Exercises 1 - Solutions

1 PL validity

For each of the following propositional logic formulae determine whether it is valid or not. If it is valid prove it, otherwise give a counterexample. Note that $A \to B \to C$ is parsed as $A \to (B \to C)$.

(i) $(P \wedge Q) \to P \to Q$

(ii) $(P \to Q) \vee (P \wedge \neg Q)$

(iii) $(P \to Q \to R) \to P \to R$

(iv) $(P \to Q \vee R) \to P \to R$

(v) $\neg(P \vee Q) \to R \to \neg R \to Q$

Solution: We can construct a *truth table* that lists all possible *assignments* of values of propositional variables. For n variables a truth table has 2^n rows. We can evaluate the formula for variable assignment and check whether it is true or false. If the result is true for *every* assignment, then the formula is *valid*. Otherwise, there is an assignment where the formula evaluates to false. This is the *counterexample*. We obtain the following results:

(i) valid

(ii) valid

(iii) counterexample: $R = \bot, P = \top, Q = \bot$

(iv) counterexample: $R = \bot, P = \top, Q = \bot$

(v) valid

2 FOL validity

For each of the following predicate logic formulae determine whether it is valid or not. If it is valid prove it, otherwise give a counterexample.

(i) $(\forall x, y.p(x, y) \to p(y, x)) \to \forall z.p(z, z)$

(ii) $\forall x, y.p(x, y) \to p(y, x) \to \forall z.p(z, z)$

(iii) $(\exists x.p(x)) \to \forall y.p(y)$

(iv) $(\forall x.p(x)) \to \exists y.p(y)$

(v) $\exists x, y.(p(x, y) \to (p(y, x) \to \forall z.p(z, z)))$

Solution: A FOL formula is valid if it is true for all possible interpretations of function and predicate symbols. To show that a formula is *not* valid, we typically find an interpretation where the formula does not hold. To show that a formula is valid, we need to consider an arbitrary interpretation of function and predicate symbols and show that in such arbitrary interpretation it is true. We can do that using any sound mathematical reasoning, for example in set theory, or using a proof system which we have agreed that contains valid reasoning steps.

(i) Not valid. Counterexample: $D = \mathbb{Z}, p = \{(x, y) | x \neq y\}$

(ii) Not valid. Counterexample: $D = \{a, b\}, p = \{(a, b), (b, a)\}$

(iii) Not valid. Counterexample: $D = \{a, b\}, p = \{a\}$

(iv) Valid. Note that this property holds because we consider only interpretations with a non-empty domain of quantifiers.

(v) Valid. To show this, consider an arbitrary interpretation of the binary predicate symbol p, in some domain of elements D. We need to show that there exist two elements to interpret x and y such that the nested implication holds. Note that the conclusion is of the form $\forall z.p(z, z)$. This is a property of the structure that we have chosen (saying that p is interpreted as a reflexive relation). This property either holds or not.

a) Suppose first that the property holds (p is reflexive). Then the nested implication will be true by setting x and y to be arbitrary elements. This is always possible, because D is non-empty.

b) Suppose now that the property does not hold. That means that there exists an element a such that $\neg p(a, a)$. We interpret x and y to be a. This makes the assumption of the implication false, so the formula holds.

3 FOL Normal forms

Put the following formulae into prenex normal form:

(i) $(\forall x.\exists y.p(x, y)) \rightarrow \forall x.p(x, x)$

(ii) $\exists z.(\forall x.\exists y.p(x, y)) \rightarrow \forall x.p(x, z)$

(iii) $\forall w.\neg(\exists x, y.\forall z.p(x, z) \rightarrow q(y, z)) \wedge \exists z.p(w, z)$

Solution:

(i) $\exists x.\forall y.\forall z.\neg p(x, y) \vee p(z, z)$

(ii) $\exists z, x.\forall y, w.\neg p(x, y) \vee p(w, z)$

(iii) $\forall w, x, y.\exists z, v.p(x, z) \wedge \neg q(y, z) \wedge p(w, v)$

4 Redundant logical connectives

a) Given $\top, \land$ and $\neg$, prove that $\bot, \lor, \to$ and $\leftrightarrow$ are redundant logical connectives. That is, show that each of $\bot$, $F_1 \lor F_2$, $F_1 \to F_2$ and $F_1 \leftrightarrow F_2$ is equivalent to a formula that uses only $F_1, F_2, \top, \land$ and $\neg$.

b) Now extend propositional formulas with a NAND operator, denoted $\bar{\land}$ and defined by

$$x \mathbin{\bar{\land}} y = \neg(x \land y)$$

Show that for each propositional formula F there exists an equivalent formula that uses $\bar{\land}$ as the only operator.

Solution a):

$\bot$**:** $\neg\top$

$F_1 \lor F_2$**:** $\neg(\neg F_1 \land \neg F_2)$

$F_1 \to F_2$**:** $\neg(F_1 \land \neg F_2)$

$F_1 \leftrightarrow F_2$**:** $\neg\Big(\neg(\neg F_1 \land \neg F_2) \land \neg(F_2 \land F_1)\Big)$

Solution b)

$\bot$**:** $\top\bar{\land}\top$

$\neg F_1$**:** $\top\bar{\land}F_1$

$F_1\land$**:** $\top\bar{\land}(F_1\bar{\land}F_2)$

$F_1 \lor F_2$**:** $(\top\bar{\land}F_1)\bar{\land}(\top\bar{\land}F_2)$

$F_1 \to F_2$**:** $F_1\bar{\land}(\top\bar{\land}F_2)$

$F_1 \leftrightarrow F_2$**:** $\Big((\top\bar{\land}F_1)\bar{\land}(\top\bar{\land}F_2)\Big)\bar{\land}(F_2\bar{\land}F_1)$

5 Complexity of normal forms

a) We define the size of a formula as the number of nodes in its syntax tree. For example, $size(P \land \neg R) = 4$, where P and R are propositional variables.

Now consider propositional formulas containing only $\land, \lor, \to, \neg$ and the recursive definition of NNF for propositional logic.

Find an integer constant K such that, for every such formula G we have:

$$size(NNF(G)) \leq K \cdot size(G)$$

Once you guess the value K, prove that the above inequation holds for this K, using mathematical induction.

b) Prove that there is no polynomial-time algorithm for transforming a propositional formula into an equivalent formula in conjunctive normal form. You do not need to use any deep results of complexity theory.

Specifically, prove that there exists an infinite family of formulas $F_1, F_2, \ldots$ such that for each n, every algorithm that transforms F_n into an equivalent CNF formula with the same set of variables needs exponential time. (Note that it is not enough to prove that one particular algorithm will take exponential time, you need to prove that every algorithm would need exponential time.)

Solution a):

We will prove that $K = 2$, i.e. $size(NNF(G)) \leq 2 \cdot size(G)$. For the base cases $\top, \bot, \neg\top, \neg\bot$ the inequality clearly holds, as the size is actually getting smaller. Now assume that F_1 and F_2 have size $<= n$ and that the inductive hypothesis holds. We will now show that it holds for expressions of size $> n$.

$\neg\neg F_1 \leftrightarrow F_1$**:** satisfies inequality as the sizes decreases

$\neg F_1 \leftrightarrow \neg F_1$**:**

$$\begin{aligned} size(NNF(G)) &= size(NNF(F_1)) + 1 \\ &\leq 2 * size(F_1) + 1 \\ &\leq 2 * size(G) = 2 * (size(F_1) + 1) \end{aligned}$$

$\neg(F_1 \wedge F_2) \leftrightarrow \neg F_1 \vee \neg F_2$**:**

$$\begin{aligned} size(NNF(G)) &= size(NNF(F_1)) + size(NNF(F_2)) + 3 \\ &\leq 2 * size(F_1) + 2 * size(F_2) + 3 \\ &\leq 2 * size(F_1) + 2 * size(F_2) + 4 = 2 * size(G) \end{aligned}$$

