Constraint-based Invariant Inference

Invariants

- **Dictionary Meaning:** A function, quantity, or **property** which remains unchanged
- Property (in our context): a predicate that holds for some, all, or no states

- Invariant is a property of a program
  - at a specific program location
  - that holds for **every** program state that **reaches** the program point

- Specifications are invariants at exit points of programs or procedures
- Also called reachability properties.

Invariants

```
x = 0
y = n
while(y > 0){
    x = x + 1
    y = y - 1
}
//invariant: x+y = n
//invariant: y>=0 => x<=n
```

Inductive Invariants

```
x = 0
y = n
//x+y = n
while(y > 0){
  //x+y = n ∧ y > 0
  x = x + 1
  //x+y = n+1
  y = y - 1
  //x+y = n
}
//invariant: x+y = n
```

- Invariant holds initially
- Invariant holds at the start of the loop

  =>

  invariant holds at the end of the loop

Not all Invariants are Inductive

```
x = 0
y = n
//y>=0 => x<=n
while(y > 0){
  //x <= n ∧ y > 0
  x = x + 1
  //x <= n+1 ∧ y > 0
  y = y - 1
  //x <= n+1 ∧ y >= 0
}
//invariant: y>=0 => x <= n
```

Invariant cannot be proved by induction

Inductive Strengthening

```
x = 0
y = n
//(y>=0 => x<=n) ∧ x+y=n
while(y > 0){
   //x<n ∧ y>0 ∧ x+y=n
   x = x + 1
   //x<=n ∧ y>0 ∧ x+y=n+1
   y = y - 1
   //x<=n ∧ y>=0 ∧ x+y=n
}
//invariant: y>=0 => x<=n
```

Implied by the stronger inductive invariant

Formulating Inductiveness

```
x = 0
y = n
while(y > 0){
  x = x + 1
  y = y - 1
} //invariant: y>=0 => x<=n
```

Generally referred to as the verification condition (VC)

$$(x = 0 \wedge y = n) \Rightarrow (y < 0 \vee x \leq n)$$

$$\big(\underbrace{(y < 0 \vee x \leq n)}_{I} \wedge \underbrace{y > 0}_{\text{Guard}} \wedge \underbrace{x' = x + 1 \wedge y' = y - 1}_{\text{Transition}}\big)$$

$$\Rightarrow (y' < 0 \vee x' \leq n)$$

Formulating Inductive Strengthening

```
x = 0
y = n
while(y > 0){
  x = x + 1
  y = y - 1
} //invariant: y>=0 => x<=n
```

$$(x = 0 \land y = n) \Rightarrow (y < 0 \lor x \leq n) \land \boldsymbol{S}$$

$$\Big(\underbrace{(y < 0 \lor x \leq n)}_{I} \land \boldsymbol{S} \land \underbrace{y > 0}_{\text{Guard}} \land \underbrace{x' = x + 1 \land y' = y - 1}_{\text{Transition}}\Big) \Rightarrow (y' < 0 \lor x' \leq n) \land \boldsymbol{S}'$$

Finding Linear Invariants [Colon et al. CAV '03]

```
x = 0
y = n
while(y > 0){
  x = x + 1
  y = y - 1
} //invariant: y>=0 => x<=n
```

Perhaps could be called a parametric VC

$$(x = 0 \land y = n) \Rightarrow (y < 0 \lor x \leq n\,) \land \boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{c} \leq \boldsymbol{0}$$

$$\underbrace{(y < 0 \lor x \leq n)}_{I} \land \boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{c} \leq \boldsymbol{0} \land \underbrace{y > 0}_{\text{Guard}} \land \underbrace{x' = x + 1 \land y' = y - 1}_{\text{Transition}}$$

$$\Rightarrow (y' < 0 \lor x' \leq n\,) \land \boldsymbol{ax'} + \boldsymbol{by'} + \boldsymbol{c} \leq \boldsymbol{0}$$

Finding Template Coefficients

$$(x \geq 0 \land y \geq n) \Rightarrow \boldsymbol{ax + by + c < 0}$$

Find values for a,b,c s.t. the formula becomes valid

$$\boldsymbol{A \Rightarrow B \equiv \neg(A \land \neg B)}$$

$$x \geq 0 \land y \geq n \land \boldsymbol{ax + by + c \geq 0}$$

Find values for a,b,c s.t. the formula becomes unsatisfiable

Farkas' Lemma: A conjunction of linear inequalities is unsatisfiable iff we can derive **1 <= 0** by performing the following operations:

- Multiplying the inequalities by a non-negative constant
- Adding two inequalities
- Adding (or subtracting) a non-negative constant to one side

Farkas' Lemma Example

$$x \geq 0 \wedge y \geq n \wedge 2x + 2y - 2n + 3 \leq 0$$

$$x + 0y + 0n + 0 \geq 0$$
$$0x + y - n + 0 \geq 0$$
$$-2x - 2y + 2n - 3 \geq 0$$

Multiply first and second equations by 2,
Add 2 to RHS of last equation
and add them

$$-1 \geq 0$$

Farkas' Lemma: A conjunction of linear inequalities (over reals) is unsatisfiable iff we can derive **1 <= 0** by performing the following operations:

- Multiplying the inequalities by a non-negative constant
- Adding two inequalities
- Adding (or subtracting) a non-negative constant to one side

Automating Coefficient Finding

$x \geq 0 \wedge y - n \geq 0 \wedge 2x + 2y - 2n + 3 \leq 0$

Prove unsat

$$\lambda_1 x \geq 0$$
$$\lambda_2 y - \lambda_2 n \geq 0$$
$$-2\lambda_3 x - 2\lambda_3 y + 2\lambda_3 n - 3\lambda_3 \geq 0$$

Multiplying by unknown non-negative values

Adding the inequalities

$$(\lambda_1 - 2\lambda_3)x + (\lambda_2 - 2\lambda_3)y + (2\lambda_3 - \lambda_2)n - 3\lambda_3 \geq 0$$

Adding an unknown non-neg value

$$(\lambda_1 - 2\lambda_3)x + (\lambda_2 - 2\lambda_3)y + (2\lambda_3 - \lambda_2)n - 3\lambda_3 + \lambda \geq 0$$

$$\equiv -1 \geq 0$$

Equate to 1 <= 0

Automating Coefficient Finding [Cont.]

$$(\lambda_1 - 2\lambda_3)x + (\lambda_2 - 2\lambda_3)y + (2\lambda_3 - \lambda_2)n - 3\lambda_3 + \lambda \ \geq 0$$

$$\equiv -1 \geq 0$$

$$\Downarrow$$

$$\begin{aligned} \lambda_1 - 2\lambda_3 &= 0 \\ \lambda_2 - 2\lambda_3 &= 0 \\ 2\lambda_3 - \lambda_2 &= 0 \\ -3\lambda_3 + \lambda &= -1 \end{aligned}$$

Every solution for the constraints will make the inequalities unsatisfiable

$$\lambda_1 = 2\,,\ \lambda_2 = 2\,,$$
$$\lambda_3 = 1, \lambda = 2$$

Template-based Invariant Inference

$$x \geq 0 \wedge y - n \geq 0 \wedge \boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{c} \geq \boldsymbol{0}$$

Find values for a,b,c s.t. the formula becomes unsatisfiable

$$\lambda_1 x \geq 0$$
$$\lambda_2 y - \lambda_2 n \geq 0$$
$$\lambda_3 ax + \lambda_3 by + \lambda_3 c \geq 0$$

Multiplying by unknown non-negative values

Adding the inequalities

$$(\lambda_1 + \lambda_3 a)x + (\lambda_2 + \lambda_3 b)y - \lambda_2 n + \lambda_3 c \geq 0$$

Adding an unknown non-neg value

$$(\lambda_1 + \lambda_3 a)x + (\lambda_2 + \lambda_3 b)y - \lambda_2 n + \lambda_3 c + \lambda_4 \geq 0$$

$$\equiv -1 \geq 0$$

Equate to 1 <= 0

Farkas' Constraints [Cont.]

$$(\lambda_1+\lambda_3 a)x + (\lambda_2 + \lambda_3 b)y - \lambda_2 n + \lambda_3 c + \lambda_4 \geq 0$$

$$\equiv -1 \geq 0$$

$$\Downarrow$$

$$\begin{gathered}\lambda_1 + \lambda_3 a = 0 \\ \lambda_2 + \lambda_3 b = 0 \\ -\lambda_2 = 0 \\ \lambda_3 c + \lambda_4 = -1\end{gathered}$$

Every solution for the constraints will make the inequalities unsatisfiable

$b = 0, a = -1, c = -1,$
$\lambda_1 = 1\ , \lambda_2 = 0\ ,$
$\lambda_3 = 1, \lambda_4 = 0$

In summary

- We had a formula of the form: $\mathrm{A}[\mathbf{x}] \wedge B[\boldsymbol{a}, \boldsymbol{x}] \Rightarrow C[\boldsymbol{a}, \boldsymbol{x}]$
- We wanted to find a value for $\boldsymbol{a}$ that will make the implication hold for all $\mathbf{x}$
- In other words, we are trying to find a satisfiable assignment for a quantified formula.
- Farkas' Lemma converts it to satisfiability of quantifier-free non-linear real constraints

Limitations

The Farkas' Lemma approach provides a way to find linear invariants for programs that

- do not have many disjunctions
- do not have functions
- do not have data structures
- do not have nonlinear arithmetic

Further Reading and Software

We developed an approach that addresses some of these limitations.

For more details see:

"Symbolic Resource Bounds Inference For Functional Programs", CAV 2014: pdf , slides

An extension of Leon (a slightly old version) that supports templates:

Orb : http://lara.epfl.ch/w/rbound

- More Related Works
  - "Linear invariant generation using non-linear constraint solving.", Colon et al., CAV 2003
  - "Program analysis as constraint solving.", S. Gulwani et al., PLDI 2008
  - "Constraint solving for interpolation.", A.Rybalchenko et al., VMCAI 2007
  - "Non-linear loop invariant generation using grobner bases." Sankaranarayanan et al., POPL 2004