Synthesis, Analysis, and Verification

SAV 2013

Acceleration

Filip Konečný

filip.konecny@epfl.ch

Verification Conditions for Programs with Loops

Recap: Verification Conditions (VC)

- Given:
  - a program P with mutable variables $\mathbf{x}$ – transforms initial values to final values
  - a specification $\psi(\mathbf{x}, \mathbf{x}')$ that describes how the program should transform initial values to final values. $\mathbf{x}$ refer to initial values, $\mathbf{x}'$ refer to final values.
- Check if P conforms to ψ
  - compute relation $R_P(\mathbf{x}, \mathbf{x}')$ – summary of P which precisely captures how P transforms initial values to final values
  - check if $R_P(\mathbf{x}, \mathbf{x}') \rightarrow \psi(\mathbf{x}, \mathbf{x}')$ is valid
- The above can be done in certain cases
  - Presburger statements, no loops

```
S ::= if (*) S else S | assume(F) | havoc(x) | S;S | ...
```

- R_P was defined inductively on the structure of P
  - for every statement S, relation R_S captures the effects of executing S
- Can we do better?

VCs for Programs with Loops

❖ Let us extend the syntax by a loop construct:

```
while(*) body
```

- semantics: iterate the body any number of times (possibly zero times) and then continue with the next statement
- the following snippets are equivalent

```
while(condition) {
  body
}
```

```
while(*) {
  assume(condition)
  body
}
assume(!condition)
```

```
S ::= while (*) S | if (*) S else S | assume(F) | havoc(x) | S;S | ...
```

❖ Effect of a loop = effects of executing its body any number of times.

$$R_{loop} \stackrel{def}{=} R^*_{body}$$

captured by the reflexive and transitive closure of R_{body}

❖ When can we compute R^*_{body} (when can we accelerate R_{body})?

VCs for Programs with Loops

- Which relations are accelerable?
  - difference bounds, octagonal, and finite monoid affine relations

All of them are fragments of Presburger arithmetics. Given a relation R from one of these classes, R^* is a computable Presburger formula.

VCs for Programs with Loops

❖ Which relations are accelerable?

- difference bounds, octagonal, and finite monoid affine relations

All of them are fragments of Presburger arithmetics. Given a relation R from one of these classes, R^* is a computable Presburger formula.

❖ Example. Consider a program P and a specification $\psi(x, x') \equiv x' \geq 0$

```
S1: x = 10
L:  while (*)
S2:   x = x+1
```

$$
\begin{aligned}
R_{S_1} &\equiv x' = 10 \\
R_{S_2} &\equiv x' = x + 1
\end{aligned}
$$

VCs for Programs with Loops

❖ Which relations are accelerable?

- difference bounds, octagonal, and finite monoid affine relations

All of them are fragments of Presburger arithmetics. Given a relation R from one of these classes, R^* is a computable Presburger formula.

❖ Example. Consider a program P and a specification $\psi(x, x') \equiv x' \geq 0$

```
S1: x = 10
L:  while (*)
S2:   x = x+1
```

$$R_{S_1} \equiv x' = 10$$
$$R_{S_2} \equiv x' = x + 1$$

$$R_L \equiv R^*_{S_2} \equiv x' \geq x$$

VCs for Programs with Loops

- Which relations are accelerable?
  - difference bounds, octagonal, and finite monoid affine relations

All of them are fragments of Presburger arithmetics. Given a relation R from one of these classes, R^* is a computable Presburger formula.

- Example. Consider a program P and a specification $\psi(x, x') \equiv x' \geq 0$

```
S1: x = 10
L:  while (*)
S2:   x = x+1
```

$$\begin{aligned} R_{S_1} &\equiv x' = 10 \\ R_{S_2} &\equiv x' = x + 1 \end{aligned}$$

$$\begin{aligned} R_L &\equiv R_{S_2}^* \equiv x' \geq x \\ R_P &\equiv R_{S_1} \circ R_L \equiv x' \geq 10 \end{aligned}$$

VCs for Programs with Loops

- Which relations are accelerable?
  - difference bounds, octagonal, and finite monoid affine relations

All of them are fragments of Presburger arithmetics. Given a relation R from one of these classes, R^* is a computable Presburger formula.

- Example. Consider a program P and a specification $\psi(x, x') \equiv x' \geq 0$

```
S1: x = 10
L:  while (*)
S2:   x = x+1
```

$$\begin{array}{rcl} R_{S_1} & \equiv & x' = 10 \\ R_{S_2} & \equiv & x' = x + 1 \end{array}$$

$$\begin{array}{rcccl} R_L & \equiv & R^*_{S_2} & \equiv & x' \geq x \\ R_P & \equiv & R_{S_1} \circ R_L & \equiv & x' \geq 10 \\ R_P & \Rightarrow & \psi \text{ is valid} & & \end{array}$$

Difference Bounds Relations

Difference Bounds Relations

- Conjunctions of atomic propositions $x - y \leq c$, $c \in \mathbb{Z}$

Difference Bounds Relations

- Conjunctions of atomic propositions $x - y \leq c$, $c \in \mathbb{Z}$
  - Example 1:
    $(x' = x + 1) \equiv (x' \leq x + 1) \wedge (x + 1 \leq x') \equiv (x' - x \leq 1) \wedge (x - x' \leq -1)$

