Boolean Satisfiability and SAT Solvers

Philippe Suter

SAV, April 16th, 2013

Boolean Satisfiability

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- The original NP-complete problem.
  – "As hard and any other problem in NP."

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- The original NP-complete problem.
  - "As hard and any other problem in NP."
- S. Cook, *The complexity of theorem proving procedures*, STOC 1971.

SAT in Practice

- Ubiquitous in hardware/circuit design
  - E.g. equivalence checking.
- Search/AI problems
  - E.g. reduce Sudoku to SAT.
  - Dependency management in Eclipse.
- Software verification
  - By itself, and as part of the SMT stack.

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$\varphi \equiv a \land (\neg b \lor c)$

a	b	c	φ
T	T	T	T
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T	F	T	T
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- Obviously not very efficient. SAT solving is all about making this enumeration "smart".

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- SAT solving (almost) always applies to formulas normalized to *conjunctive normal form* (CNF).

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$$\{ \{ a, \overline{b}, c \}, \{ \overline{a}, c, d, \overline{e} \}, \{ b, \overline{d}, e \} \}$$

Going to Clauses

- SAT solving (almost) always applies to formulas normalized to *conjunctive normal form* (CNF).

$(a \vee \neg b \vee c) \wedge (\neg a \vee c \vee d \vee \neg e) \wedge (b \vee \neg d \vee e)$

$\{ \{ a, \overline{b}, c \}, \{ \overline{a}, c, d, \overline{e} \}, \{ b, \overline{d}, e \} \}$

$(a + \overline{b} + c)(\overline{a} + c + d + \overline{e})(b + \overline{d} + e)$

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$(a + \overline{b} + c)(\overline{a} + c + d + \overline{e})(b + \overline{d} + e)$

Note that a truth table is a kind of *disjunctive* normal form (DNF).

Going to Clauses

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  - …but that introduces an exponential blowup…
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- You could in principle use distributivity and De Morgan’s laws to convert any formula to CNF.
  - …but that introduces an exponential blowup…
  - …and you might as well convert to DNF, then.
- Instead, we use an encoding based on *introducing new variables*.

Tseitin's Encoding

- Idea: rewrite φ into ψ that is *equisatisfiable*.

$\varphi \equiv (a \land (\neg b \lor (c \land d))$

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- Idea: rewrite φ into ψ that is *equisatisfiable*.

p_2

$$\varphi \equiv (a \wedge (\neg b \vee (c \wedge d))$$

p_1

$$p_1 \Leftrightarrow c \wedge d$$

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$\neg c \vee \neg d \vee p_1$

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$p_1 \Leftrightarrow c \land d$

$\neg p_1 \lor c$

$\neg p_1 \lor d$

$\neg c \lor \neg d \lor p_1$

$p_2 \Leftrightarrow \neg b \lor p_1$

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$p_2 \Leftrightarrow \neg b \lor p_1$

$\neg p_2 \lor \neg b \lor p_1$

$b \lor p_2$

$\neg p_1 \lor p_2$

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$p_1 \Leftrightarrow c \wedge d$

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$\neg c \vee \neg d \vee p_1$

$p_2 \Leftrightarrow \neg b \vee p_1$

$\neg p_2 \vee \neg b \vee p_1$

$b \vee p_2$

$\neg p_1 \vee p_2$

$$\psi \equiv a \wedge p_2$$

Solving clauses: 2-SAT

- We can assume w.l.o.g. that each clause has at least two literals.
- What if all clauses have *exactly* two literals?

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c ∨ a

¬c ∨ b

¬b

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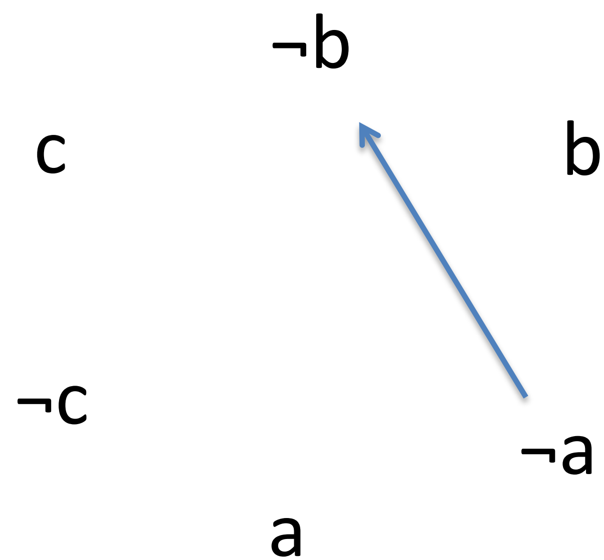

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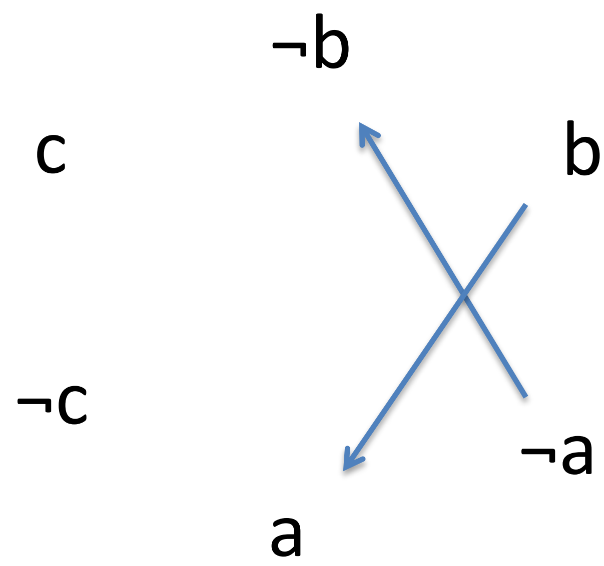

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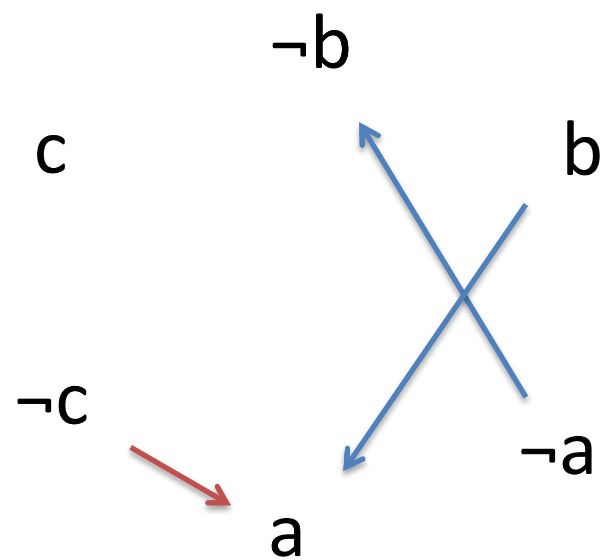

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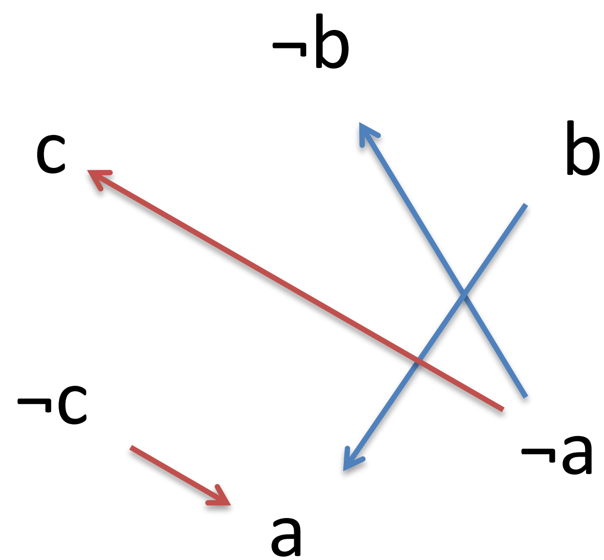

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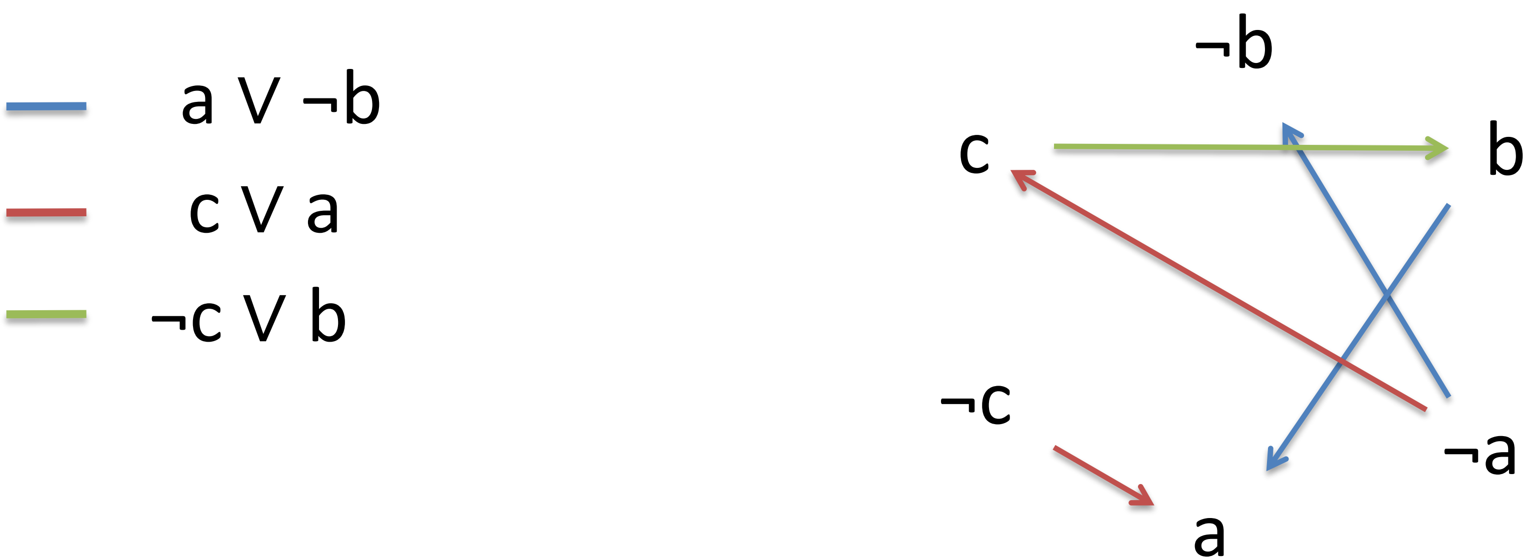

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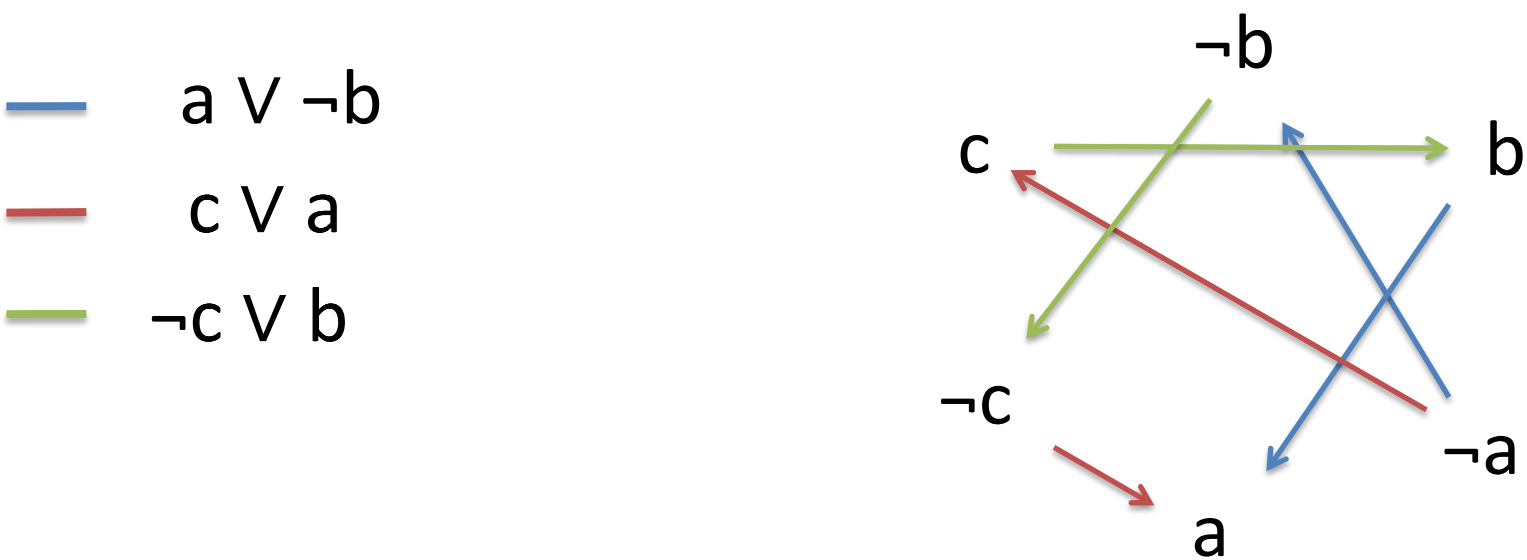

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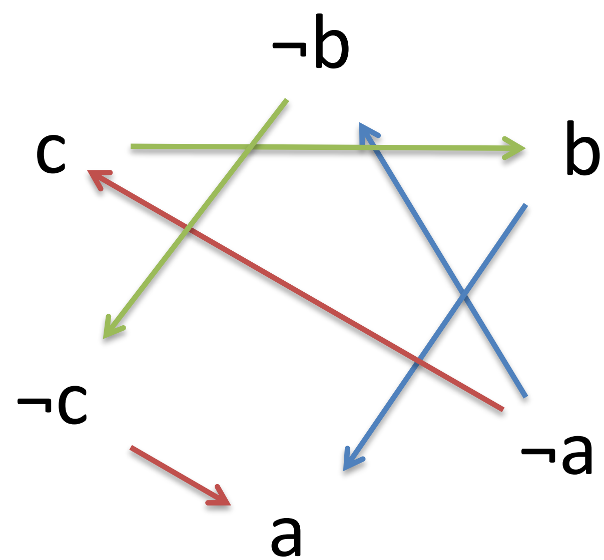
SAT



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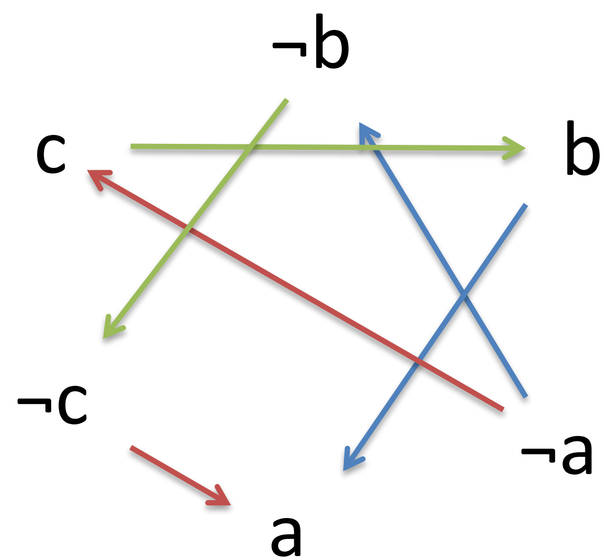
SAT



You can solve 2-SAT in polynomial time.
Some of the techniques for 2-SAT are used in general SAT solvers.

