Lecturecise 13
Abstract Interpretation - solutions to exercises

2013

Exercise

Let $C[0,1]$ be the set of continuous functions from $[0,1]$ to the reals. Define $\leq$ on $C[0,1]$ by $f \leq g$ if and only if $f(a) \leq g(a)$ for all $a \in [0,1]$.

i) Show that $\leq$ is a partial order and that $C[0,1]$ with this order forms a lattice.

ii) Does an analogous statement hold if we consider the set of differentiable functions from $[0,1]$ to the reals? That is, instead of requiring the functions to be continuous, we require them to have a derivative on the entire interval. (The order is defined in the same way.)

Solution

i) To show $C[0,1]$ is a partial order, show it is reflexive, antisymmetric and transitive (easy). To show that it is a lattice, we need to show that every two elements f and g have a least upper bound and a greatest lower bound. If f and g are comparable, one is the upper bound and the other is the lower bound. If not, then we can always find a piece-wise defined continuous function that is the upper bound. This function will be equal to f or g on parts of the domain where f and g do not intersect, (which one depends on which bound we want and whichever function has larger values on that part of the domain.) On intersections, we switch to define the next piece by the other function.

Solution

ii) If we consider differentiable functions on $[0, 1]$, then $\leq$ is still a partial order. Again, if f and g are comparable, the we choose one as the upper and the other as a lower bound. For incomparable functions, i.e. when f and g intersect, our construction from (i) fails, since at intersections the resulting function may not be differentiable.
Differentiable functions are necessarily continuous and we know that continuous functions on a closed interval are bounded, hence we can always find some upper bound that is differentiable. However, it is not possible to find a differentiable least upper bound, since no matter which upper bound you pick, you can always find one that is closer to the functions f and g and is still smooth.

Exercise

Let $A = [0,1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ be the interval of real numbers. Recall that, by definition of real numbers and complete lattice, $(A, \le)$ is a complete lattice with least lattice element 0 and greatest lattice element 1. Here $\sqcup$ is the least upper bound operator on sets of real numbers, also called //supremum// and denoted //sup// in real analysis.
Let function $f : A \to A$ be given by

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{4}x, & \text{if } x \in [0, \frac{2}{3}) \\ \\ \frac{3}{5} + \frac{1}{5}x, & \text{if } x \in [\frac{2}{3}, 1] \end{cases}$$

(It may help you to try to draw f.)

a) Prove that f is monotonic and injective (so it is strictly monotonic).

b) Compute the set of fixpoints of f.

c) Define $iter(x) = \sqcup\{f^n(x) \mid n \in \{0,1,2,\ldots\}\}$. (This is in fact equal to $\lim_{n\to\infty} f^n(x)$ when f is a monotonic bounded function.)
   Compute $iter(0)$ (prove that the computed value is correct by definition of $iter$, that is, that the value is indeed $\sqcup$ of the set of values). Is $iter(0)$ a fixpoint of f? Is $iter(iter(0))$ a fixpoint of f?

Solution

The idea:
What this question is trying to show here is that Tarski's fixed point theorem says that a least fixed point alsways exists if the function is strictly mnotonic, but does not give a way to compute it. This example shows how one can do this, even in the case of discontinuities: iterate until your function converges, then either you found a fixed point, or you did not because of a discontinuity. Then you apply f to the fixed point, which makes you jump over it, and you repeat until you hit a true fixed point.
Solution:

a) f is monotonic and injective on $[0, \frac{2}{3})$ and $(\frac{2}{3}, 1]$, since those are linear functions. $\lim_{x \to 2/3}(\frac{1}{2} + \frac{1}{4}x) = \frac{2}{3} < f(\frac{2}{3}) = \frac{11}{15}$, hence f remains strictly monotonic also across the boundary.

b) For $[0, \frac{2}{3})$, solving $x = \frac{1}{2} + \frac{1}{4}x$, we get $x = \frac{2}{3}$, which is outside of the defined domain for this piece.
For $[\frac{2}{3}, 1]$, solving $x = \frac{3}{5} + \frac{1}{5}x$, we get $x = \frac{3}{4}$, hence f has one fixed point at $\frac{3}{4}$.

Solution

c) Compute *iter*(0):
$f(0) = \frac{1}{2})$, $f^2(0) = f(\frac{1}{2}) = \frac{5}{8}) = 0.625$, $f^3(0) = f(\frac{5}{8}) = \frac{21}{32}) = 0.65625$,
$f^4(0) = f(\frac{21}{32}) = \frac{85}{128}) = 0.6640625$, $f^5(0) = f(\frac{85}{128}) = \frac{341}{512}) = 0.666015625$.
Since f is strictly monotonic and $\lim_{x \to 2/3}(\frac{1}{2} + \frac{1}{4}x) = \frac{2}{3}$, $iter(0) = \frac{2}{3}$.
$\frac{2}{3}$ is not a fixed point of f.
$iter(iter(0)) = iter(\frac{2}{3}) = \frac{3}{4}$, which is a fixed point of f
Is f ω-continuous, i.e. does it hold for any chain $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \ldots \sqsubseteq x_n \sqsubseteq \ldots$ that

$$f(\bigsqcup_{i \geq 0} x_i) = \bigsqcup_{i \geq 0} f(x_i)$$

Consider the chain given by $\lim_{x \to 2/3}$. For this chain the above property does not hold, hence f is not ω-continuous.