Difference Bounds Relations

- Conjunctions of atomic propositions $x - y \leq c$, $c \in \mathbb{Z}$
  - Example 1:
    $(x' = x + 1) \equiv (x' \leq x + 1) \wedge (x + 1 \leq x') \equiv (x' - x \leq 1) \wedge (x - x' \leq -1)$
  - Example 2:

$$R \equiv \bigwedge \begin{array}{rcr} x_1 - x_1' & \leq & 3 \\ x_1 - x_2' & \leq & -5 \\ x_2' - x_1 & \leq & 2 \\ x_1' - x_2' & \leq & -1 \end{array}$$

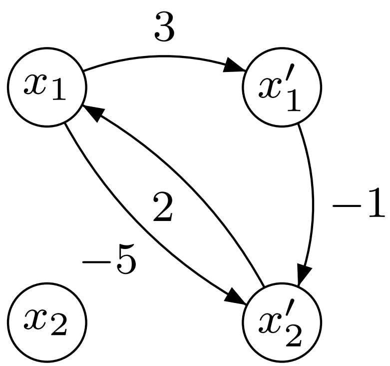
Difference Bounds Relations

❖ Conjunctions of atomic propositions $x - y \leq c$, $c \in \mathbb{Z}$

- Example 1:
  $(x' = x + 1) \equiv (x' \leq x + 1) \wedge (x + 1 \leq x') \equiv (x' - x \leq 1) \wedge (x - x' \leq -1)$

- Example 2:

$$R \equiv \bigwedge \begin{array}{rcr} x_1 - x_1' & \leq & 3 \\ x_1 - x_2' & \leq & -5 \\ x_2' - x_1 & \leq & 2 \\ x_1' - x_2' & \leq & -1 \end{array}$$



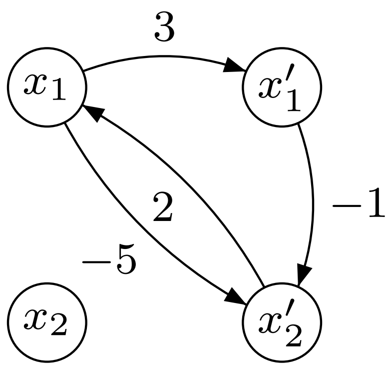
- naturally encoded as graphs

$$x - y \leq c \qquad \textbf{iff} \qquad x \xrightarrow{c} y$$

Difference Bounds Relations

- Conjunctions of atomic propositions $x - y \leq c$, $c \in \mathbb{Z}$
  - Example 1:
    $(x' = x + 1) \equiv (x' \leq x + 1) \wedge (x + 1 \leq x') \equiv (x' - x \leq 1) \wedge (x - x' \leq -1)$
  - Example 2:

$$R \equiv \bigwedge \begin{array}{rcr} x_1 - x_1' & \leq & 3 \\ x_1 - x_2' & \leq & -5 \\ x_2' - x_1 & \leq & 2 \\ x_1' - x_2' & \leq & -1 \end{array}$$



M	x_1	x_2	x_1'	x_2'
x_1	∞	∞	3	-5
x_2	∞	∞	∞	∞
x_1'	∞	∞	∞	-1
x_2'	2	∞	∞	∞

  - naturally encoded as graphs and matrices

$$x - y \leq c \qquad \textbf{iff} \qquad x \xrightarrow{c} y \qquad \textbf{iff} \qquad M_{xy} = c$$

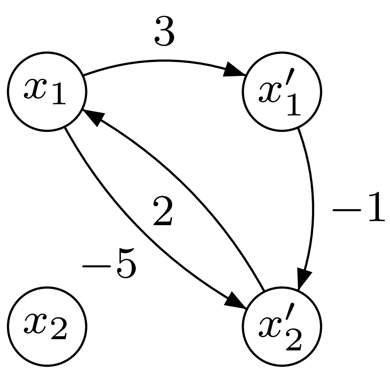
Difference Bounds Relations

❖ Conjunctions of atomic propositions $x - y \leq c$, $c \in \mathbb{Z}$

- Example 1:
  $(x' = x + 1) \equiv (x' \leq x + 1) \wedge (x + 1 \leq x') \equiv (x' - x \leq 1) \wedge (x - x' \leq -1)$

- Example 2:

$$R \equiv \bigwedge \begin{array}{rcr} x_1 - x_1' & \leq & 3 \\ x_1 - x_2' & \leq & -5 \\ x_2' - x_1 & \leq & 2 \\ x_1' - x_2' & \leq & -1 \end{array}$$



M	x_1	x_2	x_1'	x_2'
x_1	∞	∞	3	-5
x_2	∞	∞	∞	∞
x_1'	∞	∞	∞	-1
x_2'	2	∞	∞	∞

	x_1	x_2	x_1'	x_2'
x_1	0	∞	3	-5
x_2	∞	0	∞	∞
x_1'	1	∞	0	-1
x_2'	2	∞	5	0

- naturally encoded as graphs and matrices

$$x - y \leq c \qquad \textbf{iff} \qquad x \xrightarrow{c} y \qquad \textbf{iff} \qquad M_{xy} = c$$

- canonic form: (shortest paths, Floyd-Warshall)