$\neg(F_1 \vee F_2) \leftrightarrow \neg F_1 \wedge \neg F_2$**:** analogous to previous case

$F_1 \rightarrow F_2 \leftrightarrow \neg F_1 \vee F_2$

$$\begin{aligned} size(NNF(G)) &= size(NNF(F_1)) + size(NNF(F_2)) + 2 \\ &\leq 2 * size(F_1) + 2 * size(F_2) + 2 \\ &\leq 2 * size(G) \end{aligned}$$

$F_1 \wedge F_2 \leftrightarrow F_1 \wedge F_2$

$$\begin{aligned} size(NNF(G)) &= size(NNF(F_1)) + size(NNF(F_2)) + 1 \\ &\leq 2 * size(F_1) + 2 * size(F_2) + 1 \\ &\leq 2 * size(G) \end{aligned}$$

$F_1 \vee F_2 \leftrightarrow F_1 \vee F_2$**:** analogous to previous case

Solution b)

Consider formulae in disjunctive normal form with 2 atoms per conjunction, i.e.

$$(a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee \cdots \vee (a_n \wedge b_n)$$

Note that the size of the formula is linear in n, i.e. the formula has n conjuncts.

If we take a concrete algorithm, we can obtain a particular conjunctive normal form, which is of the size exponential in n. However, what we need to show is that no matter how clever CNF we choose, it will need to be exponential, because it is in CNF, and is equivalent to the original formula. Therefore, fix n, denote the original formula by F. Consider the set of all CNF formulas equivalent to F, and take one of those that have minimum length, that is, there does not exists a shorter one in terms of the number of symbols. Denote it by F'. Note that F is true precisely in those assignments for which there exists i with $1 \le i \le n$ such that both a_i and b_i are true, so this property must hold for F' as well.

F' is of the form $C_1 \wedge \ldots \wedge C_m$ where each C_j is a disjunction of literals, i.e. either propositional variables or their negations. Due to associativity, commutativity, and idempotence ($A \vee A$ is equivalent to A) of disjunction, we consider clauses to be sets of literals: duplicate literals can be eliminated, reducing the size of the formula, which would contradict its minimality. Moreover, $p \vee \neg p$ is always true, so a clause C_j containing it can be eliminated, reducing the size of the CNF, and F' would again not be minimal. Therefore, each C_i in F', for each propositional variable p, contains at most one of p or $\neg p$.

Consider the space S all functions $f : \{1, \ldots, n\} \to \{A, B\}$ where $\{A, B\}$ is just a two-element set. S contains precisely 2^n elements. For each such function consider the clause, viewed as a set of positive literals:

$$C(f) = \{a_i \mid f(i) = A\} \cup \{b_i \mid f(i) = B\}$$

Define also $\bar{C}(f) = C(f) \cup \{\neg p \mid p \notin C(f)\}$. Note that $\bar{C}(f)$ contains p or $\neg p$ for each propositional variable p. Note also that for $f, f' \in S$ we have

$$C(f) \subseteq \bar{C}(f') \ \rightarrow f = f' \quad (*)$$

The next lemma follows from the fact that F' cannot be too strong.

Lemma 1: for each C_j, there exists $f_j \in S$ such that $C(f_j) \subseteq C_j$, that is, for each i where $1 \le i \le n$, C_j contains a_i or b_i. Proof. Fix C_j and i suppose that, on the contrary, C_j contains neither a_i nor b_i. Consider the assignment where a_i and b_i are true, and all other propositional variables are assigned to be opposite of the way they occur in C_j. Then C_j is false in this assignment, so F' is false, contradicting equivalence with F.