3-SAT

- NP-complete.
  - Reduction from SAT: split longer clauses using fresh variables.

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- NP-complete.
  - Reduction from SAT: split longer clauses using fresh variables.

- (Not so relevant to SAT solving technology.)

First Approach: Resolution

$a \lor \neg b \lor f$ $\neg a \lor \neg c \lor d \lor \neg e$

First Approach: Resolution

a V ¬b V f

¬a V ¬c V d V ¬e

First Approach: Resolution

$a \lor \neg b \lor f$ $\qquad\qquad$ $\neg a \lor \neg c \lor d \lor \neg e$

$$\neg b \lor f \lor \neg c \lor d \lor \neg e$$

First Approach: Resolution

$a \vee \neg b \vee f$ $\qquad\qquad$ $\neg a \vee \neg c \vee d \vee \neg e$

$$\neg b \vee f \vee \neg c \vee d \vee \neg e$$

- Resolution eliminates one variable by producing a new clause (*resolvent*) from complementary ones.

Resolution

$(a \vee b) \wedge (a \vee \neg b) \wedge (\neg a \vee c) \wedge (\neg a \vee \neg c)$

Resolution

$(a \lor b) \land (a \lor \neg b) \land (\neg a \lor c) \land (\neg a \lor \neg c)$

Resolution

$$(a \lor b) \land (a \lor \neg b) \land (\neg a \lor c) \land (\neg a \lor \neg c)$$

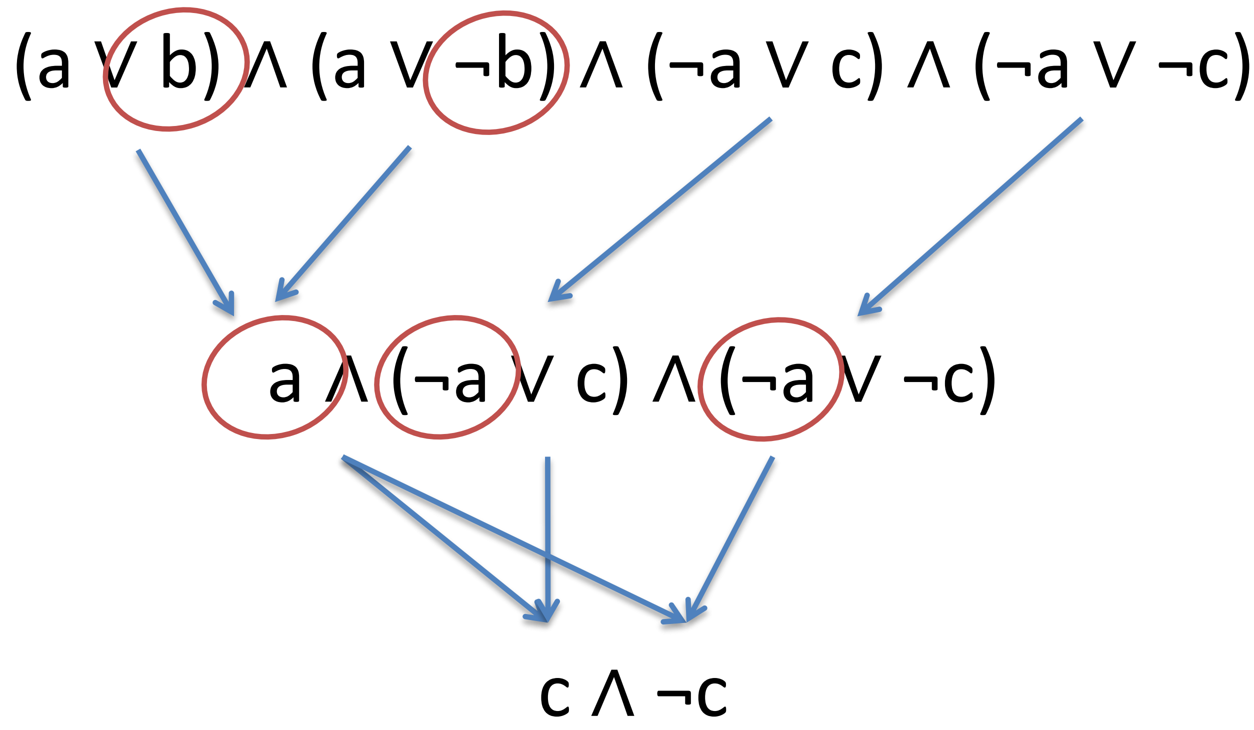
$$a \land (\neg a \lor c) \land (\neg a \lor \neg c)$$

Resolution

$$(a \lor b) \land (a \lor \neg b) \land (\neg a \lor c) \land (\neg a \lor \neg c)$$

$$a \land (\neg a \lor c) \land (\neg a \lor \neg c)$$

Resolution



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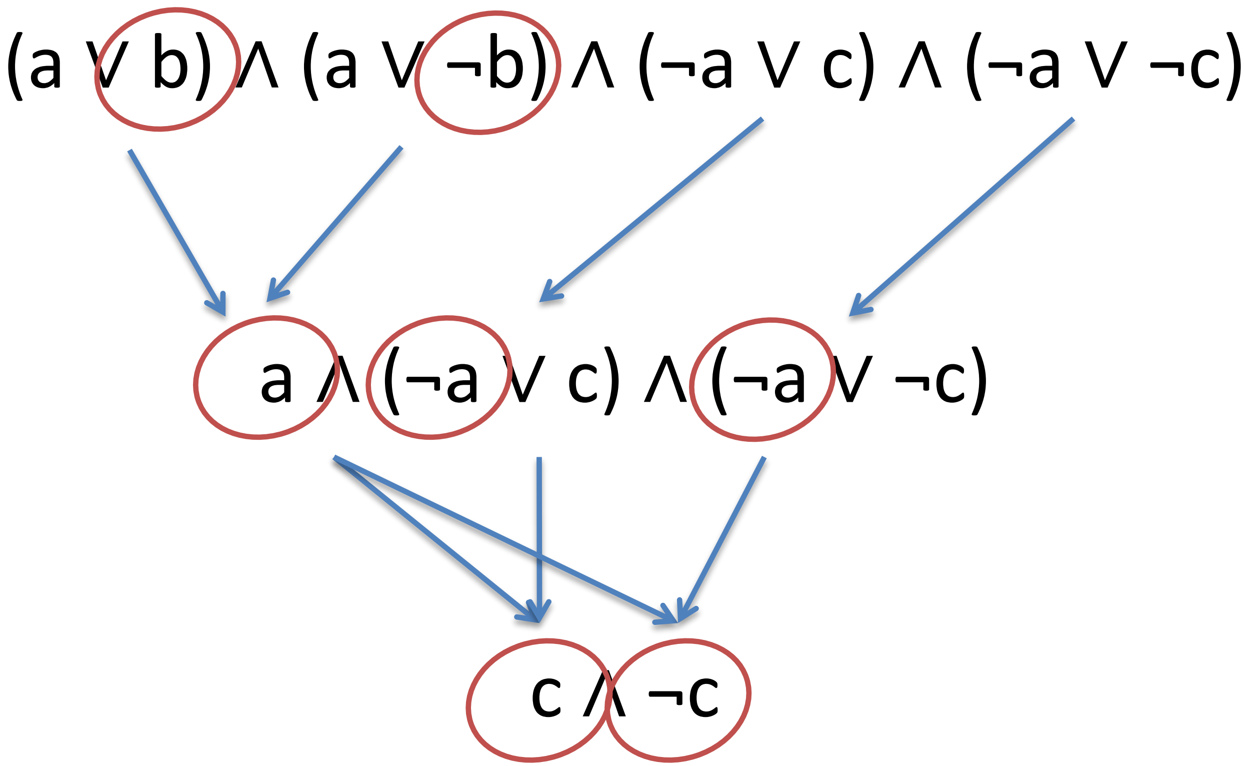

(Part of) Davis Putnam Algorithm

- (Also: when a variable appears in only one polarity, remove all clauses containing it.)
- M. Davis, H. Putnam, *A computing procedure for quantification theory*, JACM, 1960.
- Problem: space explosion!
- DP is *proof-oriented*. Current algorithms are *model-oriented*.

Backtracking Search

$$
\begin{aligned}
&(\quad\quad\ \ b \lor \neg c\,) \\
\land\ &(\neg a \lor b \lor c\,) \\
\land\ &(\neg a \lor \neg b\quad\ )
\end{aligned}
$$

Backtracking Search

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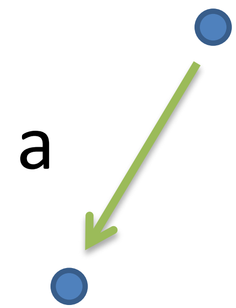

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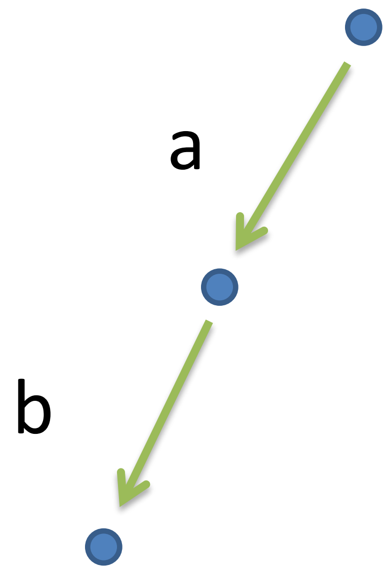
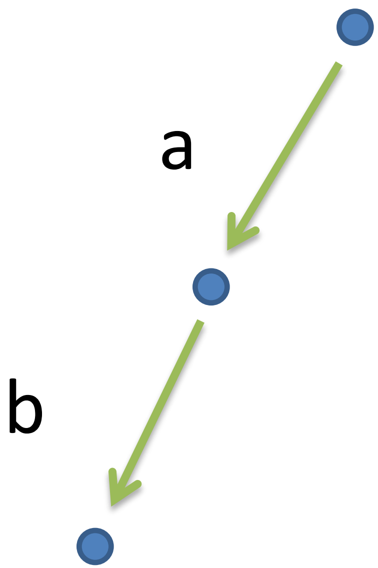


Backtracking Search

$$(\; b \lor \neg c \;)$$
$$\land\; (\; \neg a \lor b \lor c \;)$$
$$\land\; (\; \neg a \lor \neg b \;) \; !$$



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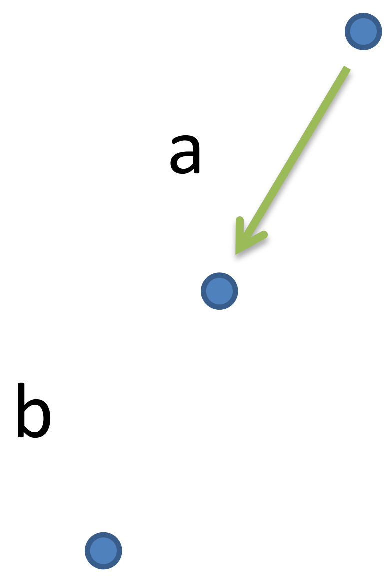

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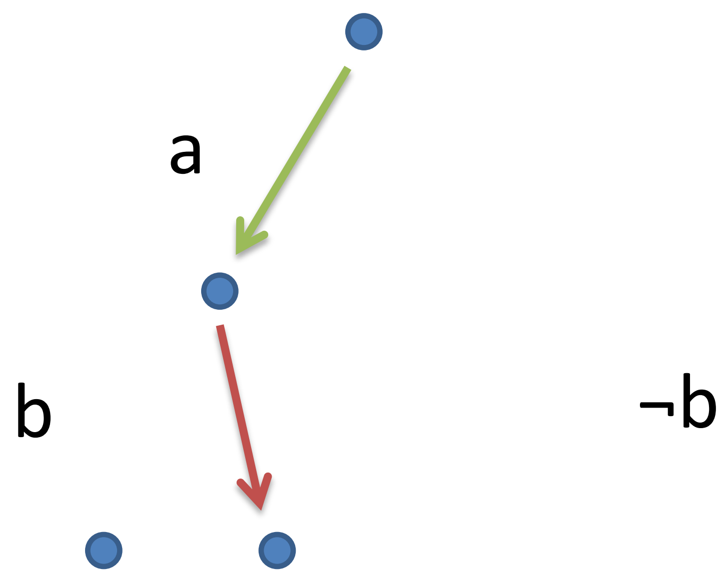