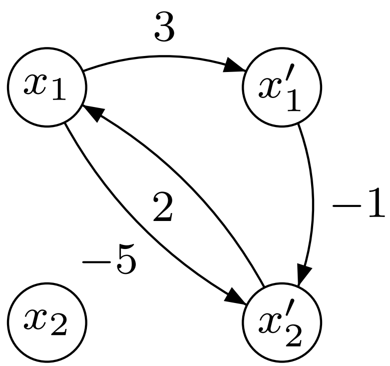
Difference Bounds Relations

❖ Conjunctions of atomic propositions $x - y \leq c$, $c \in \mathbb{Z}$

- Example 1:
  $(x' = x + 1) \equiv (x' \leq x + 1) \wedge (x + 1 \leq x') \equiv (x' - x \leq 1) \wedge (x - x' \leq -1)$

- Example 2:

$$R \equiv \bigwedge \begin{array}{rcr} x_1 - x_1' & \leq & 3 \\ x_1 - x_2' & \leq & -5 \\ x_2' - x_1 & \leq & 2 \\ x_1' - x_2' & \leq & -1 \end{array}$$



M	x_1	x_2	x_1'	x_2'
x_1	∞	∞	3	-5
x_2	∞	∞	∞	∞
x_1'	∞	∞	∞	-1
x_2'	2	∞	∞	∞

	x_1	x_2	x_1'	x_2'
x_1	0	∞	3	-5
x_2	∞	0	∞	∞
x_1'	1	∞	0	-1
x_2'	2	∞	5	0

- naturally encoded as graphs and matrices

$$x - y \leq c \qquad \textbf{iff} \qquad x \xrightarrow{c} y \qquad \textbf{iff} \qquad M_{xy} = c$$

- canonic form: (shortest paths, Floyd-Warshall)
- DB relation is satisfiable **iff** its graph encoding contains no negative cycle (can be checked by Floyd-Warshall)

Operations

- Difference bounds relations are closed under
  - intersection: $R_1(\mathbf{x}, \mathbf{x}') \wedge R_2(\mathbf{y}, \mathbf{y}')$
  - existential quantification: $\exists x \; . \; R(\mathbf{x}, \mathbf{x}')$ is a difference bounds relation

Operations

- Difference bounds relations are closed under
  - intersection: $R_1(\mathbf{x}, \mathbf{x}') \wedge R_2(\mathbf{y}, \mathbf{y}')$
  - existential quantification: $\exists x \,.\, R(\mathbf{x}, \mathbf{x}')$ is a difference bounds relation

	x_1	x_2	x'_1	x'_2
x_1	0	∞	3	-5
x_2	∞	0	∞	∞
x'_1	1	∞	0	-1
x'_2	2	∞	5	0

$\exists x_1, x_2 \,.\, R(\mathbf{x}, \mathbf{x}')$

	x'_1	x'_2
x'_1	0	-1
x'_2	5	0

Operations

❖ Difference bounds relations are closed under

- intersection: $R_1(\mathbf{x}, \mathbf{x}') \wedge R_2(\mathbf{y}, \mathbf{y}')$
- existential quantification: $\exists x \;.\; R(\mathbf{x}, \mathbf{x}')$ is a difference bounds relation

	x_1	x_2	x_1'	x_2'
x_1	0	∞	3	-5
x_2	∞	0	∞	∞
x_1'	1	∞	0	-1
x_2'	2	∞	5	0

$$\exists x_1, x_2 \;.\; R(\mathbf{x}, \mathbf{x}')$$

	x_1'	x_2'
x_1'	0	-1
x_2'	5	0

- relational composition

Operations

- Difference bounds relations are closed under
  - intersection: $R_1(\mathbf{x}, \mathbf{x}') \wedge R_2(\mathbf{y}, \mathbf{y}')$
  - existential quantification: $\exists x \,.\, R(\mathbf{x}, \mathbf{x}')$ is a difference bounds relation

	x_1	x_2	x_1'	x_2'
x_1	0	∞	3	-5
x_2	∞	0	∞	∞
x_1'	1	∞	0	-1
x_2'	2	∞	5	0

$$\exists x_1, x_2 \,.\, R(\mathbf{x}, \mathbf{x}')$$

	x_1'	x_2'
x_1'	0	-1
x_2'	5	0

  - relational composition

$$R_1(\mathbf{x}, \mathbf{x}') \circ R_2(\mathbf{x}, \mathbf{x}') \equiv \exists \mathbf{y} \,.\, R_1(\mathbf{x}, \mathbf{y}) \ \wedge \ R_2(\mathbf{y}, \mathbf{x}')$$

Encoding of $x \leq c$ and $-x \leq c$

$$R \equiv x \leq 100 \wedge x' = x + 1$$

is not a difference bounds relation, due to $x \leq 100$.

Encoding of $x \leq c$ *and* $-x \leq c$

$$R \equiv x \leq 100 \wedge x' = x + 1$$

is not a difference bounds relation, due to $x \leq 100$.

- Let x_{zero} be a fresh variable:

$$R_{zero} \equiv x - x_{zero} \leq 100 \wedge x' = x + 1 \wedge x'_{zero} = x_{zero}$$

x_{zero} is a parameter of R_{zero} since $x'_{zero} = x_{zero}$.

Encoding of $x \leq c$ and $-x \leq c$

$$R \equiv x \leq 100 \wedge x' = x + 1$$

is not a difference bounds relation, due to $x \leq 100$.

❖ Let x_{zero} be a fresh variable:

$$R_{zero} \equiv x - x_{zero} \leq 100 \wedge x' = x + 1 \wedge x'_{zero} = x_{zero}$$

x_{zero} is a parameter of R_{zero} since $x'_{zero} = x_{zero}$.