Now we derive consequences from the fact that F' cannot be too weak:

$$\forall f \in S. \exists C_j.\ C_j \subseteq \bar{C}(f)$$

To show this by contradiction, suppose that for some $f_0 \in S$ no clause C_j is contained in $\bar{C}(f_0)$, that is

$$\bigwedge_{0 \le j \le m} C_j \setminus \bar{C}(f_0) \neq \emptyset$$

Define a propositional assignment

$$(\alpha(p) = \text{true}) \iff (p \notin C(f_0))$$

Then for each $1 \le i \le n$ exactly one of the two cases holds:

1. $f_0(i) = A$, $a_i \in C(f_0)$, $\neg a_i \notin \bar{C}(f_0)$, $b_i \notin C(f_0)$, $\neg b_i \in \bar{C}(f_0)$, $\alpha(a_i) = \text{false}$, $\alpha(b_i) = \text{true}$
2. $f_0(i) = B$, $a_i \notin C(f_0)$, $\neg a_i \in \bar{C}(f_0)$, $b_i \in C(f_0)$, $\neg b_i \notin \bar{C}(f_0)$, $\alpha(a_i) = \text{true}$, $\alpha(b_i) = \text{false}$

F is false for α, because for each i either a_i or b_i is false. We claim that F' is true in α by showing that each C_j is true. Take any C_j. By Lemma 1, $C(f_j) \subseteq C_j$. If $f_j \neq f_0$ then for some k we have $f_j(k) \neq f_0(k)$, so $C(f_j)$ is true in α, which implies that C_j is also true. Now consider the case

$$C(f_0) \subseteq C_j \subsetneq \bar{C}(f_0)$$

Let the literal $l \in C_j$ be such that $l \notin \bar{C}(f_0)$, so also $l \notin C(f_0)$. We show $\alpha(l) = \text{true}$. If l is a positive literal p then $\alpha(p) = \text{true}$ by the definition of α. Let l be $\neg a_i$, so $\neg a_i \notin \bar{C}(f_0)$. Therefore the case 1 above applies and $\alpha(a_i) = \text{false}$, which means that $\alpha(l) = \text{true}$. Analogously, if l is $\neg b_i$ then from $\neg b_i \notin \bar{C}(f_0)$ case 2 above applies so again $\alpha(l) = \text{true}$.

By contradiction we have shown that for each $f \in S$, there exists C_j such that $C_j \subseteq \bar{C}(f)$. This establishes a map from an exponentially larger set S into the set of clauses C_j of the formula F'. We next argue that this map is, in fact, injective, so there are at least $|S|$ clauses in F'. Suppose that for $f, f' \in S$. Then

$$C_j \subseteq \bar{C}(f) \quad \wedge \quad C_j \subseteq \bar{C}(f')$$

Then both $C(f_j) \subseteq \bar{C}(f)$ and $C(f_j) \subseteq \bar{C}(f')$, so by $(*)$ we have $f' = f$, proving injectivity. Thus, F' has exponentially many clauses, completing the proof.

6 Relations

Prove the following or give a counterexample.

(i) $(r \cup s) \circ t = (r \circ t) \cup (s \circ t)$

(ii) $(r \cap s) \circ t = (r \circ t) \cap (s \circ t)$

(iii) $(r_1 \circ r_2)^{-1} = (r_2^{-1} \circ r_1^{-1})$

(iv) $S \bullet r = ran(\triangle_S \circ r)$

(v) If $r_1 \subseteq r_1'$ then $r_1 \circ r_2 \subseteq r_1' \circ r_2$ and $r_2 \circ r_1 \subseteq r_2 \circ r_1'$.

(vi) If $r_1 \subseteq r_1'$ then $r_1 \cup r_2 \subseteq r_1' \cup r_2$ and $r_2 \cup r_1 \subseteq r_2 \cup r_1'$.

