Lecture 7
More Recursion. Bounded Model Checking

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Summary: Least Fixpoint as Meaning of Recursion

A recursive program is a recursive definition of a relation $E(r) = r$

We define the intended meaning as $s = \bigcup_{i \geq 0} E(\emptyset)$, which satisfies $E(s) = s$ and also is the least among all relations r such that $E(r) \subseteq r$ (therefore, also the least among r for which $E(r) = r$)

We picked **least** fixpoint, so if the execution cannot terminate on a state x, then there is no x' such that $(x, x') \in s$.
This model is simple (just relations on states) though it has some limitations: let q be a program that never terminates, then

- $\rho(q) = \emptyset$ and $\rho(c \,\square\, q) = \rho(c) \cup \emptyset = \rho(c)$
  (we cannot observe optional non-termination in this model)
- also, $\rho(q) = \rho(\Delta_\emptyset)$ (assume(false)), so the absence of results due to path conditions and infinite loop are represented in the same way

Alternative: error states for non-termination (we will not pursue)

Procedure Meaning is the Least Relation

def f =
  if (x > 0) {
    x = x − 1
    f
    y = y + 2
  }

$$
\begin{aligned}
E(r_f) = \ & \Big(\Delta_{x \tilde{>} 0} \circ \big(\\
& \quad \rho(x = x - 1) \circ \\
& \quad r_f \circ \\
& \quad \rho(y = y + 2)) \\
& \Big) \cup \Delta_{x \tilde{\leq} 0}
\end{aligned}
$$

What does it mean that $E(r) \subseteq r$?

Procedure Meaning is the Least Relation

```
def f =
  if (x > 0) {
    x = x − 1
    f
    y = y + 2
  }
```

$$E(r_f) = \Big(\Delta_{x \tilde{>} 0} \circ \big(\rho(x = x - 1) \circ r_f \circ \rho(y = y + 2) \big) \Big) \cup \Delta_{x \tilde{\leq} 0}$$

What does it mean that $E(r) \subseteq r$?
Plugging r instead of the recursive call results in something that conforms to r

Justifies modular reasoning for recursive functions

To prove that recursive procedure with body E satisfies specification r, show

- $E(r) \subseteq r$
- then because procedure meaning s is least, $s \subseteq r$

Proving that recursive function meets specification

Prove that if s is the relation denoting the recursive function below, then

$$((x, y), (x', y')) \in s \rightarrow y' \geq y$$

```
def f =
  if (x > 0) {
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```

$$\begin{aligned} E(r_f) = \ & (\Delta_{x \tilde{>} 0} \circ (\\ & \quad \rho(x = x - 1) \circ \\ & \quad r_f \circ \\ & \quad \rho(y = y + 2)) \\ &) \cup \Delta_{x \tilde{\leq} 0} \end{aligned}$$

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Solution: let specification relation be $q = \{((x, y), (x', y')) \mid y' \geq y\}$

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Solution: let specification relation be $q = \{((x, y), (x', y')) \mid y' \geq y\}$
Prove $E(q) \subseteq q$ - given by a quantifier-free formula

Formula for Checking Specification

```
def f =
  if (x > 0) {
    x = x − 1
    f
    y = y + 2
  }
```

Specification: $q = \{((x,y),(x',y')) \mid y' \geq y\}$
Formula to prove, generated by representing $E(q) \subseteq q$:

$$\begin{array}{l}\big[(x > 0 \land x_1 = x - 1 \land y_1 = y \land y_2 \geq y_1 \land y' = y_2 + 2) \\ \lor(\neg(x > 0) \land x' = x \land y' = y)\big] \;\rightarrow\; y' \geq y\end{array}$$

- Because q appears as $E(q)$ and q, the condition appears twice.
- Proving $f \subseteq q$ by $E(q) \subseteq q$ is always sound, whether or not function f terminates; the meaning of f talks only about properties of terminating executions (relations can be partial)

Multiple Procedures: Functions on Pairs of Relations

Two mutually recursive procedures $r_1 = E_1(r_1, r_2), \quad r_2 = E_2(r_1, r_2)$
We extend the approach to work on pairs of relations:

$$(r_1, r_2) = (E_1(r_1, r_2), E_2(r_1, r_2))$$

Define $\bar{E}(r_1, r_2) = (E_1(r_1, r_2), E_2(r_1, r_2))$, let $\bar{r} = (r_1, r_2)$. We define semantics of procedures as the least solution of

$$\bar{E}(\bar{r}) = \bar{r}$$

where $(r_1, r_2) \sqsubseteq (r_1', r_2')$ means $r_1 \subseteq r_1'$ and $r_2 \subseteq r_2'$
Even though pairs of relations are not sets but pairs of sets, we can define set-like operations on them, e.g.

$$(r_1, r_2) \sqcup (r_1', r_2') = (r_1 \cup r_1',\ r_2 \cup r_2')$$

The entire theory works when we have a partial order $\sqsubseteq$ with some "good properties". (**Lattice** elements are a generalization of sets.)

Multiple Procedures: Least Fixedpoint and Consequences

Two mutually recursive procedures $r_1 = E_1(r_1, r_2), \quad r_2 = E_2(r_1, r_2)$
For $E(r_1, r_2) = (E_1(r_1, r_2), E_2(r_1, r_2))$, semantics is

$$(s_1, s_2) = \bigsqcup_{i \geq 0} \bar{E}^i(\emptyset, \emptyset)$$

It follows that for any c_1, c_2 if

$$E_1(c_1, c_2) \subseteq c_1 \text{ and } E_2(c_1, c_2) \subseteq c_2$$

then $s_1 \subseteq c_1$ and $s_2 \subseteq c_2$.

Induction-like principle: To prove that mutually recursive relations satisfy two contracts, prove those contracts for the relation body definitions in which recursive calls are replaced by those contracts.

Replacing Calls by Contracts: Example