❖ Any iteration of R_{zero} which starts with $x_{zero} = 0$ corresponds to an iteration of R.

$$\begin{array}{rcl} R & \equiv & R_{zero}[x_{zero} := 0, x'_{zero} := 0] \\ R^* & \equiv & R^*_{zero}[x_{zero} := 0, x'_{zero} := 0] \end{array}$$

Encoding of $x \leq c$ and $-x \leq c$

$$R \equiv x \leq 100 \wedge x' = x + 1$$

is not a difference bounds relation, due to $x \leq 100$.

❖ Let x_{zero} be a fresh variable:

$$R_{zero} \equiv x - x_{zero} \leq 100 \wedge x' = x + 1 \wedge x'_{zero} = x_{zero}$$

x_{zero} is a parameter of R_{zero} since $x'_{zero} = x_{zero}$.

❖ Any iteration of R_{zero} which starts with $x_{zero} = 0$ corresponds to an iteration of R.

$$\begin{array}{rcl} R & \equiv & R_{zero}[x_{zero} := 0, x'_{zero} := 0] \\ R^* & \equiv & R^*_{zero}[x_{zero} := 0, x'_{zero} := 0] \end{array}$$

❖ Constructing R_{zero} from a conjunction of atoms of the form $x - y \leq c$, $x \leq c$, $-x \leq c$

- let x_{zero} be a fresh variable
- replace $x \leq c$ with $x - x_{zero} \leq c$
- replace $-x \leq c$ with $x_{zero} - x \leq c$
- add a constraint $x'_{zero} = x_{zero}$

We can without loss of generality consider relations like $x \leq 100 \wedge x' = x + 1 \wedge y' = 100$.

$*$-consistent Relations

- A relation $R(\mathbf{x}, \mathbf{x}')$ is consistent (satisfiable) **iff** $\models R(\nu, \nu')$ for some valutaions ν, ν' of $\mathbf{x}, \mathbf{x}'$.
- A relation R is $*$-consistent **iff** R^i is consistent for all $i \geq 0$.
  - $*$-inconsistent **iff** not $*$-consistent

$*$-consistent Relations

- A relation $R(\mathbf{x}, \mathbf{x}')$ is consistent (satisfiable) **iff** $\models R(\nu, \nu')$ for some valutaions ν, ν' of $\mathbf{x}, \mathbf{x}'$.
- A relation R is $*$-consistent **iff** R^i is consistent for all $i \geq 0$.
  - $*$-inconsistent **iff** not $*$-consistent
- Example 1: Relation $0 \leq x \leq 100 \wedge x' = x + 1$ is $*$-inconsistent

$$
\begin{array}{lclcll}
R^1 & \equiv & 0 \leq x \leq 100 & \wedge & x' = x + 1 & \\
R^2 & \equiv & 0 \leq x \leq 99 & \wedge & x' = x + 2 & \\
R^3 & \equiv & 0 \leq x \leq 98 & \wedge & x' = x + 3 & \\
\ldots & & & & & \\
R^{100} & \equiv & 0 \leq x \leq 1 & \wedge & x' = x + 100 & \\
R^{101} & \equiv & 0 \leq x \leq 0 & \wedge & x' = x + 101 & \\
R^{102} & \equiv & 0 \leq x \leq -1 & \wedge & x' = x + 102 & \equiv \quad \text{false}
\end{array}
$$

  - there is a lower bound on term x and an upper bound on x is decreasing

$*$-consistent Relations

❖ Example 2: Relation $x \leq 100 \wedge x' = x + 1$ is $*$-consistent

$$
\begin{array}{llll}
R^1 & \equiv & x \leq 100 & \wedge \quad x' = x + 1 \\
R^2 & \equiv & x \leq 99 & \wedge \quad x' = x + 2 \\
R^3 & \equiv & x \leq 98 & \wedge \quad x' = x + 3 \\
\ldots & & & \\
R^{100} & \equiv & x \leq 1 & \wedge \quad x' = x + 100 \\
R^{101} & \equiv & x \leq 0 & \wedge \quad x' = x + 101 \\
R^{102} & \equiv & x \leq -1 & \wedge \quad x' = x + 102 \\
R^{103} & \equiv & x \leq -2 & \wedge \quad x' = x + 103 \\
\ldots & & &
\end{array}
$$

- no lower bound on x

$*$-consistent Relations

❖ Example 2: Relation $x \leq 100 \wedge x' = x + 1$ is $*$-consistent

$$
\begin{array}{llll}
R^1 & \equiv & x \leq 100 & \wedge & x' = x + 1 \\
R^2 & \equiv & x \leq 99 & \wedge & x' = x + 2 \\
R^3 & \equiv & x \leq 98 & \wedge & x' = x + 3 \\
\ldots & & & & \\
R^{100} & \equiv & x \leq 1 & \wedge & x' = x + 100 \\
R^{101} & \equiv & x \leq 0 & \wedge & x' = x + 101 \\
R^{102} & \equiv & x \leq -1 & \wedge & x' = x + 102 \\
R^{103} & \equiv & x \leq -2 & \wedge & x' = x + 103 \\
\ldots & & & &
\end{array}
$$

- no lower bound on x

Remark: If R^i is inconsistent, so is $R^{i+1}, R^{i+2}, \ldots$

$*$-consistent Relations

❖ Example 2: Relation $x \leq 100 \wedge x' = x + 1$ is $*$-consistent

$$
\begin{array}{llllll}
R^1 & \equiv & x \leq 100 & \wedge & x' = x + 1 \\
R^2 & \equiv & x \leq 99 & \wedge & x' = x + 2 \\
R^3 & \equiv & x \leq 98 & \wedge & x' = x + 3 \\
\ldots \\
R^{100} & \equiv & x \leq 1 & \wedge & x' = x + 100 \\
R^{101} & \equiv & x \leq 0 & \wedge & x' = x + 101 \\
R^{102} & \equiv & x \leq -1 & \wedge & x' = x + 102 \\
R^{103} & \equiv & x \leq -2 & \wedge & x' = x + 103 \\
\ldots
\end{array}
$$

- no lower bound on x

Remark: If R^i is inconsistent, so is $R^{i+1}, R^{i+2}, \ldots$

❖ Transitive closures for $*$-inconsistent relations (degenerate case).