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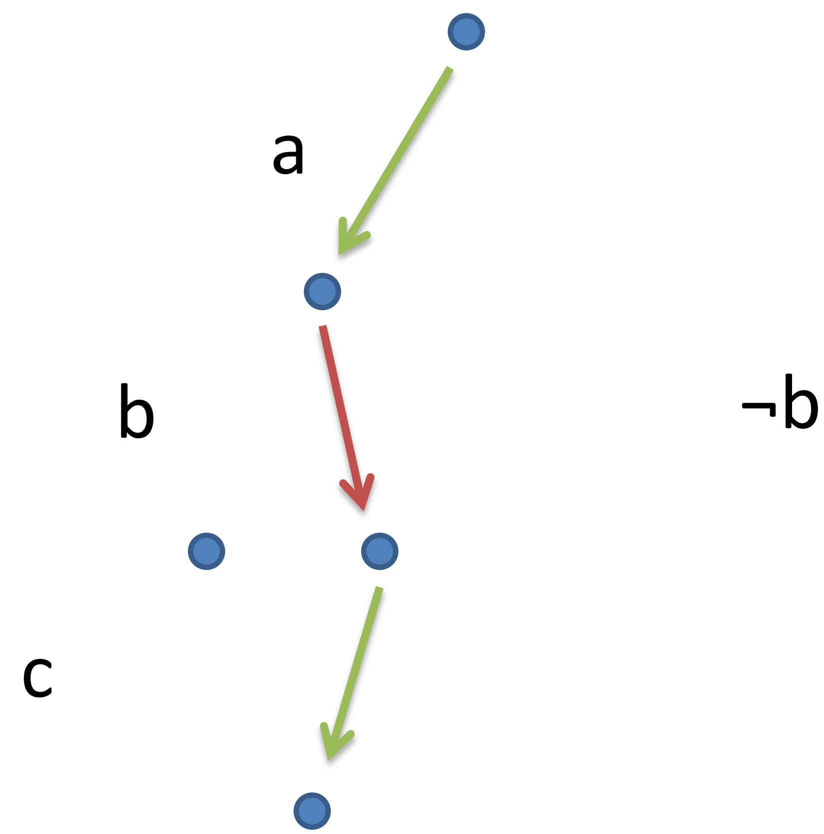

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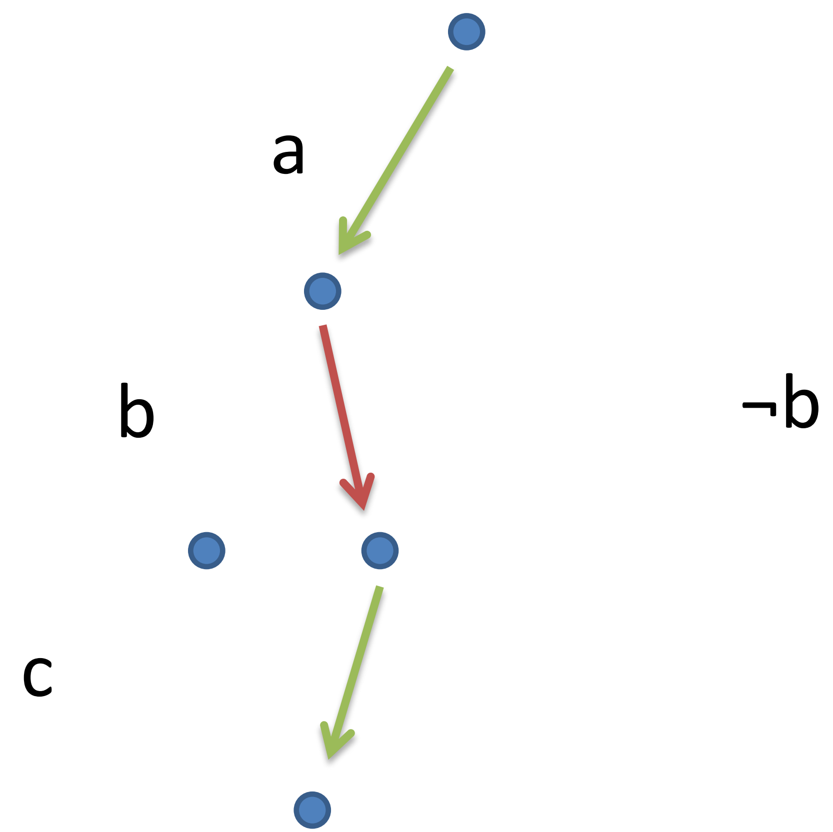

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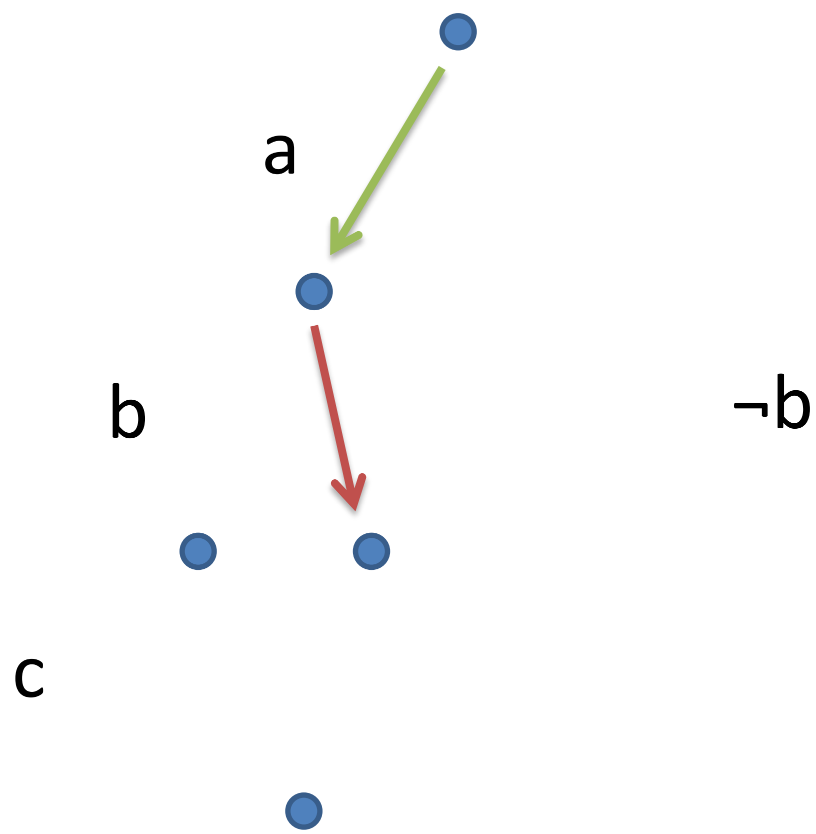

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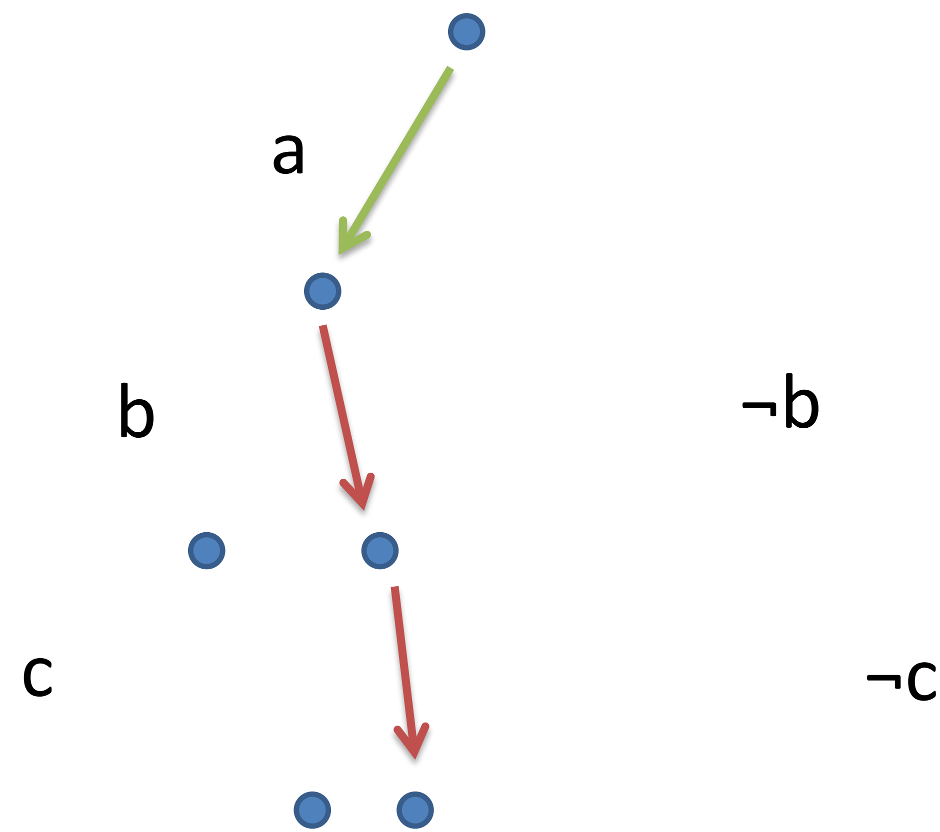

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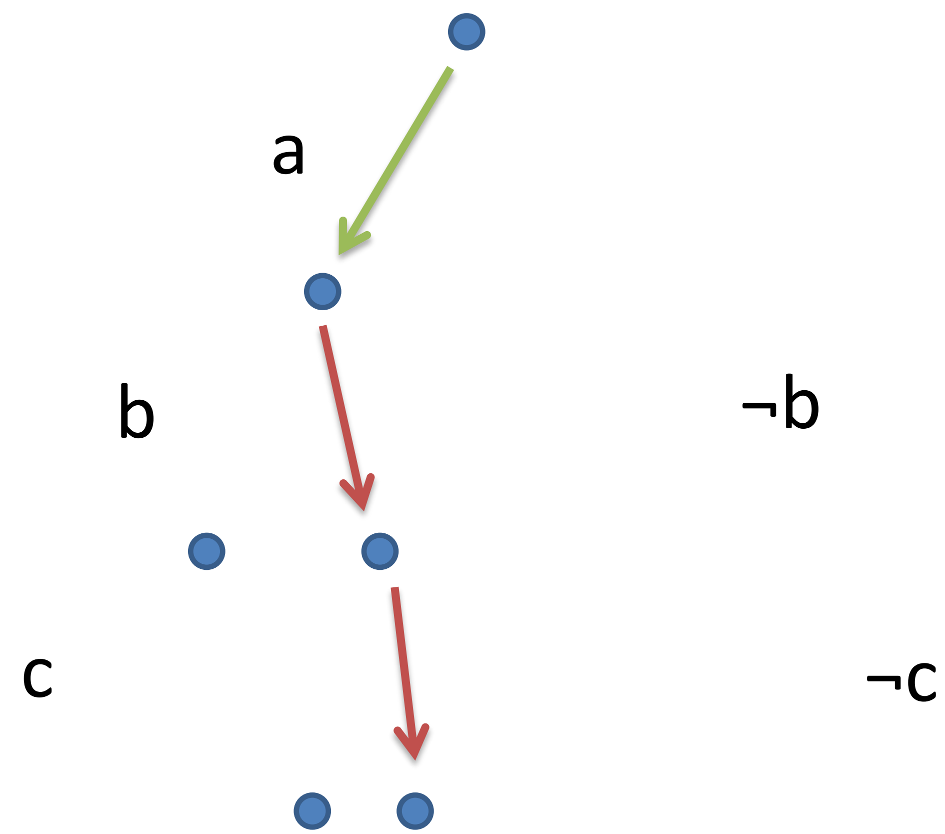

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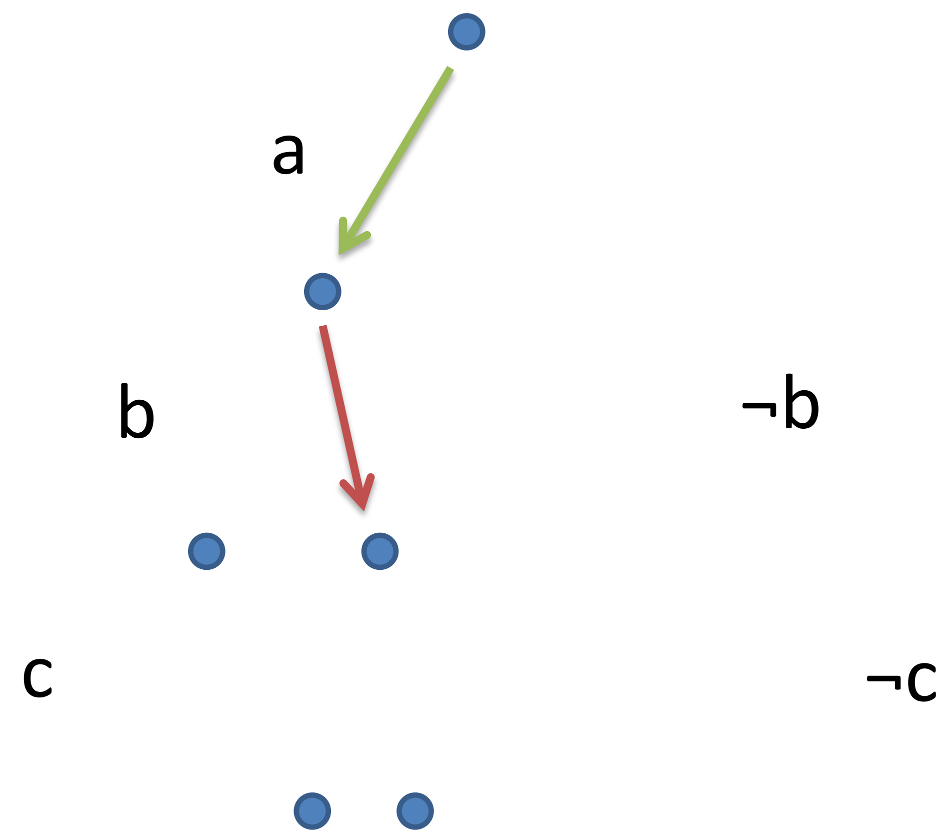