```
def r1 = {
  if (x % 2 == 1) {
    x = x − 1
  }
  y = y + 2
  r2
} ensuring(y > old(y))
```

```
def r2 = {
  if (x != 0) {
    x = x / 2
    r1
  }
} ensuring(y >= old(y))
```

Replacing Calls by Contracts: Example

```
def r1 = {
  if (x % 2 == 1) {
    x = x − 1
  }
  y = y + 2
  r2
} ensuring(y > old(y))
```

```
def r2 = {
  if (x != 0) {
    x = x / 2
    r1
  }
} ensuring(y >= old(y))
```

Reduces to checking these two non-recursive procedures:

```
def r1 = {
  if (x % 2 == 1) {
    x = x − 1
  }
  y = y + 2
  { val x0 = x; y0 = y
    havoc(x,y)
    assume(y >= y0) }
} ensuring(y > old(y))
```

```
def r2 = {
  if (x != 0) {
    x = x / 2
    val x0 = x; y0 = y
    havoc(x,y)
    assume(y > y0)
  }
} ensuring(y >= old(y))
```

Bounded Model Checking and k-Induction

Concrete program semantics and verification

For each program there is a (monotonic, ω-continuous) function $F : C^n \to C^n$ such that

$$\bar{c}_* = \bigcup_{n \geq 0} F^n(\emptyset, \ldots, \emptyset)$$

describes the set of reachable states for each program point.
(Safety) verification can be stated as saying that the semantics remains within the set of good states G, that is $c_* \subseteq G$, or

$$\left(\bigcup_{n \geq 0} F^n(\emptyset, \ldots, \emptyset) \right) \subseteq G$$

which is equivalent to

$$\forall n.\ F^n(\emptyset, \ldots, \emptyset) \subseteq G$$

Unfolding for Counterexamples: Bounded Model Checking

$$\forall n.\ F^n(\emptyset, \ldots, \emptyset) \subseteq G$$

The above condition is false iff there exists k and $\bar{c} \in C^n$ such that

$$\bar{c} \in F^k(\emptyset, \ldots, \emptyset) \land \bar{c} \notin G$$

For a fixed k this can often be expressed as a quantifier-free formula.
Example: replace a loop $([c]s) * [!c]$ with finite unrolding $([c]s)^k[!c]$
Specifically, for $n = 1$, $S = \mathbb{Z}^2$, $C = 2^S$, and $F : C \to C$ describes the program: x=0;while(*)x=x+y

$$F(B) = \{(x, y) \mid x = 0\} \cup \{(x + y, y) \mid (x, y) \in B\}$$

We have $F(\emptyset) = \{(x, y) \mid x = 0\} = \{(0, y) \mid y \in \mathbb{Z}\}$

$$F^2(\emptyset) = \{(0, y) \mid y \in \mathbb{Z}\} \cup \{(y, y) \mid y \in \mathbb{Z}\}$$

$$F^3(\emptyset) = \{(x, y) \mid x = 0 \lor x = y \lor x = 2 * y\}$$

Formula for Bounded Model Checking

Let $P_B(x, y)$ be a formula in Presburger arithmetic such that $B = \{(x, y) \mid P_B(x, y)\}$ then the formula

$$x = 0 \vee (\exists x_0, y_0 . x = x_0 + y_0 \wedge y = y_0 \wedge P_B(x_0, y_0))$$

describes $F(B)$. Suppose the set $F^k(B)$ can be described by a PA formula P_k. If G is given by a formula P_G then the program can reach error in k steps iff

$$P_k \wedge \neg P_G$$

is satisfiable.
Suppose P_G is $x \leq y$. For $k = 3$ we obtain

$$(x = 0 \vee x = y \vee x = 2 * y) \wedge \neg(x \leq y)$$

By checking satisfiability of the formula we obtain counterexample values $x = -1, y = -2$.

Bounded Model Checking Algorithm

$B = \emptyset$
while $(*)$ {
  checksat(!($B \subseteq G$)) **match**
    case Assignment(v) **=>** **return** Counterexample(v)
    case Unsat **=>**
      $B' = F(B)$
      if $(B' \subseteq B)$ **return** Valid
      else $B = B'$
}

Good properties

- subsumes testing up to given depth for all possible initial states
- for a buggy program k, can be small, tools can find many bugs fast
- a semi-decision procedure for finding all error inputs

Bounded Model Checking is Bounded

Bad properties

- can prove correctness only if $F^{n+1}(\emptyset) = F^n(\emptyset)$ for a finite n
- errors after initializations of long arrays require unfolding for large n. This program requires unfolding past all loop iterations, even if the property does not depend on the loop:

```
i = 0
z = 0
while (i < 1000) {
  a(i) = 0
}
y = 1/z
```

- For large k formula F^k becomes large, so deep bugs are hard to find

Unfolding for Proving Correctness: k-Induction

$$\text{Goal:} \quad \forall n.\ F^n(\emptyset, \dots, \emptyset) \subseteq G \tag{1}$$

Suppose that, for some $k \geq 1$

$$F^k(G) \subseteq G \tag{2}$$

By induction on p, for every $p \geq 1$,

$$F^{pk}(G) \subseteq G$$

By monotonicity of F, if $n \leq pk$ then

$$F^n(\bar{\emptyset}) \subseteq F^{pk}(\bar{\emptyset}) \subseteq F^{pk}(G) \subseteq G$$

Therefore, (1) holds.
Algorithm: check (2) for increasing $k \in \{1, 2, \dots\}$

Summary: Using F^k for Proofs and Counterexamples

Exact semantics is: $\bigcup_{n\geq 0} F^n(\bar{\emptyset})$
Specification is G
If for some k:

- $\neg(F^k(\bar{\emptyset}) \subseteq G)$ then we prove that specification **does not** hold (and there is a "k-step" execution in $G \subseteq F^k(\bar{\emptyset})$ showing this)
- $F^k(G) \subseteq G$, then we prove that specification **holds** by showing that it holds in all base cases up to k and assuming it holds for all recursive steps at depth k and deeper (k-induction)

Least fixedpoint of F^k is the same as least fixedpoint of F: $F^i(\bar{\emptyset}) \subseteq F^{ki}(\bar{\emptyset})$, so $\bigcup$ gives same result as sequences are monotonic.
Each F^k defines the program with the meaning same as F but syntactically more obvious as k grows and we unfold more.

k-induction Algorithm

Prove or find counterexample for:

$$\forall n.\ F^n(\emptyset, \ldots, \emptyset) \subseteq G$$

$Fk = F$
while $(*)$ {
  checksat$(!(Fk(G) \subseteq G))$ **match**
    case Unsat **=>** **return** Valid
    case Assignment(v0) **=>**
      checksat$(!(Fk(\emptyset) \subseteq G))$ **match**
        case Assignment(v) **=>** **return** Counterexample(v)
        case Unsat **=>** $Fk = Fk \circ F'$ // unfold one more
}

$F'(c)$ can be $F(c)$ or, thanks to previous checks, $F(c) \cap G$
Save work: preserve solver state in checksats across different k
Lucky test: if $(!(lfp(F)(initState(v0)) \subseteq G))$ return Counterexample(v0)

Divergence in k-Induction

$Fk = F$
while $(*)$ {
  checksat(!($Fk(G) \subseteq G$)) **match**
    case Unsat => **return** Valid
    case Assignment(v0) =>
      checksat(!($Fk(\emptyset) \subseteq G$)) **match**
        case Assignment(v) => **return** Counterexample(v)
        case Unsat => $Fk = Fk \circ F'$ // unfold one more
}

Subsumes bounded model checking, so finds all counterexamples
Often cannot find proofs when $lfp(F) \subseteq G$. Then G may be too weak to be inductive, $(F')^n(G)$ may remain too weak:

$$F^n(\bar{\emptyset}) \subseteq lfp(F) \subseteq (F')^n(G) \subseteq F^n(G)$$

Need weakening of $F^n(\emptyset)$ or strengthening of $(F')^n(G)$

Approximate Postconditions

Suppose we did not find counterexample yet and we have sequence

$$c_0 \subseteq c_1 \subseteq \ldots c_k \subseteq G$$

where $c_i = F^i(\bar{\emptyset})$, so $F(c_i) = c_{i+1}$
Instead of simply increasing k, we try to obtain larger values by finding another sequence a_i satisfying $a_i \subseteq a_{i+1}$ and

$$F(a_i) \subseteq a_{i+1}$$

for $0 \leq i \leq k$, and with $a_k \subseteq G$.
$c_0 \subseteq a_0$ and, by induction, $c_i \subseteq a_i$
If $a_{i+1} = a_i$ for some i, then $F(a_i) = a_i$ so

$$lfp(F) \subseteq a_i \subseteq a_k \subseteq G$$

so we have proven $lfp(F) \subseteq G$, i.e., program satisfies spec.
We can also dually require $a_{i-1} \subseteq F(a_i)$, ensuring $a_i \subseteq F^{k-i}(G)$.

Abstract Interpretation

A Method for Constructing Inductive Invariants

Basic idea of abstract interpretation

Abstract interpretation is a way to infer properties of program computations.
Consider the assignment: $z = x + y$.

Interpreter:

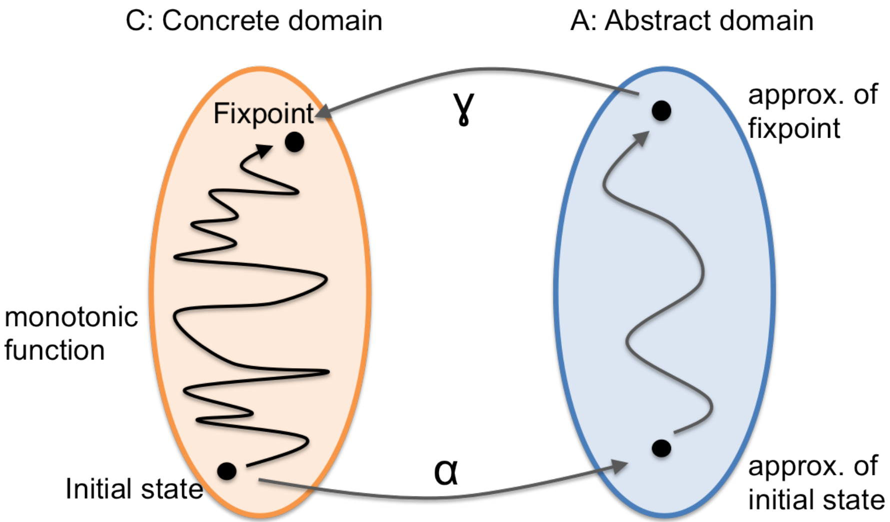
$$\begin{pmatrix} x : 10 \\ y : -2 \\ z : 3 \end{pmatrix} \xrightarrow{z=x+y} \begin{pmatrix} x : 10 \\ y : -2 \\ z : 8 \end{pmatrix}$$

Abstract interpreter:

$$\begin{pmatrix} x \in & [0,10] \\ y \in & [-5,5] \\ z \in & [0,10] \end{pmatrix} \xrightarrow{z=x+y} \begin{pmatrix} x \in & [0,10] \\ y \in & [-5,5] \\ z \in & [-5,15] \end{pmatrix}$$

Each abstract state represents a set of concrete states

Program Meaning is a Fixpoint. We Approximate It.



γ maps abstract states to concrete states

Proving through Fixpoints of Approximate Functions

Meaning of a program (e.g. a relation) is a least fixpoint of F.
Given specification s, the goal is to prove $\mathbf{lfp(F)} \subseteq \mathbf{s}$

- if $F(s) \subseteq s$ then $lfp(F) \subseteq s$ and we are done
- $lfp(F) = \bigcup_{k \geq 0} F^k(\emptyset)$, but that is too hard to compute because it is infinite union unless, by some luck, $F^{n+1}(\emptyset) = F^n$ for some n

Instead, we search for an inductive strengthening of s: find s' such that:

- $F(s') \subseteq s'$ (s' is inductive). If so, theorem says $lfp(F) \subseteq s'$
- $s' \subseteq s$ (s' implies the desired specification). Then $lfp(F) \subseteq s' \subseteq s$

How to find s'? Iterating F is hard, so we try some simpler function $F_\#$

- suppose $F_\#$ is *approximation*: $F(r) \subseteq F_\#(r)$ for all r
- we can find s' such that: $F_\#(s') \subseteq s'$ (e.g. $s' = F_\#^{n+1}(\emptyset) = F_\#^n(\emptyset)$)

Then: $F(s') \subseteq F_\#(s') \subseteq s' \subseteq s$
Abstract interpretation: automatically construct $F_\#$ from F (and sometimes s)

Programs as control-flow graphs

