delta_2 = \Delta_{\sim\tilde{F}}$ we have

$$\begin{aligned} r_L &= (\Delta_1 \circ r_c \circ r_L) \cup \Delta_2 \\ &= (\Delta_1 \circ r_c \circ ((\Delta_1 \circ r_c \circ r_L) \cup \Delta_2)) \cup \Delta_2 \\ &= \Delta_2 \cup \\ &\quad (\Delta_1 \circ r_c) \circ \Delta_2 \cup \\ &\quad (\Delta_1 \circ r_c)^2 \circ r_L \end{aligned}$$

Unfolding Loops

$$\begin{aligned} r_L = & \Delta_2 \cup \\ & (\Delta_1 \circ r_c) \circ \Delta_2 \cup \\ & (\Delta_1 \circ r_c)^2 \circ \Delta_2 \cup \\ & (\Delta_1 \circ r_c)^3 \circ r_L \end{aligned}$$

We prove by induction that for every $n \geq 0$,

$$(\Delta_1 \circ r_c)^n \circ \Delta_2 \subseteq r_L$$

So, $\bigcup_{n \geq 0} ((\Delta_1 \circ r_c)^n \circ \Delta_2) \subseteq r_L$, that is

$$\left(\bigcup_{n \geq 0} (\Delta_1 \circ r_c)^n \right) \circ \Delta_2 \subseteq r_L$$

We do not wish to have unnecessary elements in relation, so we try

$$r_L = (\Delta_1 \circ r_c)^* \circ \Delta_2$$

and this does satisfy $r_L = (\Delta_1 \circ r_c \circ r_L) \cup \Delta_2$, so we define

$$\rho(\mathbf{while}(F)c) = (\Delta_{\tilde{F}} \circ \rho(c))^* \circ \Delta_{\sim \tilde{F}}$$

Why loop semantics satisfies the condition

We defined

$$r_L = (\Delta_1 \circ r_c)^* \circ \Delta_2$$

Show that $(\Delta_1 \circ r_c \circ r_L) \cup \Delta_2$ equals r_L , as we expect from recursive definition of a while loop.

Why loop semantics satisfies the condition

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Show that $(\Delta_1 \circ r_c \circ r_L) \cup \Delta_2$ equals r_L , as we expect from recursive definition of a while loop.

Using property $r^* = \Delta \cup r \circ r^*$ we have

$$\begin{aligned} r_L &= (\Delta_1 \circ r_c)^* \circ \Delta_2 \\ &= [\Delta \cup (\Delta_1 \circ r_c \circ (\Delta_1 \circ r_c)^*)] \circ \Delta_2 \\ &= \Delta_2 \cup [\Delta_1 \circ r_c \circ (\Delta_1 \circ r_c)^* \circ \Delta_2] \\ &= \Delta_2 \cup \Delta_1 \circ r_c \circ r_L \end{aligned}$$

Using Loop Semantics in Example

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is:

$$(\Delta_{x \gtrsim 0} \circ \rho(x = x - y))^* \circ \Delta_{\neg(x \gtrsim 0)}$$

Compute each relation:

$$\begin{aligned}\Delta_{x \gtrsim 0} &= \{((x, y), (x, y)) \mid x > 0\} \\ \Delta_{\neg(x \gtrsim 0)} &= \{((x, y), (x, y)) \mid x \leq 0\} \\ \rho(x = x - y) &= \{((x, y), (x - y, y)) \mid x, y \in \mathbb{Z}\} \\ \Delta_{x \gtrsim 0} \circ \rho(x = x - y) &= \\ (\Delta_{x \gtrsim 0} \circ \rho(x = x - y))^k &= \\ (\Delta_{x \gtrsim 0} \circ \rho(x = x - y))^* &= \\ \rho(L) &= \end{aligned}$$

Semantics of a Program with Loop

Compute and simplify relation for this program:

$x = 0$	
while ($y > 0$) {	$\rho(x = 0) \circ$
$x = x + y$	$(\Delta_{y \gtrsim 0} \circ \rho(x = x + y; y = y - 1))^* \circ$
$y = y - 1$	$\Delta_{y \lesssim 0}$
}	

Approximate Semantics of Loops

Instead of computing exact semantics, it can be sufficient to compute approximate semantics.