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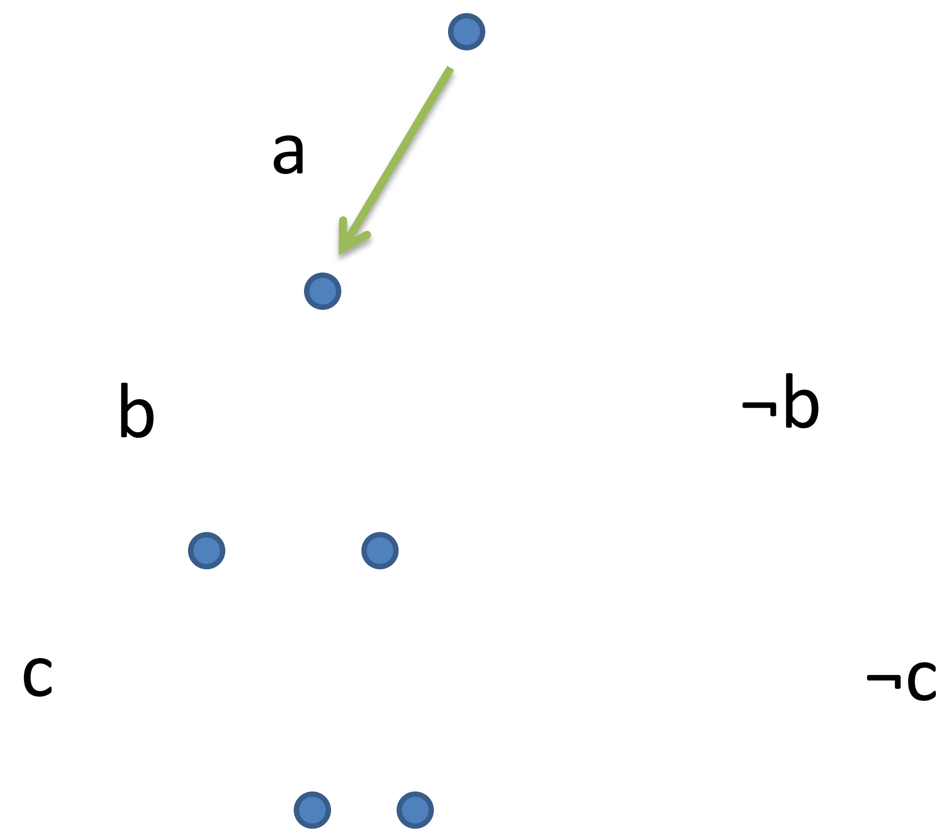

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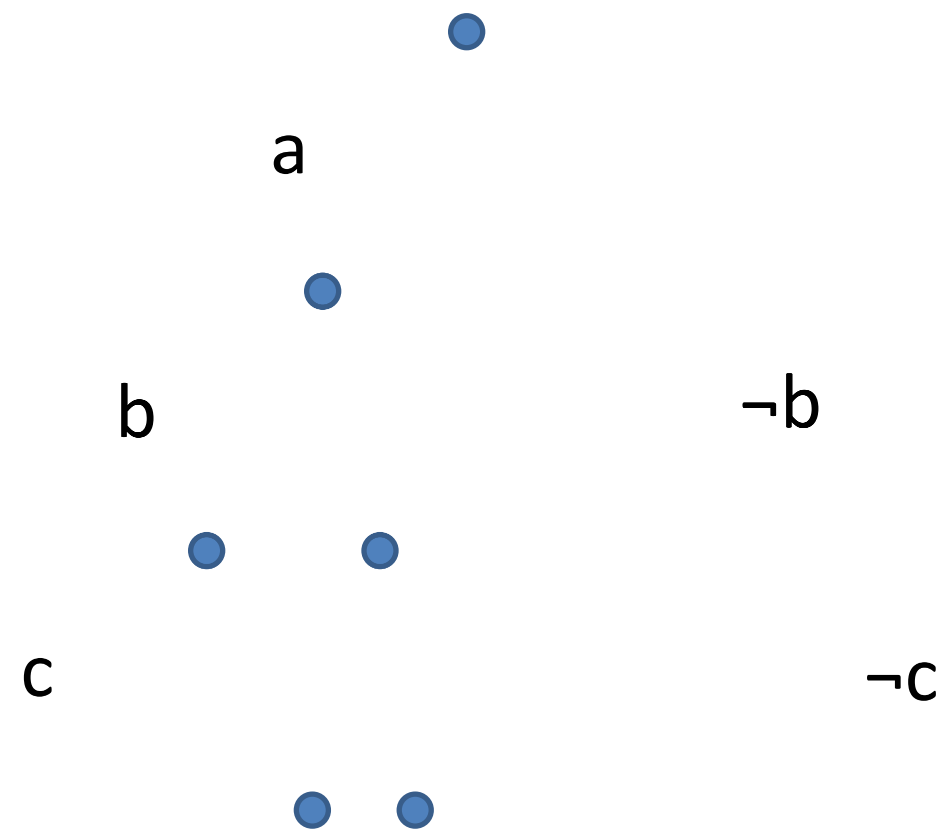

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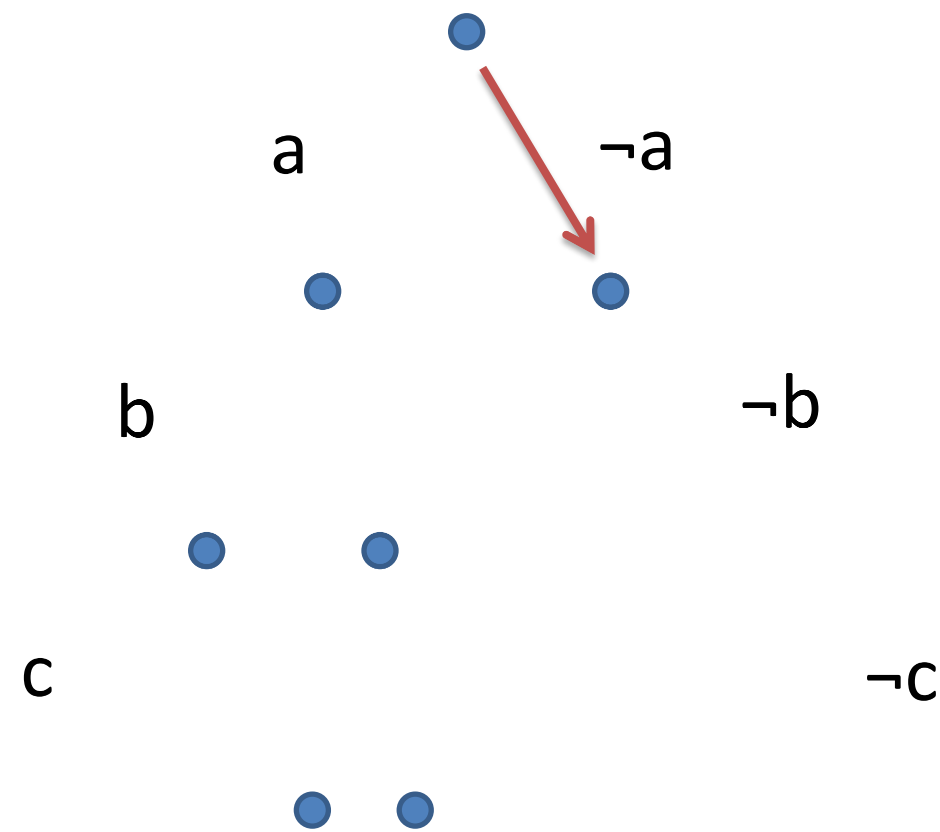

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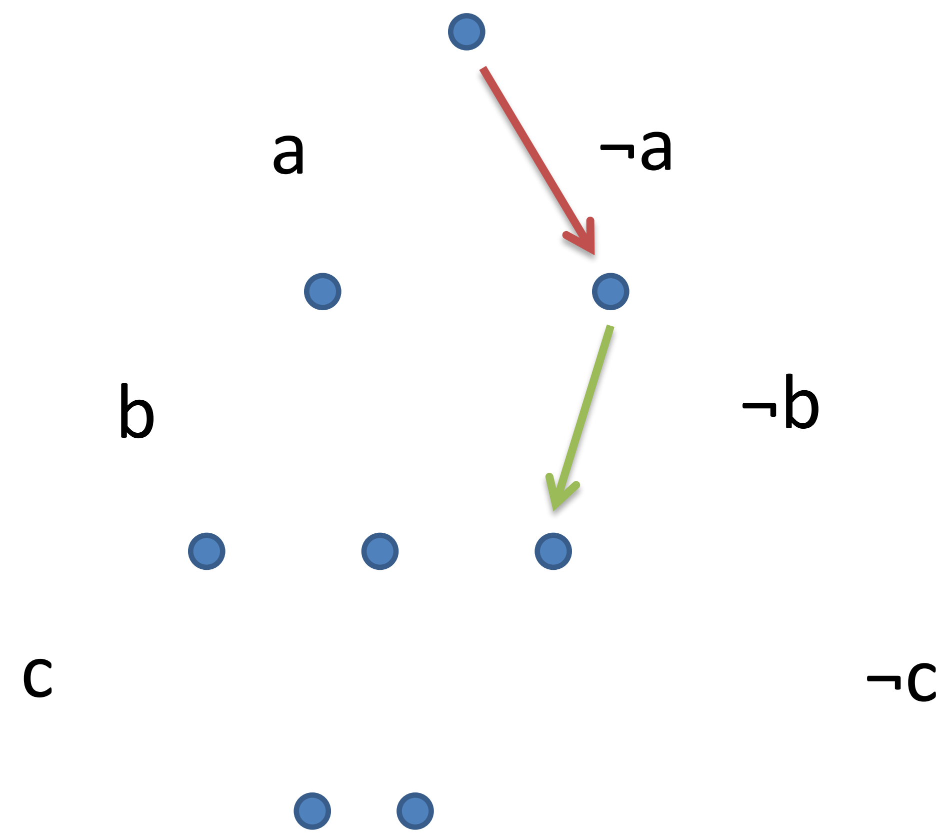

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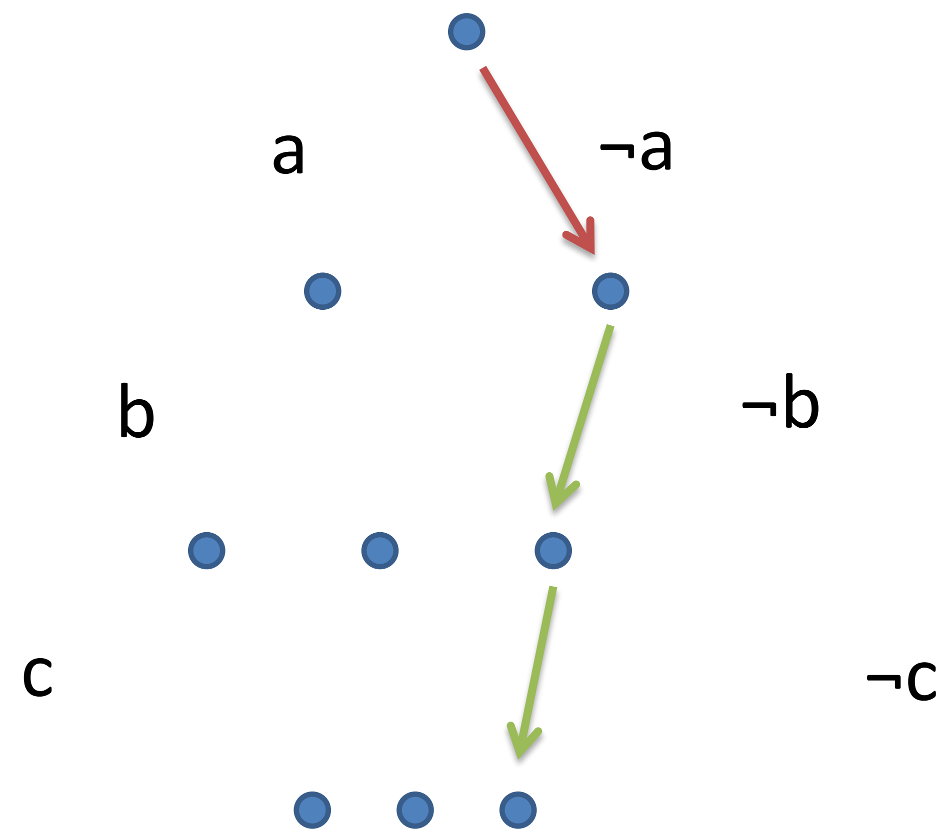

Boolean Constraint Propagation

- *"When all but one literal are falsified, it becomes implied."*

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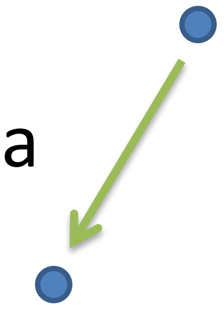
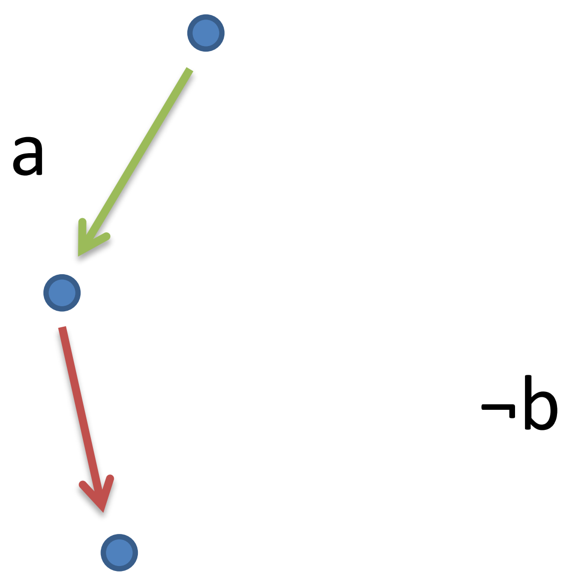
\end{aligned}
$$