```
//a
i = 0;
    //b
while (i < 10) {
  //d
  if (i > 1)
    //e
    i = i + 3;
  else
    //f
    i = i + 2;
  //g
}
//c
```

One possible corresponding control-flow graph is:

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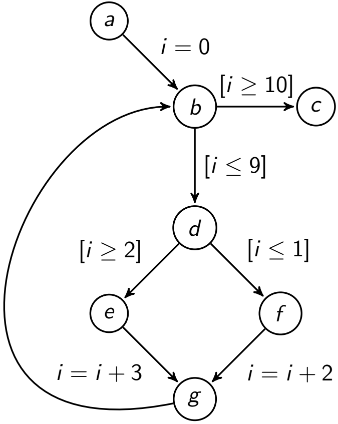

Sets of states at each program point

Suppose that

- program state is given by the value of the integer variable i
- initially, it is possible that i has any value

Compute the set of states at each vertex in the CFG.

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//a
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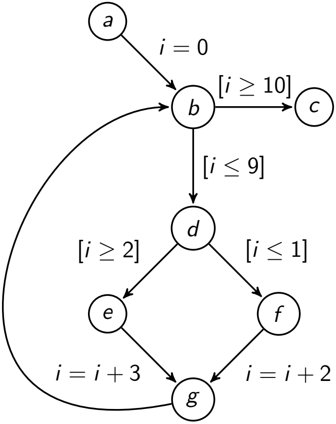

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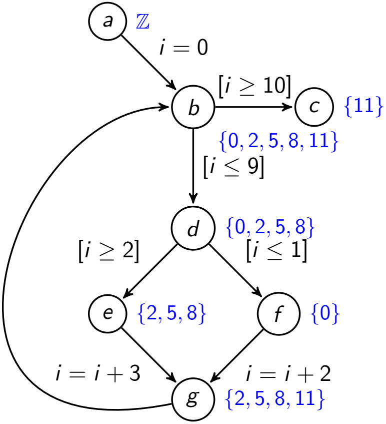


Sets of states at each program point

Running the Program

One way to describe the set of states for each program point: for each initial state, run the CFG with this state and insert the modified states at appropriate points.

Reachable States as A Set of Recursive Equations

If c is the label on the edge of the graph, let $\rho(c)$ denotes the relation between initial and final state that describes the meaning of statement. For example,

$$\begin{aligned}
&\rho(i = 0) = \{(i, i') \mid i' = 0\}\\
&\rho(i = i + 2) = \{(i, i') \mid i' = i + 2\}\\
&\rho(i = i + 3) = \{(i, i') \mid i' = i + 3\}\\
&\rho([i < 10]) = \{(i, i') \mid i' = i \wedge i < 10\}
\end{aligned}$$

Sets of states at each program point

We will write $T(S, c)$ (transfer function) for the image of set S under relation $\rho(c)$. For example,

$$T(\{10, 15, 20\}, i = i + 2) = \{12, 17, 22\}$$

General definition can be given using the notion of strongest postcondition

$$T(S, c) = sp(S, \rho(c))$$

If [p] is a condition (assume(p), coming from 'if' or 'while') then

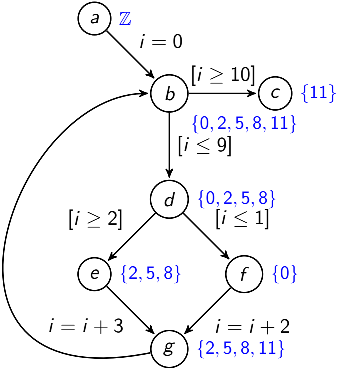
$$T(S, [p]) = \{x \in S \mid p\}$$

If an edge has no label, we denote it skip. So, $T(S, skip) = S$.

Reachable States as A Set of Recursive Equations

Now we can describe the meaning of our program using recursive equations:

$$
\begin{aligned}
S(a) &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
S(b) &= T(S(a), i = 0) \cup T(S(g), \mathit{skip}) \\
S(c) &= T(S(b), [\neg(i < 10)]) \\
S(d) &= T(S(b), [i < 10]) \\
S(e) &= T(S(d), [i > 1]) \\