Solution:

(i)

$$\begin{aligned} (r \cup s) \circ t \quad &\Leftrightarrow \quad \{(x,z) | \exists y.((x,y) \in r \vee (x,y) \in s) \wedge (y,z) \in t\} && (1) \\ &\Leftrightarrow \quad \{(x,z) | (\exists y.(x,y) \in r \wedge (y,z) \in t) \vee (\exists y.(x,y) \in s \wedge (y,z) \in t)\} && (2) \\ &\Leftrightarrow \quad (r \circ t) \cup (s \circ t) && (3) \end{aligned}$$

(ii) This is not true, one possible counterexample is:
$r = \{(a,b),(b,c),(c,d)\}$ $\quad s = \{(b,c),(a,a),(a,d)\}$ and $t = \{(d,d),(b,c),(c,b),(a,c)\}$

(iii)

$$\begin{aligned} (r_1 \circ r_2)^{-1} \quad &\Leftrightarrow \quad \{(x,z) | \exists y.(x,y) \in r_1 \wedge (y,z) \in r_2\}^{-1} && (4) \\ &\Leftrightarrow \quad \{(z,x) | (\exists y.(x,y) \in r_1 \wedge (y,z) \in r_2\} && (5) \\ &\Leftrightarrow \quad \{(z,x) | (\exists y.(y,x) \in r_1^{-1} \wedge (z,y) \in r_2^{-1}\} && (6) \\ &\Leftrightarrow \quad r_2^{-1} \circ r_1^{-1} && (7) \end{aligned}$$

(iv)

$$\begin{aligned}
& e \in ran(\triangle_S \circ r) && (8)\\
\Leftrightarrow\quad & e \in ran(\{(p,q) | \exists w.(p,w) \in \triangle_S \wedge (w,q) \in r\} && (9)\\
\Leftrightarrow\quad & e \in \{q | \exists p.\exists w.(p,w) \in \triangle_S \wedge (w,q) \in r\} && (10)\\
\Leftrightarrow\quad & e \in \{q | \exists p.p \in S \wedge (p,q) \in r\} && (11)\\
\Leftrightarrow\quad & \exists p.p \in S \wedge (p,e) \in r && (12)
\end{aligned}$$

Now, $e \in S \bullet r \quad \Leftrightarrow \quad \exists p.p \in S \wedge (p,e) \in r$, so the two expressions are equivalent, which proves set equality.

(v)

$$\begin{aligned}
(x,y) \in r_1 \circ r_2 &\leftrightarrow \exists w.(x,w) \in r_1 \wedge (w,y) \in r_2 && (13)\\
&\rightarrow \exists w.(x,w) \in r_1' \wedge (w,y) \in r_2 && (14)\\
&\leftrightarrow (x,y) \in r_1' \circ r_2 && (15)
\end{aligned}$$

The other statement is analogous.

(vi)

$$\begin{aligned}
(x,y) \in r_1 \cup r_2 &\leftrightarrow (x,y) \in r_1 \vee (x,y) \in r_2 && (16)\\
&\rightarrow (x,y) \in r_1' \vee (x,y) \in r_2 && (17)\\
&\leftrightarrow (x,y) \in r_1' \cup r_2 && (18)
\end{aligned}$$

7 Composition of partial functions

Given two partial functions r_1 and r_2, show that $r = r_1 \circ r_2$ is also a partial function.

Solution:

Suppose r_1 and r_2 are partial functions. Hence, $\forall x, y, z.(x,y) \in r_1 \wedge (x,z) \in r_1 \rightarrow y = z$ and similarly for r_2. To show that $r_1 \circ r_2$ is a partial function, we have to show the following:

$$\forall x, y, z.(x,y) \in r_1 \circ r_2 \wedge (x,z) \in r_1 \circ r_2 \rightarrow y = z$$

Let x, y, z be such that $(x,y), (x,z) \in r_1 \circ r_2$. Then there exist y_1 such that $(x, y_1) \in r_1$ and $(y_1, y) \in r_2$ and, analogously, z_1 such that $(x, z_1) \in r_1$ and $(z_1, z) \in r_2$. Because r_1 is a partial function, $y_1 = z_1$. Thus we have $(y_1, y), (y_1, z) \in r_2$. Because r_2 is partial function, we have $y = z$, as desired.

8 Transitive relations

Given a relation $r \subseteq A \times A$, prove that r is transitive if and only if $r \circ r \subseteq r$.