Boolean Constraint Propagation

- *"When all but one literal are falsified, it becomes implied."*

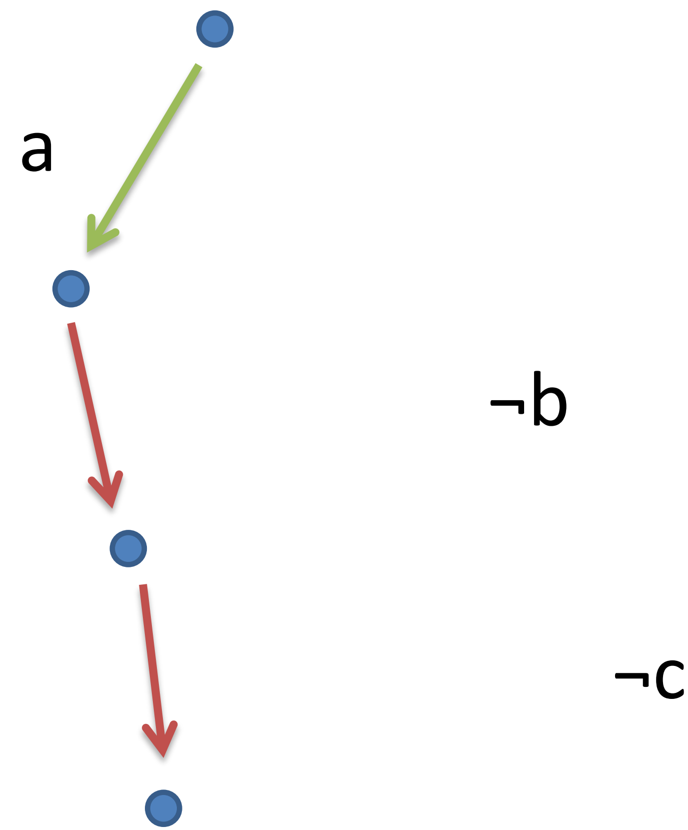
$$
\begin{aligned}
& (b \lor \neg c) \\
\land\ & (\neg a \lor b \lor c) \\
\land\ & (\neg a \lor \neg b)
\end{aligned}
$$



Boolean Constraint Propagation

- *"When all but one literal are falsified, it becomes implied."*

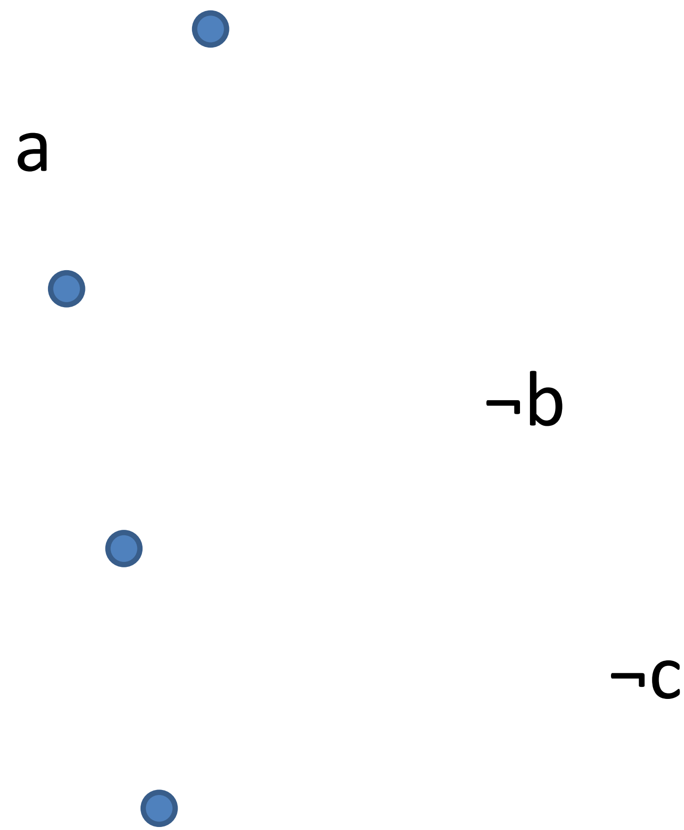
$$
\begin{aligned}
& (\qquad\quad b \lor \lnot c\ ) \\
\land\ & (\ \lnot a \lor b \lor c\ ) \\
\land\ & (\ \lnot a \lor \lnot b\qquad\ )
\end{aligned}
$$



Boolean Constraint Propagation

- *"When all but one literal are falsified, it becomes implied."*

$$
\begin{aligned}
& (\quad\quad\;\; b \lor \neg c\,) \\
\land\, & (\,\neg a \lor b \lor c\,) \\
\land\, & (\,\neg a \lor \neg b\,)
\end{aligned}
$$



Boolean Constraint Propagation

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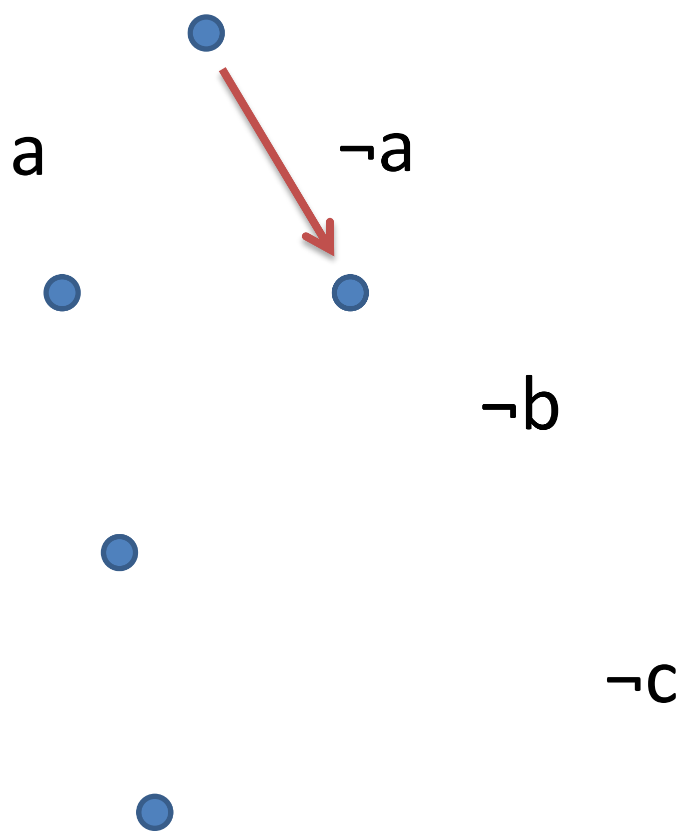
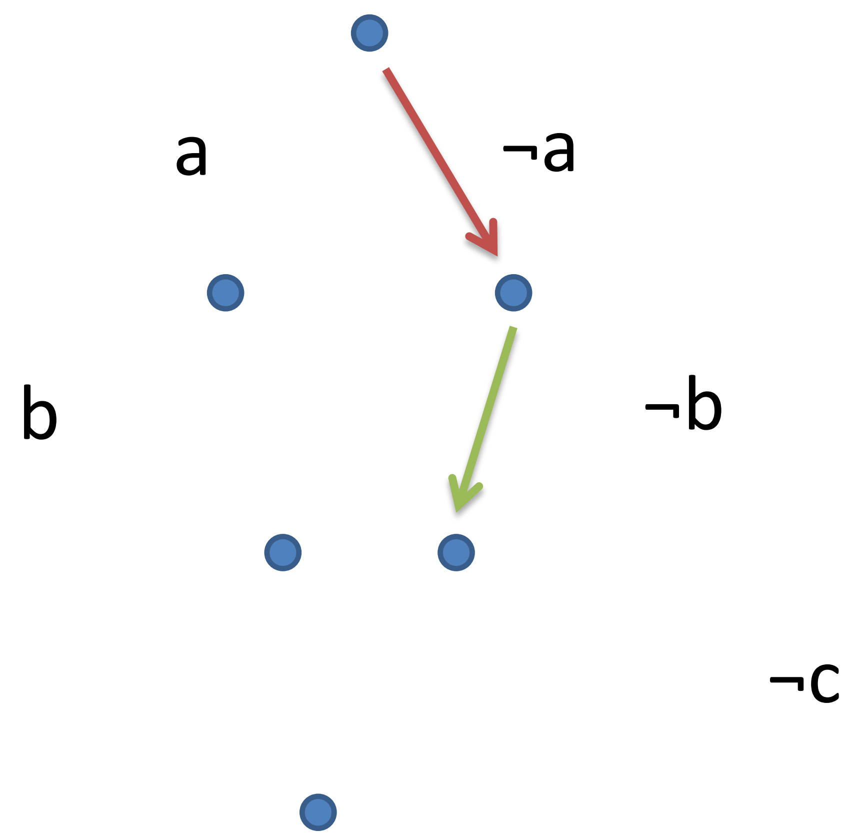
\land\ & (\neg a \lor b \lor c) \\
\land\ & (\neg a \lor \neg b)
\end{aligned}
$$



Boolean Constraint Propagation

- *"When all but one literal are falsified, it becomes implied."*

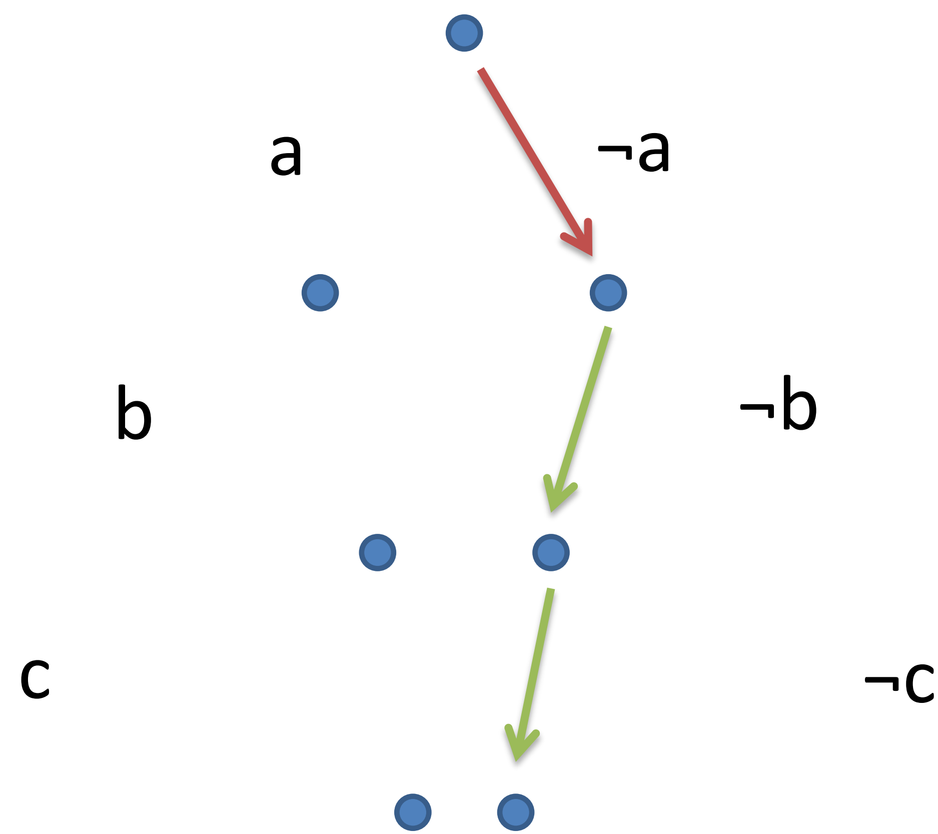
$$
\begin{aligned}
& (\quad\quad\quad\quad b \lor \neg c\ ) \\
\land\ & (\ \neg a \lor \ \ b \lor \ \ \ c\ ) \\
\land\ & (\ \neg a \lor \neg b \quad\quad\quad)
\end{aligned}
$$



Boolean Constraint Propagation

- *"When all but one literal are falsified, it becomes implied."*

$$
\begin{aligned}
& (\quad\quad\ \ b \vee \neg c\,) \\
\wedge\ & (\,\neg a \vee b \vee c\,) \\
\wedge\ & (\,\neg a \vee \neg b\quad\ \ )
\end{aligned}
$$



Two-watched-literal Scheme for BCP

- BCP can cut the search tree dramatically…
- …but checking each clause for potential implications is expensive.

- Observation: as long as at least two literals in a clause are “not false”, that clause does not imply any new literal.
- Idea: for each clause, try to maintain that invariant.

Cutting Deeper: Learning

- Idea: compute new clauses that are logically implied, and that may trigger more BCP.

- Use an *implication graph*. When a conflict is derived, look for a *small explanation*.