S(f) &= T(S(d), [\neg(i > 1)]) \\
S(g) &= T(S(e), i = i + 3) \\
&\quad \cup T(S(f), i = i + 2)
\end{aligned}
$$



Our solution is the unique **least** solution of these equations

The problem:

These exact equations are as difficult to compute as running the program on all possible input states. Instead, we consider **approximate** descriptions of these sets of states.

A Large Analysis Domain: All Intervals of Integers

For every $L, U \in \mathbb{Z}$ interval:

$$\{x \mid L \leq x \wedge x \leq U\}$$

This domain has infinitely many elements, but is already an approximation of all possible sets of integers.

Smaller Domain: Finitely Many Intervals

We continue with the same example but instead of allowing to denote all possible sets, we will allow sets represented by expressions

$$[L, U]$$

which denote the set $\{x \mid L \leq x \wedge x \leq U\}$.
Example: $[0, 127]$ denotes integers between 0 and 127.

- L is the lower bound and U is the upper bound, with $L \leq U$.
- to ensure that we have only a few elements, we let

$$L, U \in \{\text{MININT}, -128, 1, 0, 1, 127, \text{MAXINT}\}$$

- $[\text{MININT}, \text{MAXINT}]$ denotes all possible integers, denote it $\top$
- instead of writing $[1, 0]$ and other empty sets, we will always write $\perp$

So, we only work with a finite number of sets $1 + \binom{7}{2} = 22$.
Denote the family of these sets by D (domain).

New Set of Recursive Equations

We want to write the same set of equations as before, but because we have only a finite number of sets, we must approximate. We approximate sets with possibly larger sets.

$$
\begin{aligned}
S^{\#}(a) &= \top \\
S^{\#}(b) &= T^{\#}(S^{\#}(a), i = 0) \\
&\sqcup\ T^{\#}(S^{\#}(g), skip) \\
S^{\#}(c) &= T^{\#}(S^{\#}(b), [\neg(i < 10)]) \\
S^{\#}(d) &= T^{\#}(S^{\#}(b), [i < 10]) \\
S^{\#}(e) &= T^{\#}(S^{\#}(d), [i > 1]) \\
S^{\#}(f) &= T^{\#}(S^{\#}(d), [\neg(i > 1)]) \\
S^{\#}(g) &= T^{\#}(S^{\#}(e), i = i + 3) \\
&\sqcup\ T^{\#}(S^{\#}(f), i = i + 2)
\end{aligned}
$$

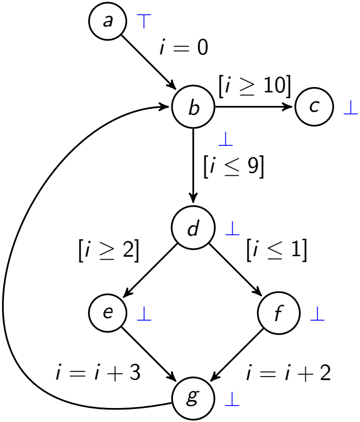
- $S_1 \sqcup S_2$ denotes the approximation of $S_1 \cup S_2$: it is the set that contains both S_1 and S_2, that belongs to D, and is otherwise as small as possible. Here $[a, b] \sqcup [c, d] = [min(a, c), max(b, d)]$