Solution:

$$\begin{aligned}
r \circ r \subseteq r &\Leftrightarrow \forall x, y.(\exists w.(x,w) \in r \wedge (w,y) \in r) \rightarrow (x,y) \in r && (19)\\
&\Leftrightarrow \forall x, y.\neg(\exists w.(x,w) \in r \wedge (w,y) \in r) \vee (x,y) \in r && (20)\\
&\Leftrightarrow \forall x, y.\forall w.\neg((x,w) \in r \wedge (w,y) \in r) \vee (x,y) \in r && (21)\\
&\Leftrightarrow \forall x, y, w.((x,w) \in r \wedge (w,y) \in r) \rightarrow (x,y) \in r && (22)\\
&\Leftrightarrow \text{r is transitive} && (23)
\end{aligned}$$

9 Symmetric relations

Recall that a relation $r \subseteq A \times A$ is symmetric if $\forall x, y \in A.\ (x, y) \in r \to (y, x) \in r$.
Now let r be an arbitrary relation. Prove that $r^{-1} \circ (r \cup r^{-1})^* \circ r$ is symmetric.

Solution:

$r \cup r^{-1}$ is symmetric

$$
\begin{aligned}
(x, y) \in r \cup r^{-1} \quad &\leftrightarrow \quad (x, y) \in r \vee (x, y) \in r^{-1} && (24)\\
&\leftrightarrow \quad (x, y) \in r \vee (y, x) \in r && (25)\\
&\leftrightarrow \quad (y, x) \in r \cup r^{-1} && (26)
\end{aligned}
$$

if r is symmetric, so is $r \circ r$

$$
\begin{aligned}
(x, y) \in r \circ r \quad &\to \exists z.(x, z) \in r \wedge (z, y) \in r && (27)\\
&\to \exists z.(y, z) \in r \wedge (z, x) \in r && (28)\\
&\to (y, x) \in r \circ r && (29)
\end{aligned}
$$

Hence it follows that $(r \cup r^{-1})^m$ is symmetric.

Now let $(x, y) \in r^{-1} \circ (r \cup r^{-1})^* \circ r$. Then there exists a, b such that $(x, a) \in r^{-1}$ and $(a, b) \in (r \cup r^{-1})^m$ and $(b, y) \in r$. By symmetry and definition of inverse, $(a, x) \in r$, $(b, a) \in (r \cup r^{-1})^m$ and $(y, b) \in r^{-1}$. Hence, $(y, x) \in r^{-1} \circ (r \cup r^{-1})^* \circ r$, i.e. the relation is symmetric.

10 Transitive closure

Recall that we define the powers of a relation $r \subseteq A \times A$ as follows:

$$r^0 = \triangle_A, \quad r^1 = r, \quad \text{and} \quad r^{n+1} = r^n \circ r$$

We showed that the *reflexive and transitive closure* $r^* = \bigcup_{n \geq 0} r^n$ is the smallest reflexive and transitive relation on A containing r. Show that for any relation r on a set A, $(r \cup r^{-1})^*$ is the least equivalence relation containing r. Precisely, show that

(i) $(r \cup r^{-1})^*$ is an equivalence relation, and

(ii) if t is an equivalence relation containing r, then $(r \cup r^{-1})^* \subseteq t$.

Solution:
Let $s = (r \cup r^{-1})^* = \bigcup_{n \geq 0} (r \cup r^{-1})^n$.

(i) To show s is an equivalence relation, show that it is reflexive ($\triangle_A \subseteq s$), symmetric $s^{-1} \subseteq s$ and transitive $s \circ s \subseteq s$.

- Since $\triangle_A = (r \cup r^{-1})^0 \subseteq s$, s is reflexive.
- $s^{-1} \Leftrightarrow [(r \cup r^{-1})^*]^{-1} \Leftrightarrow [(r \cup r^{-1})^{-1}]^* \Leftrightarrow [r \cup r^{-1}]^*$ Since $s^{-1} = s$ we have $s^{-1} \subseteq s$.