Learning

$$
\begin{aligned}
& (a \lor d) \\
\land\ & (a \lor \neg c \lor \neg h) \\
\land\ & (a \lor h \lor \neg m) \\
\land\ & (b \lor k) \\
\land\ & (\neg g \lor \neg c \lor i) \\
\land\ & (\neg g \lor h \lor \neg i) \\
\land\ & (g \lor h \lor \neg j) \\
\land\ & (g \lor j \lor \neg m)
\end{aligned}
$$

Learning

$$
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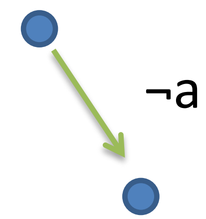
\land\ & (g \lor h \lor \neg j) \\
\land\ & (g \lor j \lor \neg m)
\end{aligned}
$$



Learning

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\begin{aligned}
& (a \lor d) \\
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\land\ & (a \lor h \lor \neg m) \\
\land\ & (b \lor k) \\
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\land\ & (\neg g \lor h \lor \neg i) \\
\land\ & (g \lor h \lor \neg j) \\
\land\ & (g \lor j \lor \neg m)
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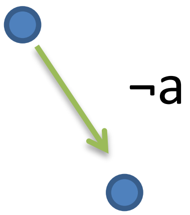
$$



Learning

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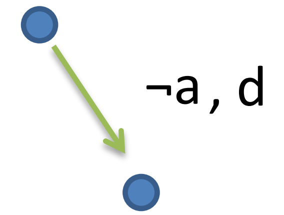
\land\ & (g \lor j \lor \neg m)
\end{aligned}
$$



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\land\ & (g \lor h \lor \neg j) \\
\land\ & (g \lor j \lor \neg m)
\end{aligned}
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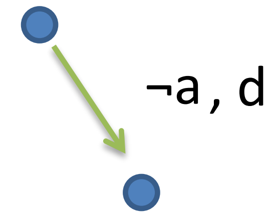

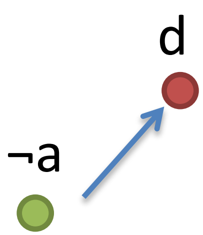

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\land\ & (b \lor k) \\
\land\ & (\neg g \lor \neg c \lor i) \\
\land\ & (\neg g \lor h \lor \neg i) \\
\land\ & (g \lor h \lor \neg j) \\
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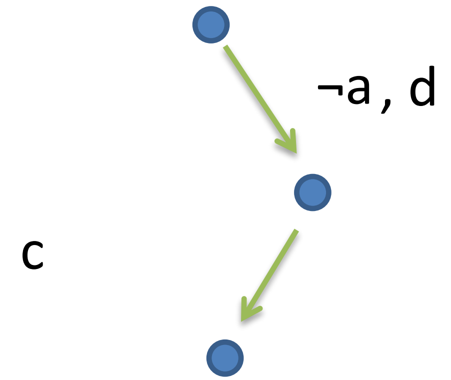

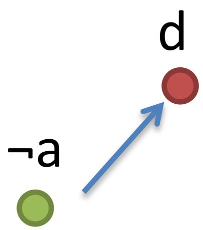

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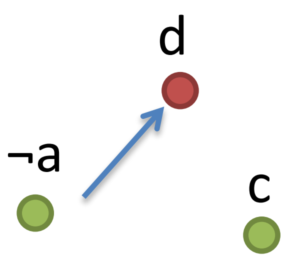

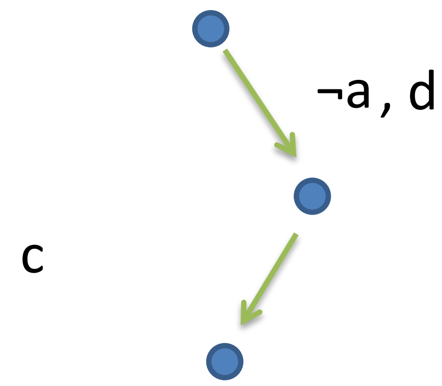

Learning

$(a \lor d)$
$\land\ (a \lor \neg c \lor \neg h)$
$\land\ (a \lor h \lor \neg m)$
$\land\ (b \lor k)$
$\land\ (\neg g \lor \neg c \lor i)$
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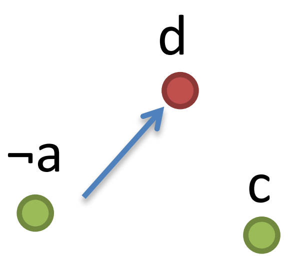

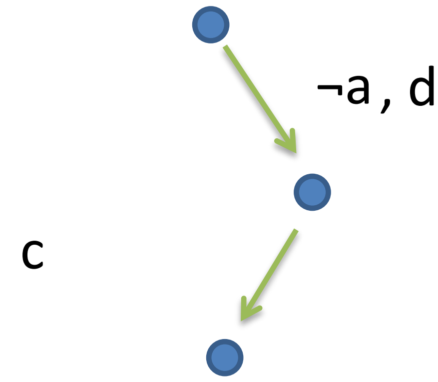

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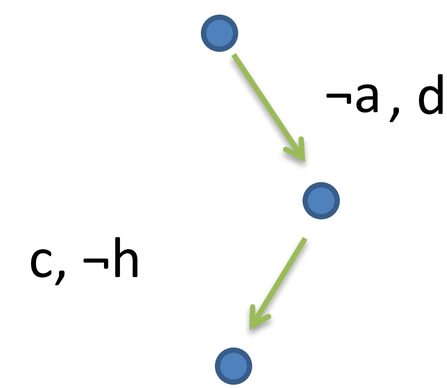

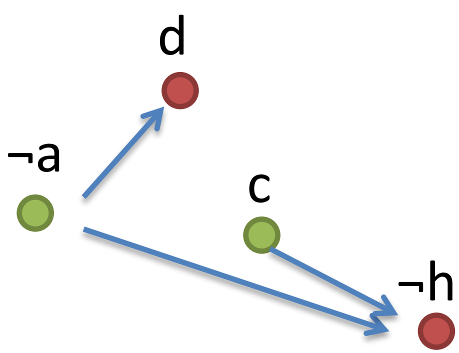

Learning

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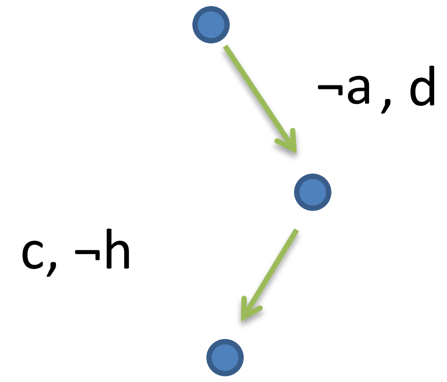

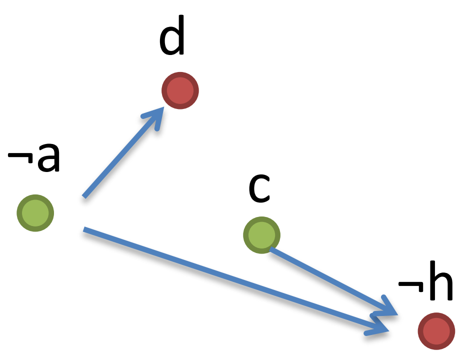

Learning

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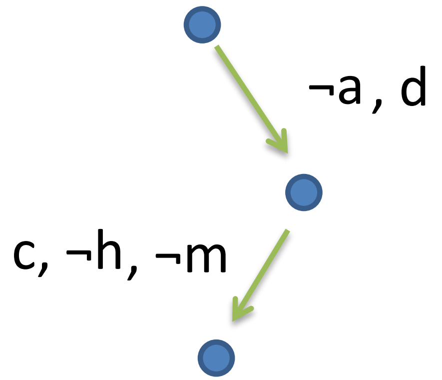
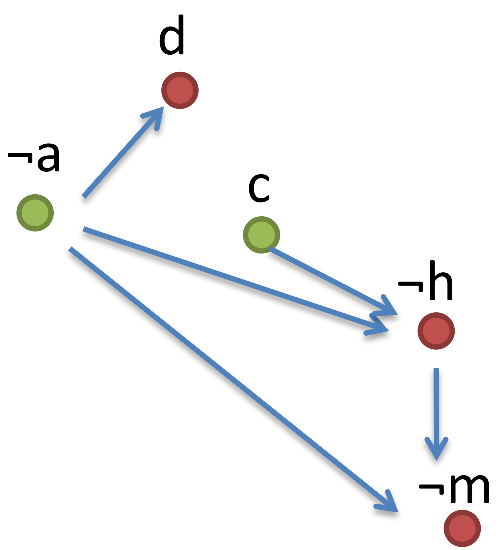
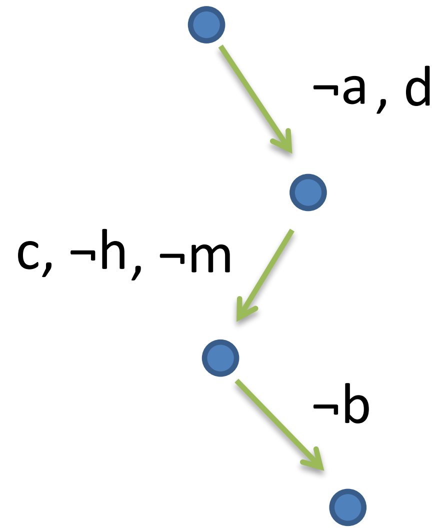
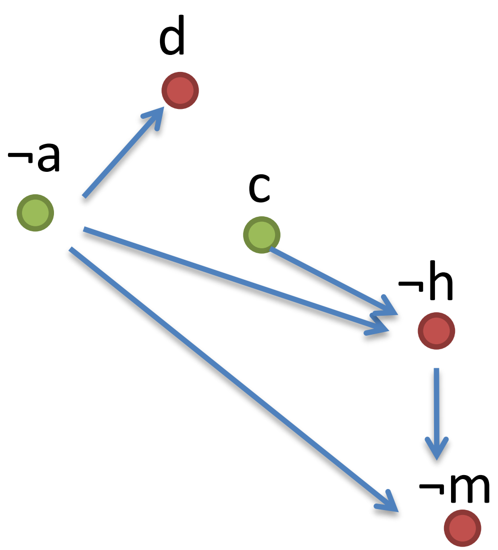




Learning

(a ∨ d)
∧ (a ∨ ¬c ∨ ¬h)
∧ (a ∨ h ∨ ¬m)
∧ (b ∨ k)
∧ (¬g ∨ ¬c ∨ i)
∧ (¬g ∨ h ∨ ¬i)
∧ (g ∨ h ∨ ¬j)
∧ (g ∨ j ∨ ¬m)





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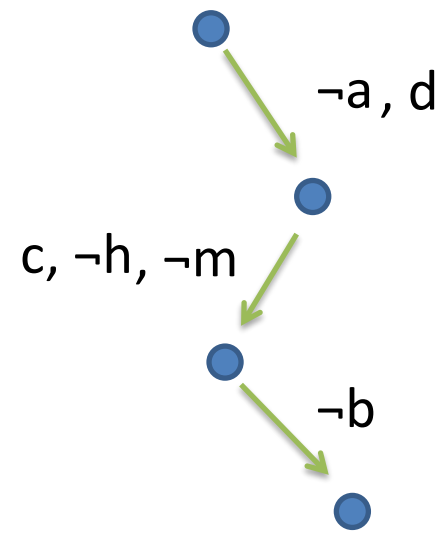

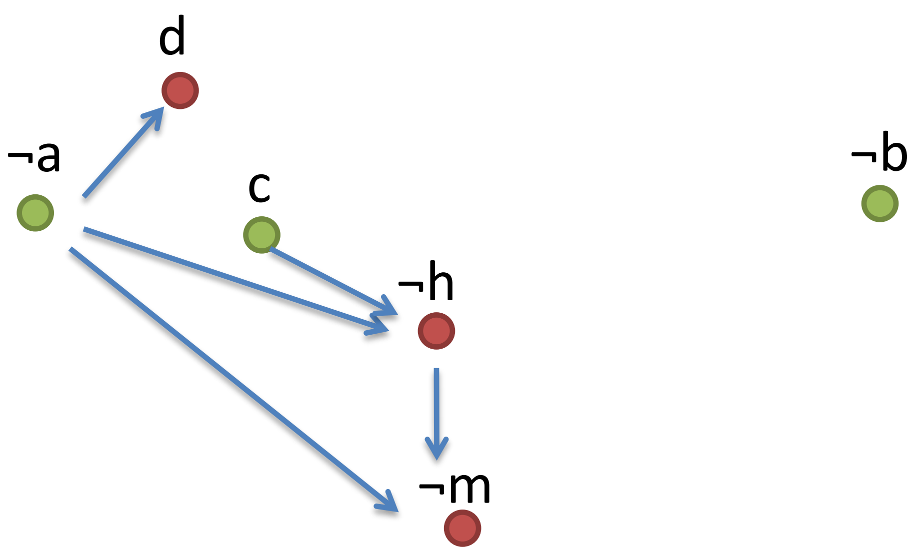

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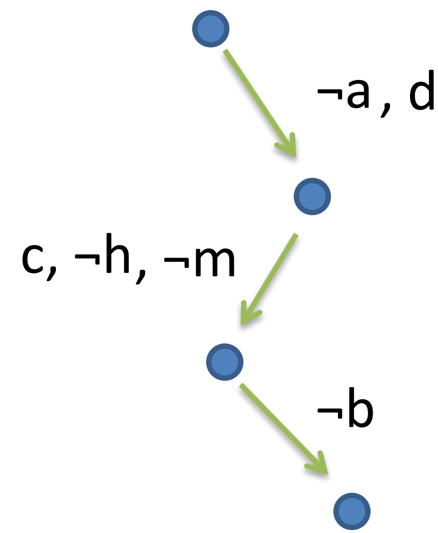