- if $R^k \Leftrightarrow false$ for some $k > 0$, then R^* can be expressed as

$$\bigvee_{i=0}^{k-1} R^i$$

Towards Transitive Closures

Transitive Closures

❖ Given a relation R, consider an infinite sequence of powers

$$R^0, R^1, R^2, R^3, R^4, \ldots$$

- R^* is a disjuction of elements in this sequence

❖ Example. Consider relation R defined as $x' = y + 1 \land y' = x$

- iterating R from $x = 0 \land y = 10$

	0	1	2	3	4	5	6
x	0	11	1	12	2	13	3
y	10	0	11	1	12	2	13

- infinite sequence of powers

R^0	R^1	R^2	R^3	R^4	R^5	R^6	
$x' = x$	$x' = y + 1$	$x' = x + 1$	$x' = y + 2$	$x' = x + 2$	$x' = y + 3$	$x' = x + 3$	$\ldots$
$y' = y$	$y' = x$	$y' = y + 1$	$y' = x + 1$	$y' = y + 2$	$y' = x + 2$	$y' = y + 3$	$\ldots$

Transitive Closures

R^0	R^1	R^2	R^3	R^4	R^5	R^6	
$x' = x$	$x' = y + 1$	$x' = x + 1$	$x' = y + 2$	$x' = x + 2$	$x' = y + 3$	$x' = x + 3$	$\ldots$
$y' = y$	$y' = x$	$y' = y + 1$	$y' = x + 1$	$y' = y + 2$	$y' = x + 2$	$y' = y + 3$	$\ldots$

Even powers can be described by a formula

$$\bigvee_{i=0}^{\infty} R^{2i} \Leftrightarrow \exists \ell \geq 0 \, . \, x' = x + \ell \wedge y' = y + \ell$$

The formula $x' = x + \ell \wedge y' = y + \ell$ is a closed form of $\{R^{2i}\}_{i=0}^{\infty}$.

Transitive Closures

R^0	R^1	R^2	R^3	R^4	R^5	R^6	
$x' = x$	$x' = y + 1$	$x' = x + 1$	$x' = y + 2$	$x' = x + 2$	$x' = y + 3$	$x' = x + 3$	$\ldots$
$y' = y$	$y' = x$	$y' = y + 1$	$y' = x + 1$	$y' = y + 2$	$y' = x + 2$	$y' = y + 3$	$\ldots$

Even powers can be described by a formula

$$\bigvee_{i=0}^{\infty} R^{2i} \Leftrightarrow \exists \ell \geq 0 \,.\, x' = x + \ell \wedge y' = y + \ell$$

The formula $x' = x + \ell \wedge y' = y + \ell$ is a closed form of $\{R^{2i}\}_{i=0}^{\infty}$.
Then,

$$\begin{array}{rcl} \bigvee_{i=0}^{\infty} R^{2i+1} & \Leftrightarrow & \bigvee_{i=0}^{\infty} (R^{2i} \circ R) \\ & \Leftrightarrow & \left(\bigvee_{i=0}^{\infty} R^{2i}\right) \circ R \\ & \Leftrightarrow & \left(\exists \ell \geq 0 \,.\, x' = x + \ell \wedge y' = y + \ell\right) \circ R \end{array}$$

Transitive Closures

R^0	R^1	R^2	R^3	R^4	R^5	R^6	
$x' = x$	$x' = y + 1$	$x' = x + 1$	$x' = y + 2$	$x' = x + 2$	$x' = y + 3$	$x' = x + 3$	$\ldots$
$y' = y$	$y' = x$	$y' = y + 1$	$y' = x + 1$	$y' = y + 2$	$y' = x + 2$	$y' = y + 3$	$\ldots$

Even powers can be described by a formula

$$\bigvee_{i=0}^{\infty} R^{2i} \Leftrightarrow \exists \ell \geq 0 \, . \, x' = x + \ell \wedge y' = y + \ell$$

The formula $x' = x + \ell \wedge y' = y + \ell$ is a closed form of $\{R^{2i}\}_{i=0}^{\infty}$.
Then,

$$\begin{array}{rcl} \bigvee_{i=0}^{\infty} R^{2i+1} & \Leftrightarrow & \bigvee_{i=0}^{\infty} (R^{2i} \circ R) \\ & \Leftrightarrow & \left(\bigvee_{i=0}^{\infty} R^{2i}\right) \circ R \\ & \Leftrightarrow & \left(\exists \ell \geq 0 \, . \, x' = x + \ell \wedge y' = y + \ell\right) \circ R \end{array}$$

$$R^* \quad \Leftrightarrow \quad \bigvee \begin{array}{l} \left(\exists \ell \geq 0 \, . \, x' = x + \ell \wedge y' = y + \ell\right) \\ \left(\exists \ell \geq 0 \, . \, x' = x + \ell \wedge y' = y + \ell\right) \circ R \end{array}$$