- $$\begin{aligned}
s \circ s &\Leftrightarrow (r \cup r^{-1})^* \circ (r \cup r^{-1})^* && (30)\\
&= \bigcup_{n \geq 0} (r \cup r^{-1})^n \circ \bigcup_{m \geq 0} (r \cup r^{-1})^m && (31)\\
&= \bigcup_{n,m \geq 0} (r \cup r^{-1})^{n+m} && (32)\\
&= \bigcup_{k \geq 0} (r \cup r^{-1})^k && (33)\\
&= s && (34)
\end{aligned}$$

  Since $s \circ s = s$ then also $s \circ s \subseteq s$.

(ii) Let t be an arbitrary equivalence relation containing r, hence t satisfies $\triangle_A \cup r \cup (t \circ t) \cup t^{-1} \subseteq t$. We show that for every n, $(r \cup r^{-1})^n \subseteq t$ by induction on n.

n = 0 $(r \cup r^{-1})^0 = \triangle_A \subseteq t$ holds by reflexivity of t

n = 1 $r \cup r^{-1} \subseteq t$ holds since every t contains r and is symmetric.

Inductive step. Suppose $(r \cup r^{-1})^n \subseteq t$. From $(r \cup r^{-1}) \subseteq t$ then by monotonicity of composition $(r \cup r^{-1})^{n+1} \subseteq t \circ t \subseteq t$.

Hence $s = (r \cup r^{-1})^* \subseteq t$.

11 Monotonicity of relation composition

Let $E(r_1, r_2, ..., r_n)$ be a relation composed of relations r_i with an arbitrary combination of relation composition and union, e.g. one possible expression could be $(r_1 \circ r_2) \cup r_3$. Show that this operation is monotone, that is show that for any i $r_i \subseteq r'_i \quad \rightarrow \quad E(r_1, r_2, ..., r_i, ..., r_n) \subseteq E(r_1, r_2, ..., r'_i, ..., r_n)$

Solution:

We will show this by structural induction on the expression trees. Suppose we have $r_i \subseteq r'_i$ for some fixed i. Note that when writing $E(r_1, ..., r_i, ..., r_n) \subseteq E(r_1, ..., r'_i, ..., r_n)$, we mean the subset on relations represented by the expressions.

base case The expression E is just a leaf, i.e. $E(r_1, ..., r_i, ..., r_n) = r_j$ for some j. Then either $i = j$ and $E(r_1, ..., r_i, ..., r_n) \subseteq E(r_1, ..., r'_i, ..., r_n)$ follows directly from $r_i \subseteq r'_i$. Otherwise, $i \neq j$, but in this case $r_j \subseteq r_j$ as well, so the result follows.

inductive case Now suppose monotonicity holds for two expressions E_1 and E_2. Hence, denoting the relations represented by these expressions by s_1 and s_2 respectively, $s_1 \subseteq s'_1$ and $s_2 \subseteq s'_2$, where the primed version come from replacing r_i by r'_i in E_1 and E_2. Let $E = E_1 \circ E_2$. By

$$\begin{aligned}
(x, y) \in s_1 \circ s_2 &\leftrightarrow \exists w.(x, w) \in s_1 \wedge (w, y) \in s_2 && (35)\\
&\rightarrow \exists w.(x, w) \in s'_1 \wedge (w, y) \in s_2 && (36)\\
&\leftrightarrow (x, y) \in s'_1 \circ s_2 && (37)
\end{aligned}$$

sequential composition preserves monotonicity and so $E(r_1, ..., r_i, ..., r_n) \subseteq E(r_1, ..., r'_i, ..., r_n)$.

Now let $E = E_1 \cup E_2$. Since

$$
\begin{aligned}
(x, y) \in s_1 \cup s_2 &\leftrightarrow (x, y) \in s_1 \vee (x, y) \in s_2 && (38)\\
&\rightarrow (x, y) \in s_1' \vee (x, y) \in s_2 && (39)\\
&\leftrightarrow (x, y) \in s_1' \cup s_2 && (40)
\end{aligned}
$$

union preserves monotonicity and so $E(r_1, ..., r_i, ..., r_n) \subseteq E(r_1, ..., r_i', ..., r_n)$.