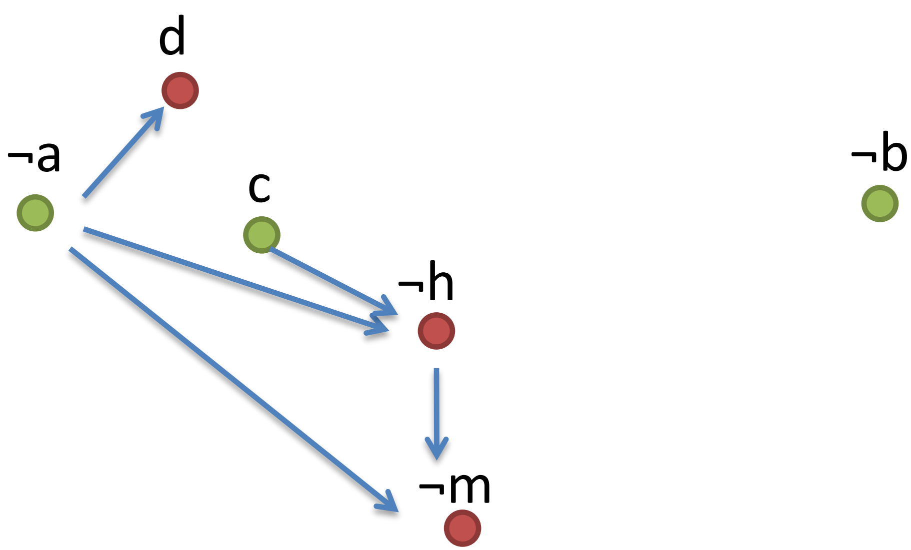

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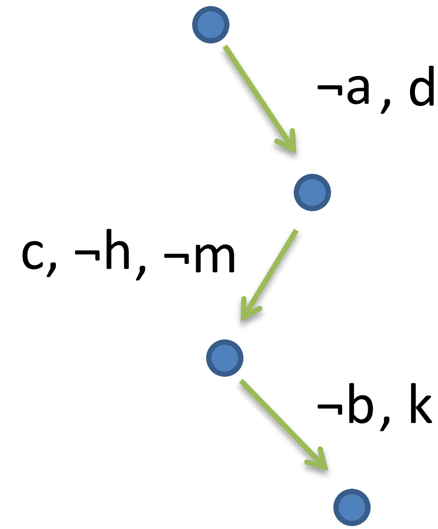

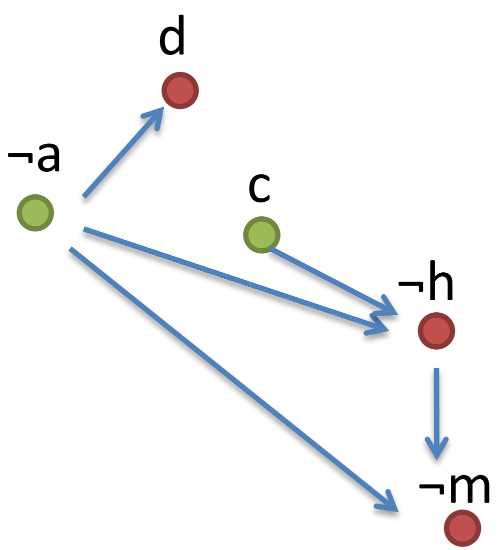

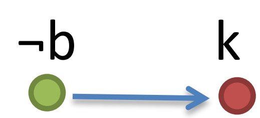

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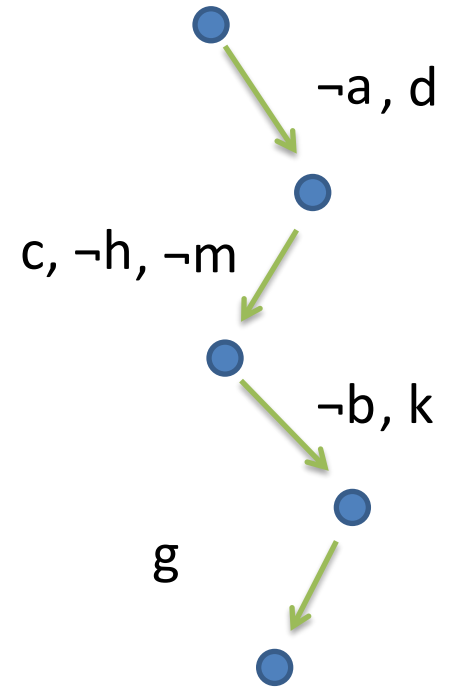

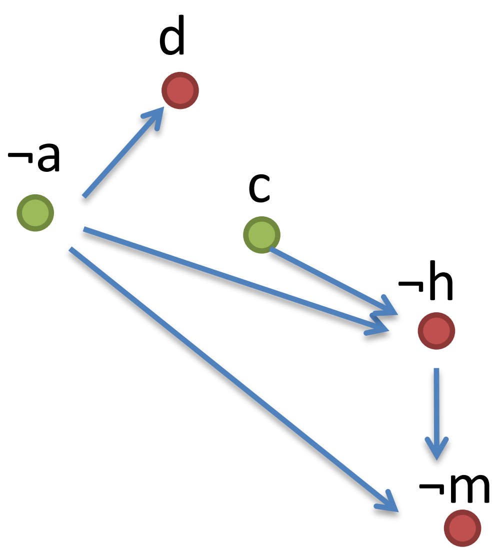

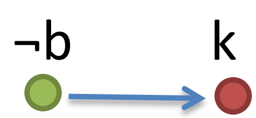

Learning

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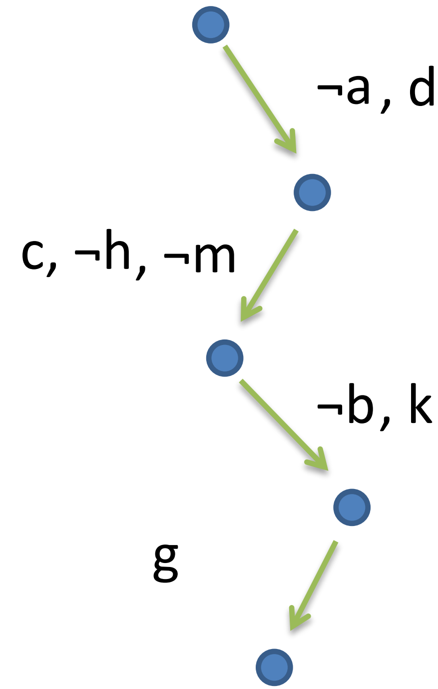

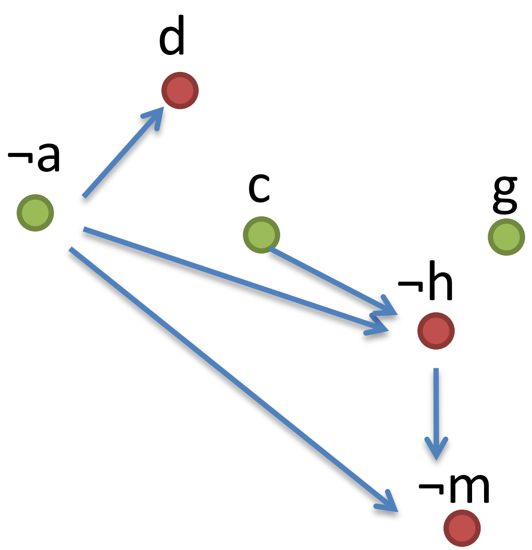

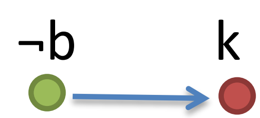

Learning

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\land\ & (g \lor j \lor \neg m)
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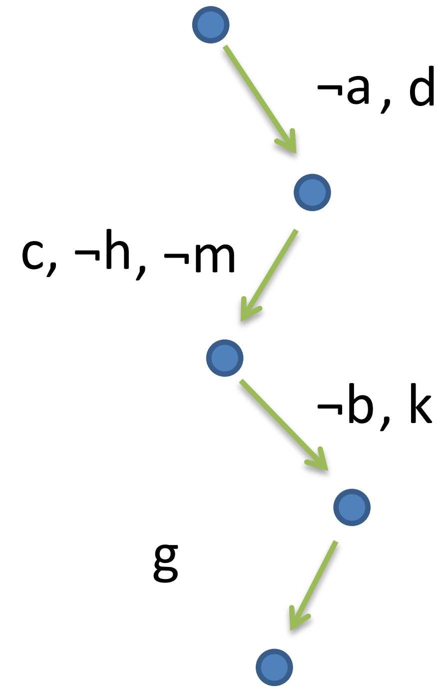

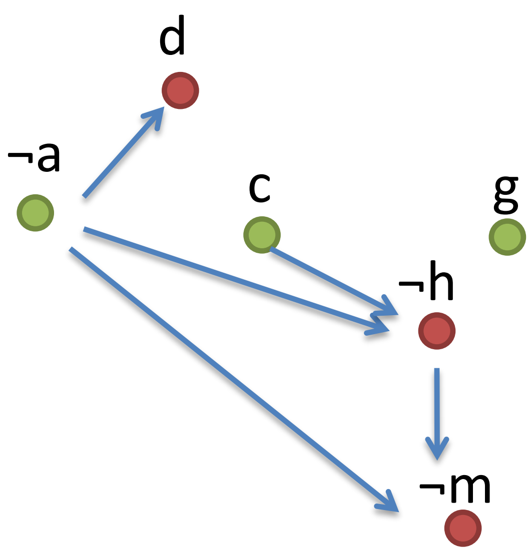

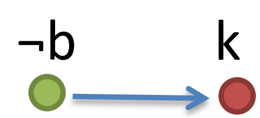

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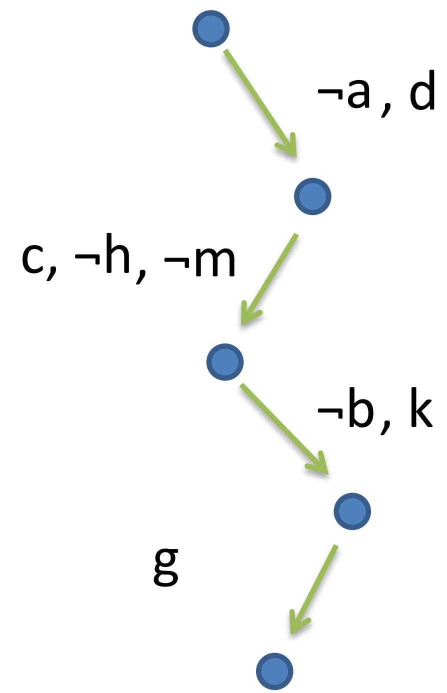

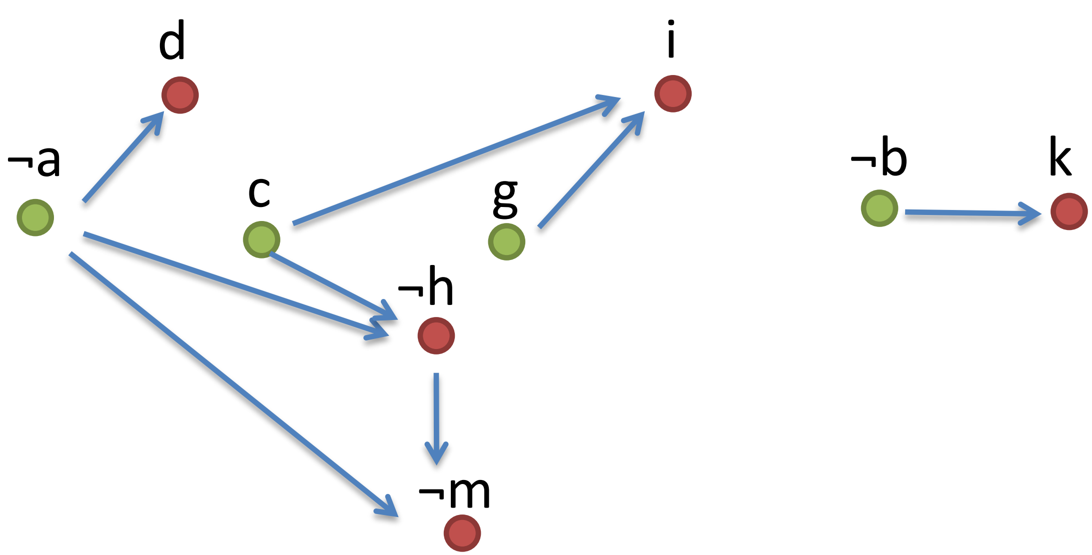

Learning

$$
\begin{aligned}
& (a \lor d) \\
\land\ & (a \lor \neg c \lor \neg h) \\
\land\ & (a \lor h \lor \neg m) \\
\land\ & (b \lor k) \\
\land\ & (\neg g \lor \neg c \lor i) \\
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\land\ & (g \lor h \lor \neg j) \\
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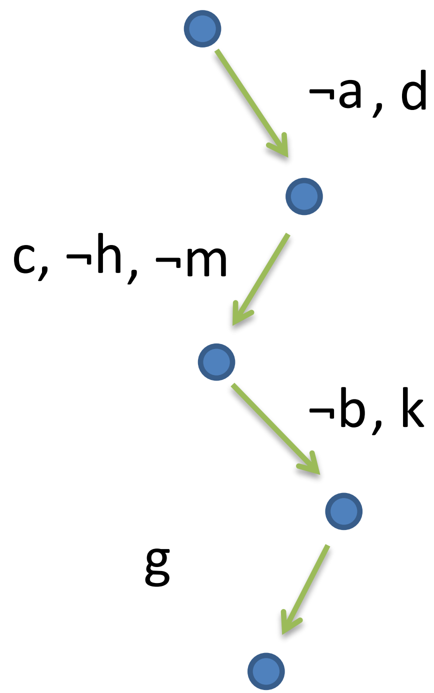
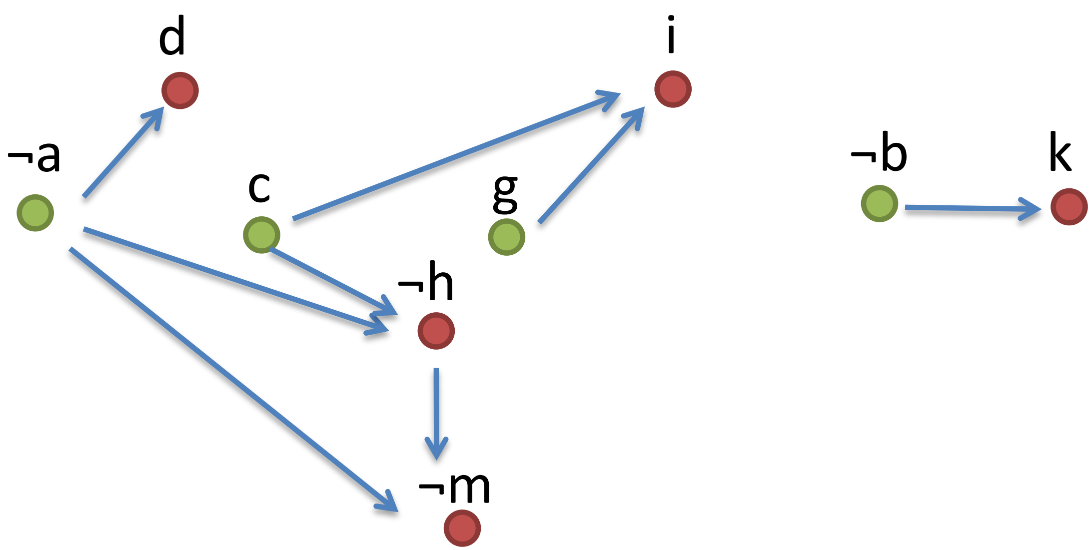
$$