Transitive Closures

Assuming a certain notion of periodicity:

$$\begin{array}{ccccccccccccc} R^0 & R^1 & \ldots & R^b & R^{b+1} & \ldots & R^{b+c} & R^{b+c+1} & \ldots & R^{b+2c} & R^{b+2c+1} & \ldots & R^{b+ic} \\ & & & = & & & = & & & = & & & = \\ & & & \widehat{R}_{b,c}(0) & & & \widehat{R}_{b,c}(1) & & & \widehat{R}_{b,c}(2) & & & \widehat{R}_{b,c}(i) \end{array}$$

Periodicity manifests itself in existence of integers $b \geq 0, c > 0$ and a formula $\widehat{R}_{b,c}(\ell)$ – closed form of $\{R^{b+ci}\}_{i=0}^{\infty}$

$$R^{b+ci} \equiv \widehat{R}_{b,c}(i) \text{ for each } i \geq 0$$

Transitive Closures

Assuming a certain notion of periodicity:

$$\begin{array}{ccccccccccccc} R^0 & R^1 & \ldots & R^b & R^{b+1} & \ldots & R^{b+c} & R^{b+c+1} & \ldots & R^{b+2c} & R^{b+2c+1} & \ldots & R^{b+ic} \\ & & & = & & & = & & & = & & & = \\ & & & \widehat{R}_{b,c}(0) & & & \widehat{R}_{b,c}(1) & & & \widehat{R}_{b,c}(2) & & & \widehat{R}_{b,c}(i) \end{array}$$

Periodicity manifests itself in existence of integers $b \geq 0, c > 0$ and a formula $\widehat{R}_{b,c}(\ell)$ – closed form of $\{R^{b+ci}\}_{i=0}^{\infty}$

$$R^{b+ci} \equiv \widehat{R}_{b,c}(i) \text{ for each } i \geq 0$$

$$\widehat{R}_{b,c}(0) \vee \widehat{R}_{b,c}(1) \vee \widehat{R}_{b,c}(2) \vee \ldots \;\equiv\; \exists \ell \geq 0 \,.\, \widehat{R}_{b,c}(\ell)$$

Transitive Closures

Assuming a certain notion of periodicity:

$$\begin{array}{ccccccccccccc} R^0 & R^1 & \dots & R^b & R^{b+1} & \dots & R^{b+c} & R^{b+c+1} & \dots & R^{b+2c} & R^{b+2c+1} & \dots & R^{b+ic} \\ & & & = & & & = & & & = & & & = \\ & & & \widehat{R}_{b,c}(0) & & & \widehat{R}_{b,c}(1) & & & \widehat{R}_{b,c}(2) & & & \widehat{R}_{b,c}(i) \end{array}$$

Periodicity manifests itself in existence of integers $b \geq 0, c > 0$ and a formula $\widehat{R}_{b,c}(\ell)$ – closed form of $\{R^{b+ci}\}_{i=0}^{\infty}$

$$R^{b+ci} \equiv \widehat{R}_{b,c}(i) \text{ for each } i \geq 0$$

$$\widehat{R}_{b,c}(0) \vee \widehat{R}_{b,c}(1) \vee \widehat{R}_{b,c}(2) \vee \dots \equiv \exists \ell \geq 0 \,.\, \widehat{R}_{b,c}(\ell)$$

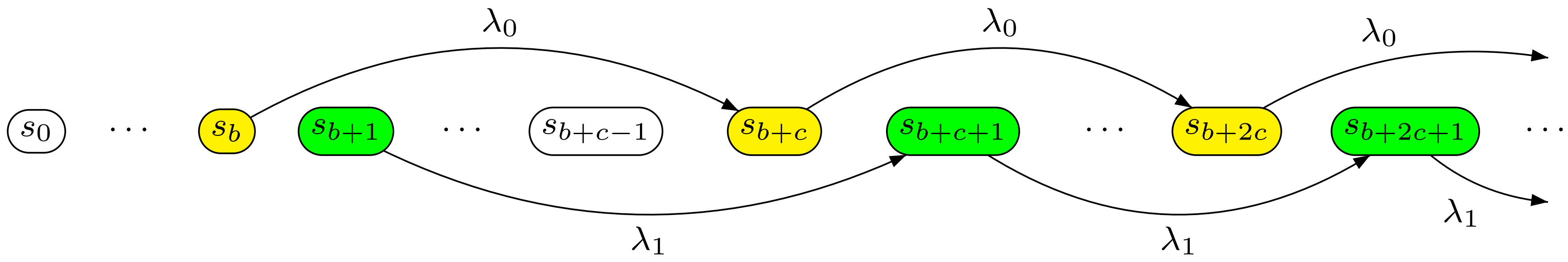
From $\widehat{R}_{b,c}(\ell)$ to the transitive closure. We can write R^* as:

$$R^* = \bigvee_{i=0}^{b-1} R^i \vee \left(\exists \ell \geq 0 \,.\, \widehat{R}_{b,c}(\ell)\right) \circ \bigvee_{i=0}^{c-1} R^i$$