- We use approximate functions $T^{\#}(S, c)$ that give a result in D.

Updating Sets

We solve the equations by starting in the initial state and repeatedly applying them.

- in the 'entry' point, we put $\top$, in all others we put $\perp$.

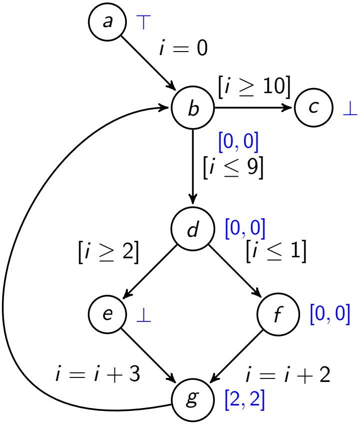
$$
\begin{aligned}
S^\#(a) &= \top \\
S^\#(b) &= T^\#(S^\#(a), i = 0) \\
&\sqcup\ T^\#(S^\#(g), skip) \\
S^\#(c) &= T^\#(S^\#(b), [\neg(i < 10)]) \\
S^\#(d) &= T^\#(S^\#(b), [i < 10]) \\
S^\#(e) &= T^\#(S^\#(d), [i > 1]) \\
S^\#(f) &= T^\#(S^\#(d), [\neg(i > 1)]) \\
S^\#(g) &= T^\#(S^\#(e), i = i + 3) \\
&\sqcup\ T^\#(S^\#(f), i = i + 2)
\end{aligned}
$$



Updating Sets

Sets after a few iterations:

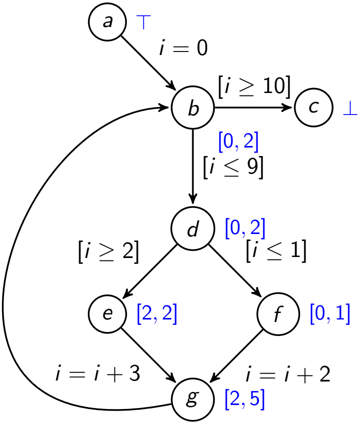
$$\begin{aligned}
S^{\#}(a) &= \top \\
S^{\#}(b) &= T^{\#}(S^{\#}(a), i = 0) \\
&\quad \sqcup\ T^{\#}(S^{\#}(g), \mathit{skip}) \\
S^{\#}(c) &= T^{\#}(S^{\#}(b), [\neg(i < 10)]) \\
S^{\#}(d) &= T^{\#}(S^{\#}(b), [i < 10]) \\
S^{\#}(e) &= T^{\#}(S^{\#}(d), [i > 1]) \\
S^{\#}(f) &= T^{\#}(S^{\#}(d), [\neg(i > 1)]) \\
S^{\#}(g) &= T^{\#}(S^{\#}(e), i = i + 3) \\
&\quad \sqcup\ T^{\#}(S^{\#}(f), i = i + 2)
\end{aligned}$$



Updating Sets

Sets after a few more iterations:

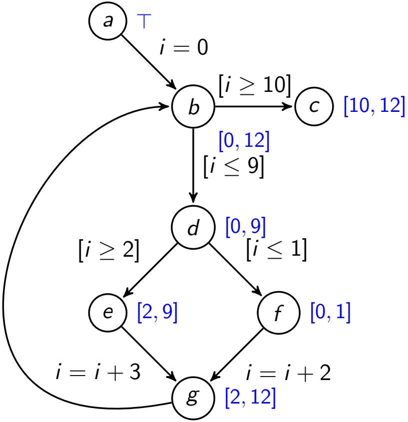
$$
\begin{aligned}
S^{\#}(a) &= \top \\
S^{\#}(b) &= T^{\#}(S^{\#}(a), i = 0) \\
&\sqcup\ T^{\#}(S^{\#}(g), \mathit{skip}) \\
S^{\#}(c) &= T^{\#}(S^{\#}(b), [\neg(i < 10)]) \\
S^{\#}(d) &= T^{\#}(S^{\#}(b), [i < 10]) \\
S^{\#}(e) &= T^{\#}(S^{\#}(d), [i > 1]) \\
S^{\#}(f) &= T^{\#}(S^{\#}(d), [\neg(i > 1)]) \\
S^{\#}(g) &= T^{\#}(S^{\#}(e), i = i + 3) \\
&\sqcup\ T^{\#}(S^{\#}(f), i = i + 2)
\end{aligned}
$$



Fixpoint Found

Final values of sets:

$$
\begin{aligned}
S^\#(a) &= \top \\
S^\#(b) &= T^\#(S^\#(a), i = 0) \\
&\sqcup\ T^\#(S^\#(g), skip) \\
S^\#(c) &= T^\#(S^\#(b), [\neg(i < 10)]) \\
S^\#(d) &= T^\#(S^\#(b), [i < 10]) \\
S^\#(e) &= T^\#(S^\#(d), [i > 1]) \\
S^\#(f) &= T^\#(S^\#(d), [\neg(i > 1)]) \\
S^\#(g) &= T^\#(S^\#(e), i = i + 3) \\
&\sqcup\ T^\#(S^\#(f), i = i + 2)
\end{aligned}
$$

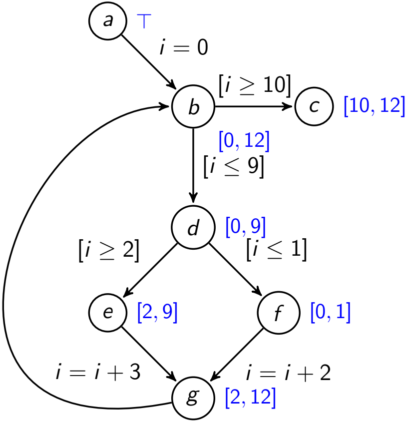


If we map intervals to sets, this is also solution of the original constraints.

Automatically Constructed Hoare Logic Proof

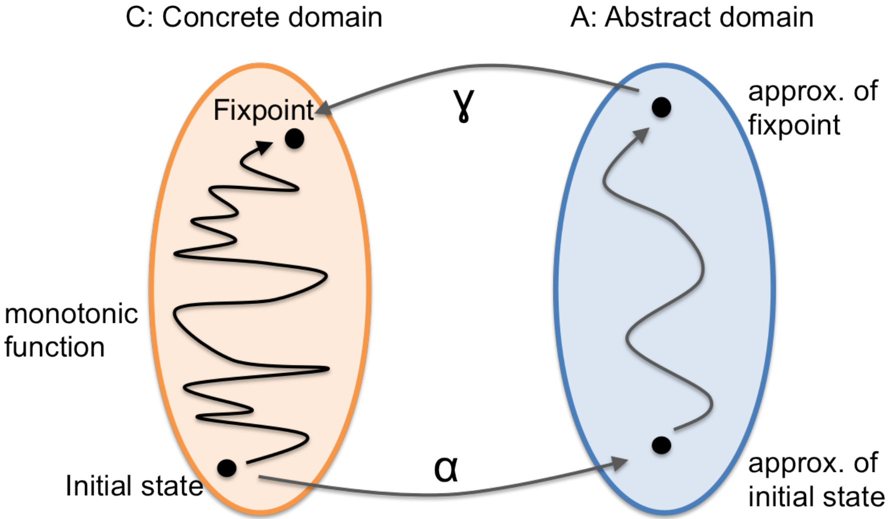
Final values of sets:

```
//a: true
i = 0;
      //b: 0 ≤ i ≤ 12
while (i < 10) {
  //d: 0 ≤ i ≤ 9
  if (i > 1)
    //e: 2 ≤ i ≤ 9
    i = i + 3;
  else
    //f: 0 ≤ i ≤ 1
    i = i + 2;
  //g: 2 ≤ i ≤ 12
}
//c: 10 ≤ i ≤ 12
```



This method constructed a sufficiently annotated program and ensured that all Hoare triples that were constructed hold

Abstract Interpretation Big Picture



Abstract Domains are Partial Orders

Program semantics is given by certain sets (e.g. sets of reachable states).

- subset relation $\subseteq$: used to compare sets
- union of states: used to combine sets coming from different executions (e.g. if statement)

Our goal is to approximate such sets. We introduce a domain of elements $d \in D$ where each d represents a set.

- $\gamma(d)$ is a set of states. γ is called **concretization function**
- given d_1 and d_2, it could happen that there is **no element** d representing union

$$\gamma(d_1) \cup \gamma(d_2) = \gamma(d)$$

  Instead, we use a set d that approximates union, and denote it $d_1 \sqcup d_2$

This leads us to review the theory of **partial orders** and **(semi)lattices**.

Partial Orders

Partial ordering relation is a binary relation $\leq$ that is reflexive, antisymmetric, and transitive, that is, the following properties hold for all x, y, z:

- $x \leq x$
- $x \leq y \wedge y \leq x \rightarrow x = y$
- $x \leq y \wedge y \leq z \rightarrow x \leq z$

If A is a set and $\leq$ a binary relation on A, we call the pair $(A, \leq)$ a **partial order**.

Given a partial ordering relation $\leq$, the corresponding **strict ordering relation** $x < y$ is defined by $x \leq y \wedge x \neq y$ and can be viewed as a shorthand for this conjunction.

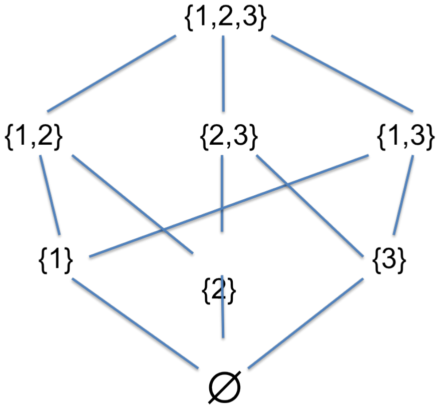
- Orders on integers, rationals, reals are all special cases of partial orders called *linear orders*.
- Given a set U, let A be any set of subsets of U, that is $A \subseteq 2^U$. Then $(A, \subseteq)$ is a partial order.

Example: Let $U = \{1, 2, 3\}$ and let $A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}\}$. Then $(A, \subseteq)$ is a partial order. We can draw it as a *Hasse diagram*.

Hasse diagram

presents the relation as a directed graph in a plane, such that

- the direction of edge is given by which nodes is drawn above
- transitive and reflexive edges are not represented (they can be derived)



Extreme Elements in Partial Orders

Given a partial order $(A, \leq)$ and a set $S \subseteq A$, we call an element $a \in A$

- **upper bound** of S if for all $a' \in S$ we have $a' \leq a$
- **lower bound** of S if for all $a' \in S$ we have $a \leq a'$
- **minimal element** of S if $a \in S$ and there is no element $a' \in S$ such that $a' < a$
- **maximal element** of S if $a \in S$ and there is no element $a' \in S$ such that $a < a'$
- **greatest element** of S if $a \in S$ and for all $a' \in S$ we have $a' \leq a$
- **least element** of S if $a \in S$ and for all $a' \in S$ we have $a \leq a'$
- **least upper bound** (lub, supremum, join, $\sqcup$) of S if a is the least element in the set of all upper bounds of S
- **greatest lower bound** (glb, infimum, meet, $\sqcap$) of S if a is the greatest element in the set of all lower bounds of S

Taking $S = A$ we obtain minimal, maximal, greatest, least elements for the entire partial order.

Extreme Elements in Partial Orders

Notes

- minimal element need not exist: $(0, 1)$ interval of rationals
- there may be multiple minimal elements: $\{\{a\}, \{b\}, \{a, b\}\}$
- if minimal element exists, it need not be least: above example
- there are no two distinct least elements for the same set
- least element is always glb and minimal
- if glb belongs to the set, then it is always least and minimal
- for relation $\subseteq$ on sets, *glb* is intersection, *lub* is union (not all families of sets are closed under $\cap$, $\cup$)

Least upper bound (lub, supremum, join, $\sqcup$)

Denoted $lub(S)$, least upper bound of S is an element M, if it exists, such that M is the least element of the set

$$U = \{x \mid x \text{ is upper bound on } S\}$$

In other words:

- M is an upper bound on S
- for every other upper bound M' on S, we have that $M \leq M'$

Note: this is the same definition as supremum in real analysis.

Least upper bound (glb, infimum, meet, $\sqcap$)

$a_1 \sqcup a_2$ denotes $lub(\{a_1, a_2\})$

$$(\ldots(a_1 \sqcup a_2)\ldots) \sqcup a_n \quad \text{is in fact } lub(\{a_1, \ldots, a_n\})$$