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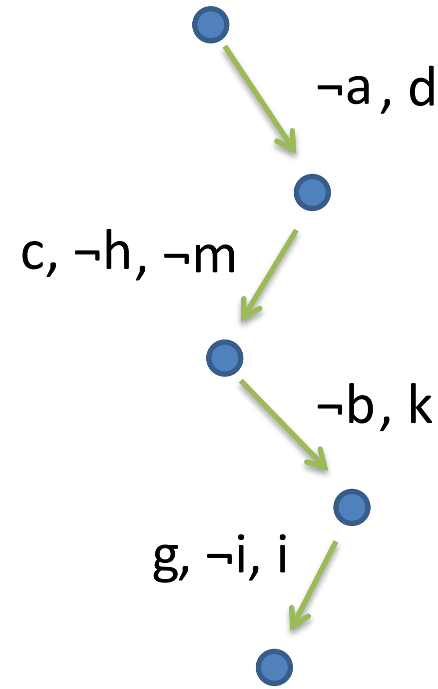

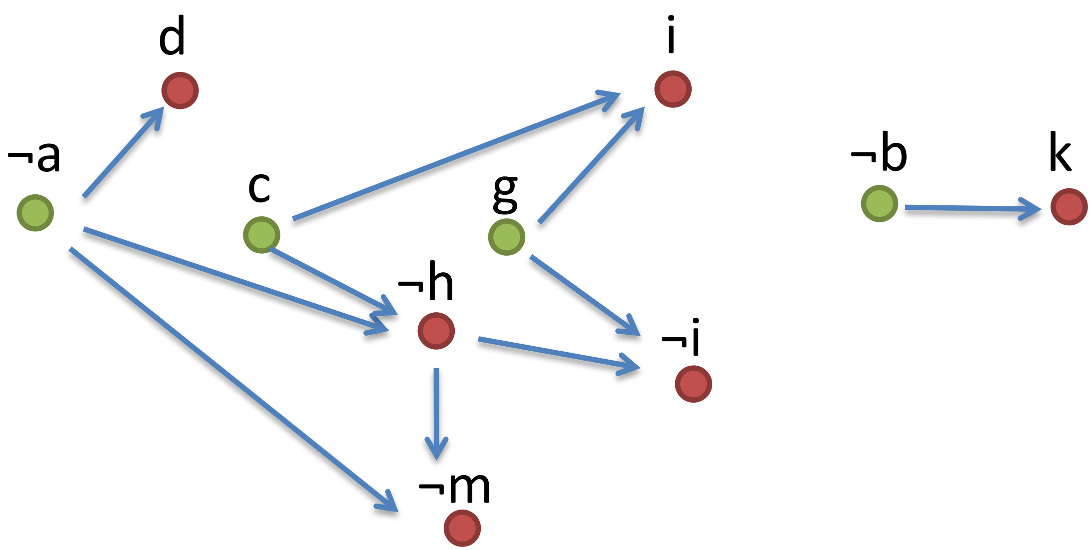

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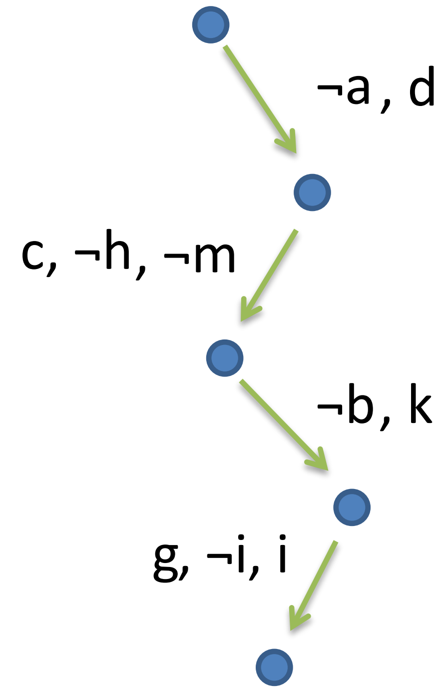

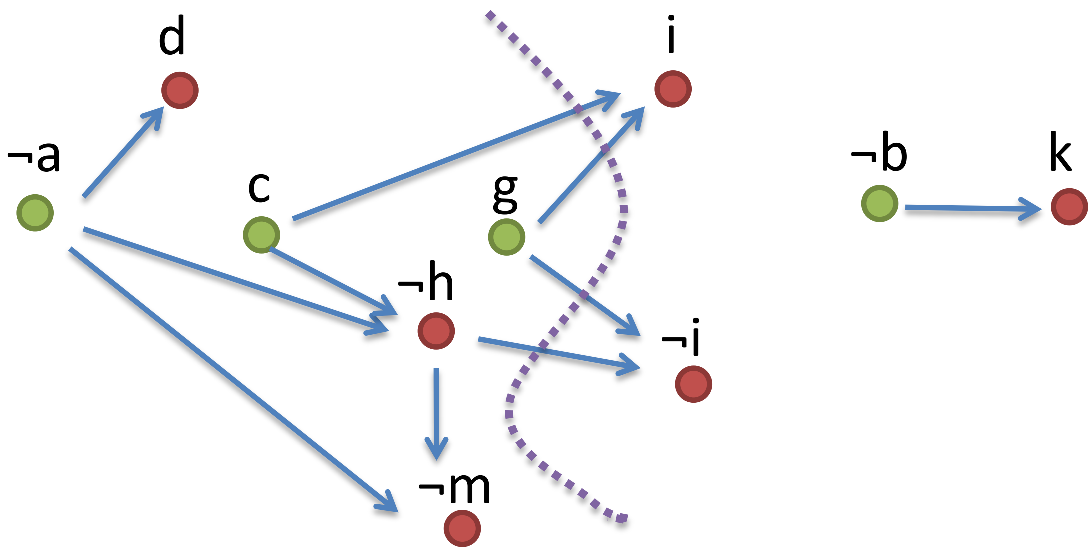

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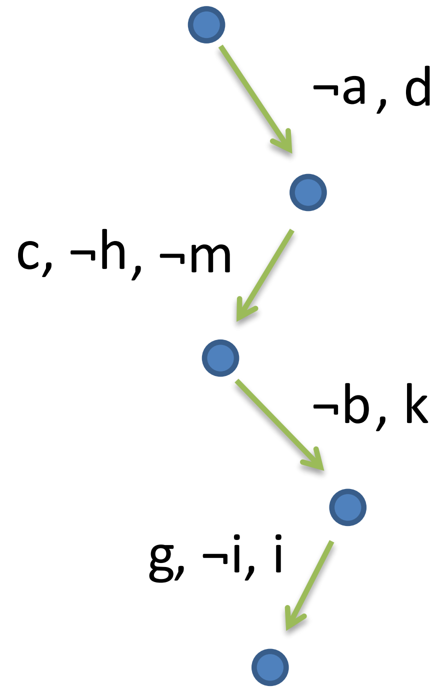

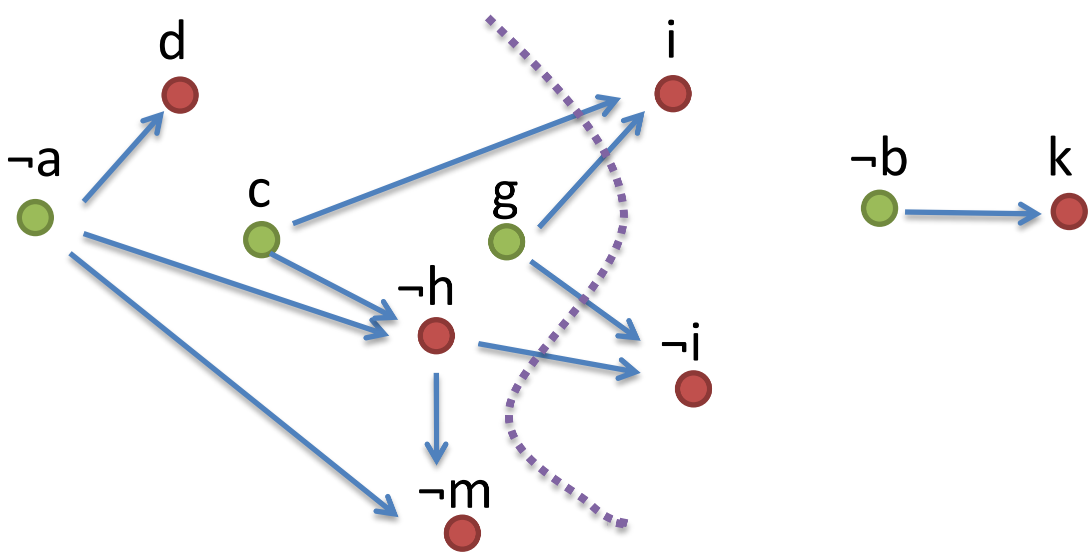

$\neg(c \land g \land \neg h)$

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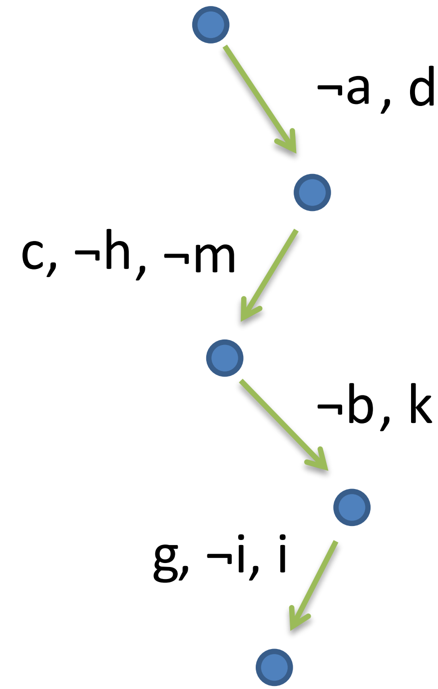

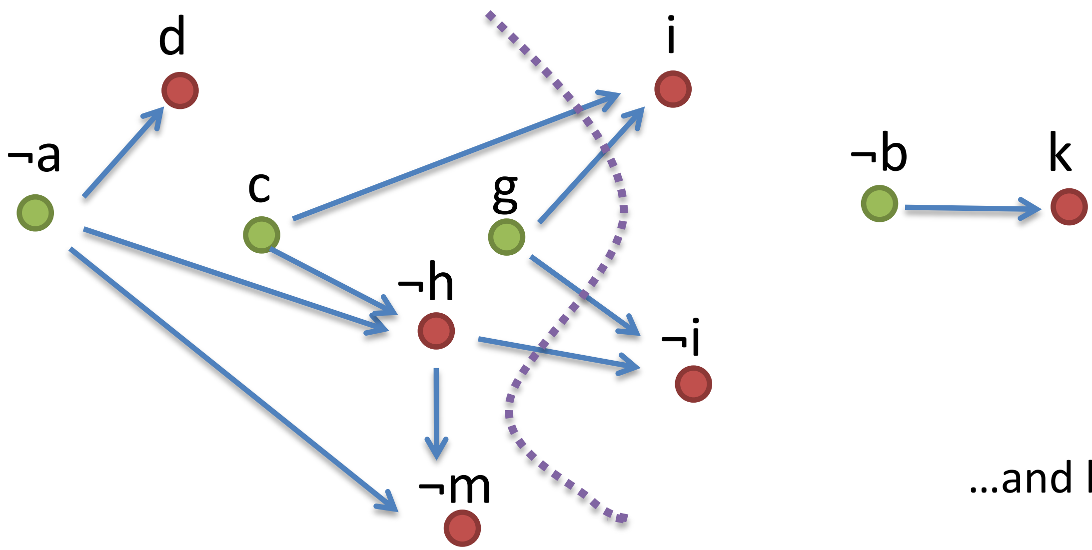

$\neg(c \land g \land \neg h)$

…and backtrack to c, then assert ¬g !

Learning

- Learning has a dramatically positive impact.

- Learning also makes *restarts* possible:
  - Idea: after some number of literal assignments, drop the assignment stack and restart from zero.
  - Goal: avoid locally difficult subtrees.
  - Clauses encode previous knowledge and make new search faster.

Picking Variable Assignments

- Potential strategies:
  - Fixed ordering,
  - Frequency based,
  - “Maximal impact”.

Picking Variable Assignments

- Potential strategies:
  - Fixed ordering,
  - Frequency based,
  - “Maximal impact”.

- Overall favorite are activity-based heuristics:
  - Pick variables that you have seen a lot in conflicts.
  - Decay weights to favor recent conflicts.
  - Cheap to compute/update.

More Engineering…

- SAT dirty little secret: the enormous impact of preprocessing.
  - Problems are generated automatically (“compiled”); many redundancies, symmetry, etc.
  - Preprocessors look for subsumed clauses, equivalent clauses, etc.
  - Typically, run with timeout, then DPLL search.

More Engineering…

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- Parallel SAT
  - State-of-the-art is to run instances with different parameters in parallel.

Beyond SAT

- SMT solvers
  - Idea: use a SAT solver for the propositional structure, and theory solvers for conjunction of literals.

- QBF
  - SAT with quantifiers. PSPACE complete.