Periodic Relations

Periodic Sequences

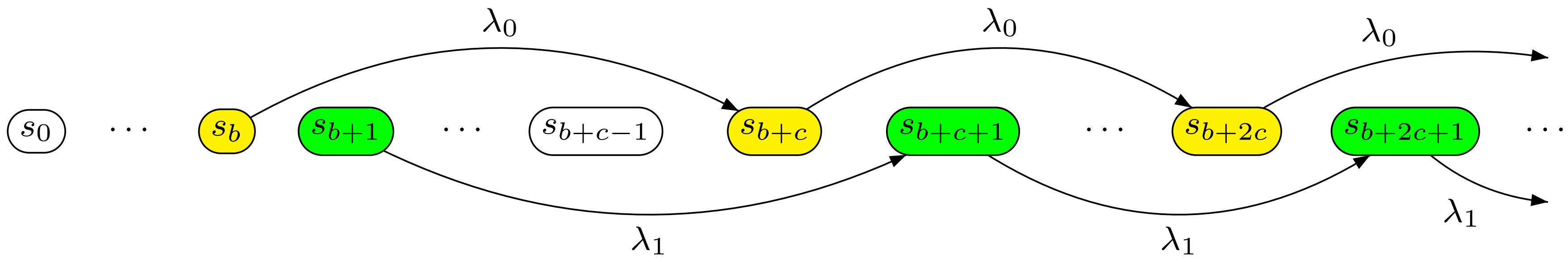
For some $b \geq 0, c \geq 1$, $\lambda_0, \ldots, \lambda_{c-1} \in \mathbb{Z}_\infty$



The smallest b, c and $\lambda_0, \lambda_1, \ldots, \lambda_{c-1}$ for which the above holds are called the prefix, period and rates of $\{s_k\}_{k=0}^{\infty}$, respectivelly.

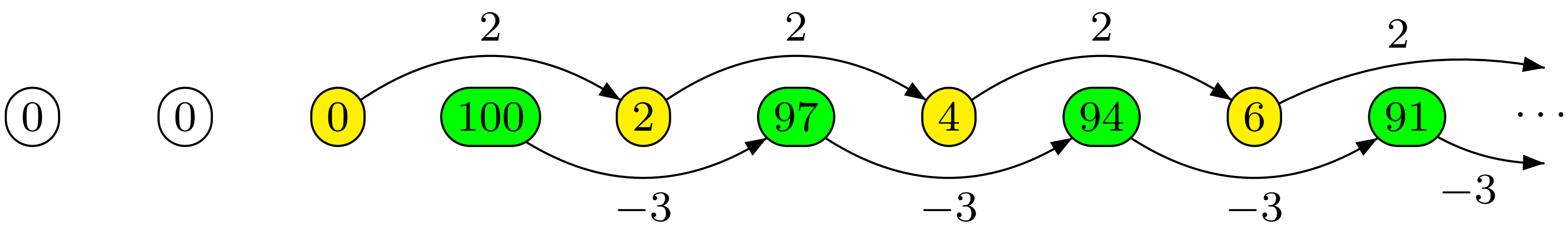
Periodic Sequences

For some $b \geq 0, c \geq 1, \lambda_0, \ldots, \lambda_{c-1} \in \mathbb{Z}_\infty$



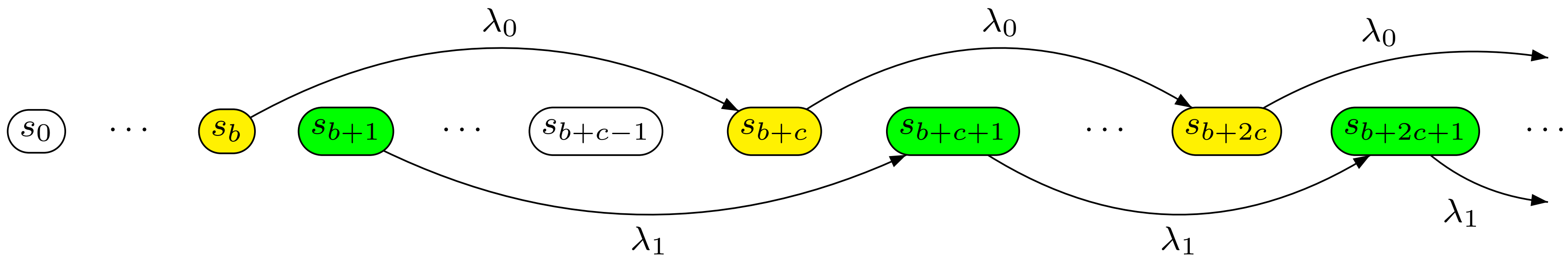
The smallest b, c and $\lambda_0, \lambda_1, \ldots, \lambda_{c-1}$ for which the above holds are called the prefix, period and rates of $\{s_k\}_{k=0}^{\infty}$, respectivelly.

Example. $b = 2, c = 2, \lambda_0 = 2, \lambda_1 = -3$.