So the operation is

- associative
- commutative
- idempotent

Real Analysis

Take as S the open interval of reals $(0, 1) = \{x \mid 0 < x < 1\}$
Then

- S has no maximal element
- S thus has no greatest element
- 2, 2.5, 3,... are all upper bounds on S
- $lub(S) = 1$

Execise: subsets of U

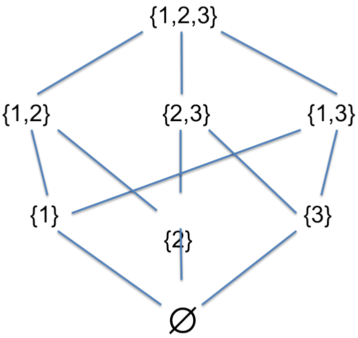
Consider

$$A = 2^U = \{S \mid S \subseteq U\} \qquad \text{and} \qquad (A, \subseteq)$$

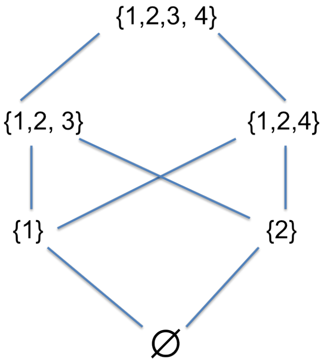
Do these exist, and if so, what are they?

- $s_1 \subseteq S, s_2 \subseteq S, lub(\{s_1, s_2\}) = ?$
- $lub(S) = ?$

Exercise: find the lub

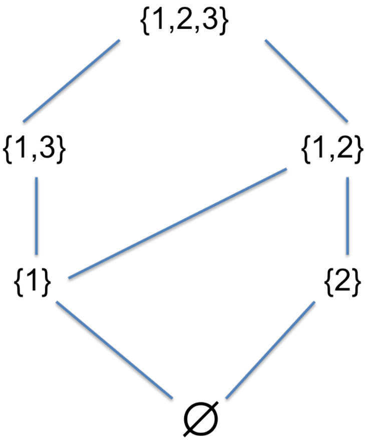


$$\{1\} \sqcup \{2\} =$$



$$\{1\} \sqcup \{2\} =$$

Does every pair of elements in this order have a least upper bound?



Dually, does it have a **greatest lower bound**?

Partial order for the domain of intervals

Domain: $D = \{\bot\} \cup \{(L, U) \mid L \in \{-\infty\} \cup \mathbb{Z}, U \in \{+\infty\} \cup \mathbb{Z}$
such that $L \leq U$.

The associated set of elements is given by the function γ:

$$\gamma : D \to 2^{\mathbb{Z}}, \qquad \gamma((L, U)) = \{x \mid L \leq x \wedge x \leq U\}$$

Lub: for $d_1, d_2 \in D$, $d_1 \sqsubseteq d_2 \quad \leftrightarrow \quad \gamma(d_1) \subseteq \gamma(d_2)$
hence

$$(L_1, U_1) \sqsubseteq (L_2, U_2) \quad \leftrightarrow \quad L_2 \leq L_1 \wedge U_1 \leq U_2$$
$$\bot \sqsubseteq d \quad \forall d \in D$$
$$(L_1, U_1) \sqcup (L_2, U_2) = (min(L_1, L_2), max(U_1, U_2))$$

Remark on constructing orders using inverse images

Suppose $\gamma : D \to C$ where C is some collection of sets.
If we define relation $\sqsubseteq$ by:

$$d_1 \sqsubseteq d_2 \iff \gamma(d_1) \subseteq \gamma(d_2)$$

then

1. $\sqsubseteq$ is reflexive
2. $\sqsubseteq$ is transitive
3. $\sqsubseteq$ is antisymmetric if and only iff γ is injective

If $\sqsubseteq$ is not antisymmetric then we can define equivalence relation

$$d_1 \sim d_2 \iff \gamma(d_1) = \gamma(d_2)$$

and then take D' to be equivalence classes of such new set.
Example: suppose we defined intervals as all possible pairs of integers (L, U). Then there would be many representations of the empty set, all those intervals where $L > U$.

Lattices

Definition: A lattice is a partial order in which every two-element set has a least upper bound and a greatest lower bound.

Lemma: In a lattice every non-empty finite set has a lub ($\sqcup$) and glb ($\sqcap$).

Lattices

Definition: A lattice is a partial order in which every two-element set has a least upper bound and a greatest lower bound.

Lemma: In a lattice every non-empty finite set has a lub ($\sqcup$) and glb ($\sqcap$).

Proof: is by induction!
Case where the set S has three elements x,y and z:
Let $a = (x \sqcup y) \sqcup z$.
By definition of $\sqcup$ we have $z \sqsubseteq a$ and $x \sqcup y \sqsubseteq a$.
Then we have again by definition of $\sqcup$, $x \sqsubseteq x \sqcup y$ and $y \sqsubseteq x \sqcup y$. Thus by transitivity we have $x \sqsubseteq a$ and $y \sqsubseteq a$.
Thus we have $S \sqsubseteq a$ and a is an upper bound.
Now suppose that there exists a' such that $S \sqsubseteq a'$. We want $a \sqsubseteq a'$ (a least upper bound):
We have $x \sqsubseteq a'$ and $y \sqsubseteq a'$, thus $x \sqcup y \sqsubseteq a'$. But $z \sqsubseteq a'$, thus $((x \sqcup y) \sqcup z) \sqsubseteq a'$.
Thus a is the lub of our 3 elements set.

Examples of Lattices

Lemma: Every linear order is a lattice.

Example: Every bounded subset of the set of real numbers has a lub. This is an axiom of real numbers, the way they are defined (or constructed from rationals).

- If a lattice has least and greatest element, then every finite set (including empty set) has a lub and glb.
- This does not imply there are lub and glb for infinite sets.
  Example: In the oder $([0, 1), \leq)$ with standard ordering on reals is a lattice, the entire set has no lub. The set of all rationals of interval $[0, 10]$ is a lattice, but the set $\{x \mid 0 \leq x \wedge x^2 < 2\}$ has no lub.

Exercises

Prove the following:

1. $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
2. $\sqcup A \sqsubseteq \sqcap B \;\Leftrightarrow\; \forall x \in A . \forall y \in B . x \sqsubseteq y$
3. Let $(A, \sqsubseteq)$ be a partial order such that every set $S \subseteq A$ has the greatest lower bound.
   Prove that then every set $S \subseteq A$ has the least upper bound.