Observation: $r_1 \subseteq r_2 \rightarrow r_1^* \subseteq r_2^*$

Suppose we only wish to show that the semantics satisfies $y' \leq y$

$x = 0$

while ($y > 0$) {

$x = x + y$

$y = y - 1$

}

$\rho(x = 0) \circ$

$(\Delta_{y \gtrsim 0} \circ \rho(x = x + y; y = y - 1))^* \circ$

$\Delta_{y \lesssim 0}$

Recursion

Example of Recursion

For simplicity assume no parameters
(we can simulate them using global variables)

<pre>def f = if (x > 0) { if (x % 2 == 0) { x = x / 2; f; y = y * 2 } else { x = x - 1; y = y + x; f } } }</pre>	$E(r_f) =$ $\Delta_{x \succ 0} \circ ($ $(\Delta_{x \% 2 = 0} \circ$ $\rho(x = x/2) \circ$ $r_f \circ$ $\rho(y = y * 2))$ \cup $(\Delta_{x \% 2 \neq 0} \circ$ $\rho(x = x - 1) \circ$ $\rho(y = y + x) \circ$ $r_f)$ $\cup \Delta_{x \preceq 0}$
---	---

Assume recursive function call denotes some relation r_f

Need to find relation r_f such that $r_f = E(r_f)$

Simpler Example of Recursion

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    f  
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$$E(r_f) = (\Delta_{x \gtrsim 0} \circ (\rho(x = x - 1) \circ r_f \circ \rho(y = y + 2))) \cup \Delta_{x \lesssim 0}$$

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What is $E(E(\emptyset))$?

$E^k(\emptyset)$?

Sequence of Bounded Recursions

Consider the sequence of relations $r_0 = \emptyset$, $r_k = E^k(\emptyset)$.

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- ▶ $r_0 \subseteq r_1$ because $\emptyset \subseteq \dots$. Moreover, we showed several lectures before that E is monotonic
- ▶ from here it follows $r_1 \subseteq r_2$ and, by induction, $r_k \subseteq r_{k+1}$

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Define

$$s = \bigcup_{k \geq 0} r_k$$

Then

$$E(s) = E\left(\bigcup_{k \geq 0} r_k\right) \stackrel{?}{=} \bigcup_{k \geq 0} E(r_k) = \bigcup_{k \geq 0} r_{k+1} = \bigcup_{k \geq 1} r_k = \emptyset \cup \bigcup_{k \geq 1} r_k = s$$

If $E(s) = s$ we say s is a **fixed point (fixpoint)** of function E

Meaning of a recursive program is fixpoint of the corresponding E

Exercise with Fixpoints of Real Functions

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$$x_{1/2} = \frac{2 \pm 4}{2}$$

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2. Compute the fixpoint that is smaller than all other fixpoints
 $x_1 = -1$ is the smallest.

Union of Finite Unfoldings is Least Fixpoint

C - a collection (set) of sets (e.g. sets of pairs, i.e. relations)

$E : C \rightarrow C$ such that for $r_0 \subseteq r_1 \subseteq r_2 \dots$

we have

$$E\left(\bigcup_i r_i\right) = \bigcup_i E(r_i)$$

(This holds when E is given in terms of \circ and \cup .) Then

$s = \bigcup_i E^i(\emptyset)$ is such that

1. $E(s) = s$ (we have shown this)
2. if r is such that $E(r) \subseteq r$ (special case: if $E(r) = r$), then $s \subseteq r$ (we show this next)

Showing that the Fixpoint is Least

$$s = \bigcup_i E^i(\emptyset)$$

Now take any r such that $E(r) \subseteq r$.

We will show $s \subseteq r$, that is

$$\bigcup_i E^i(\emptyset) \subseteq r \quad (*)$$

This means showing $E^i(\emptyset) \subseteq r$, for every i . For $i = 0$ this is just $\emptyset \subseteq r$. We proceed by induction. If $E^i(\emptyset) \subseteq r$, then by monotonicity of E

$$E(E^i(\emptyset)) \subseteq E(r) \subseteq r$$

This completes the proof of (*)

Summary: Least Fixpoint as Meaning of Recursion

A recursive program is a recursive definition of a relation $E(r) = r$

We define the intended meaning as $s = \bigcup_{i \geq 0} E^i(\emptyset)$, which satisfies $E(s) = s$ and also is the least among all relations r such that $E(r) \subseteq r$ (and therefore, also the least among those r for which $E(r) = r$)

We picked **least** fixpoint, so if the execution cannot terminate on a state x , then there is no x' such that $(x, x') \in s$

- ▶ Let q be a program that never terminates, then
- ▶ $\rho(q) = \emptyset$ and $\rho(c \sqcap q) = \rho(c) \cup \emptyset = \rho(c)$
- ▶ also, $\rho(q) = \rho(\Delta_\emptyset)$ (assume(false))