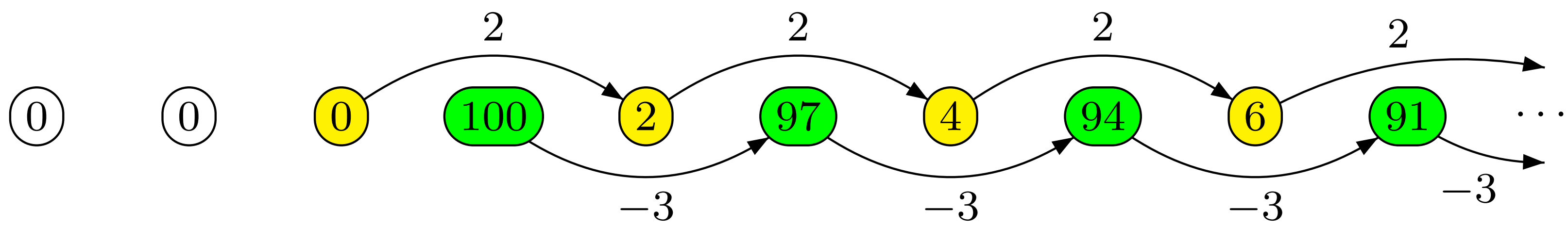
Periodic Sequences

For some $b \geq 0, c \geq 1$, $\lambda_0, \ldots, \lambda_{c-1} \in \mathbb{Z}_\infty$



The smallest b, c and $\lambda_0, \lambda_1, \ldots, \lambda_{c-1}$ for which the above holds are called the prefix, period and rates of $\{s_k\}_{k=0}^{\infty}$, respectivelly.

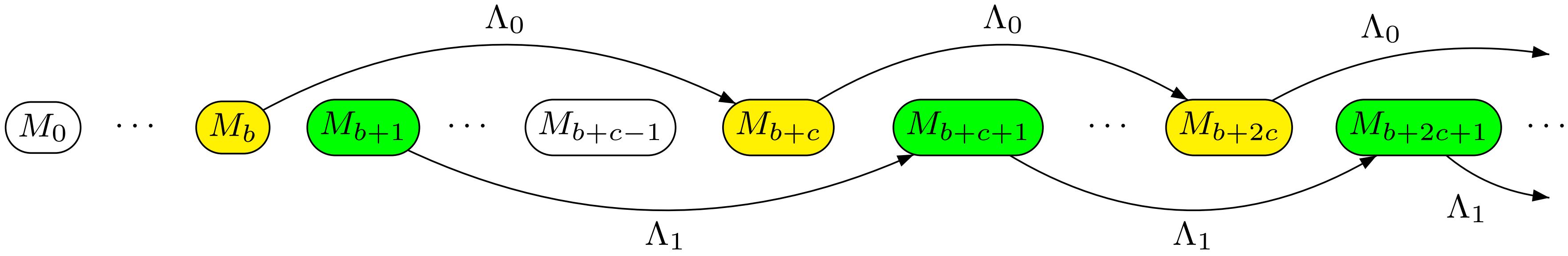
Example. $b = 2, c = 2, \lambda_0 = 2, \lambda_1 = -3$.



Another way of defining periodicity: $\forall n \geq 0 \;.\; s_{b+nc} = s_b + n \cdot \lambda_0$

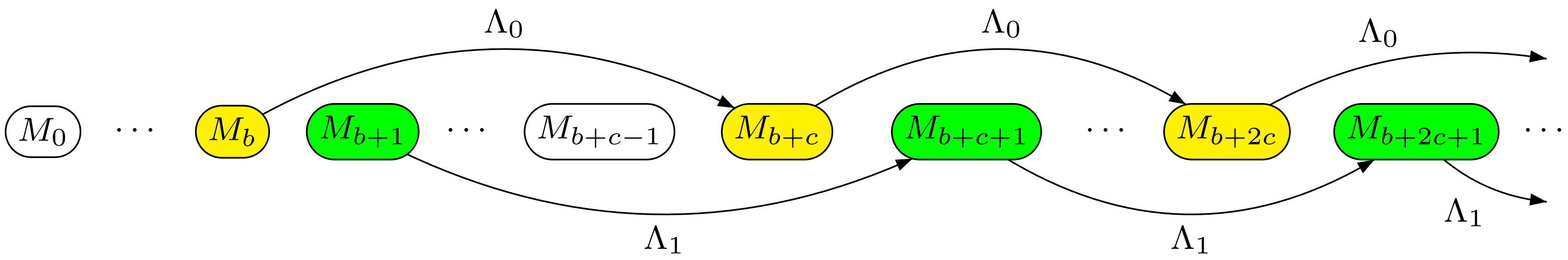
Periodic Matrix Sequences

For some $b \geq 0, c \geq 1$, $\lambda_0, \ldots, \lambda_{c-1} \in \mathbb{Z}_\infty^{m \times m}$



Periodic Matrix Sequences

For some $b \geq 0, c \geq 1, \lambda_0, \ldots, \lambda_{c-1} \in \mathbb{Z}_\infty^{m\times m}$



$$\begin{pmatrix} 0 & \infty & 0 & \infty \\ \infty & 0 & \infty & 0 \\ 0 & \infty & 0 & \infty \\ 0 & \infty & \infty & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ \infty & 0 & \infty & 0 \\ 1 & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & -2 & 0 \\ \infty & 0 & \infty & 0 \\ 2 & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 & -3 & 0 \\ \infty & 0 & \infty & 0 \\ 3 & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix} \begin{pmatrix} 0 & -3 & -4 & 0 \\ \infty & 0 & \infty & 0 \\ 4 & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix} \cdots$$

$$b = c = 1, \lambda = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Periodic Relations

A relation $R(\mathbf{x}, \mathbf{x}')$ is called periodic if and only if either:

Periodic Relations

A relation $R(\mathbf{x}, \mathbf{x}')$ is called periodic if and only if either:

- R is $*$-inconsistent

Periodic Relations

A relation $R(\mathbf{x}, \mathbf{x}')$ is called periodic if and only if either:

- R is $*$-inconsistent

  $R^0 \quad R^1 \quad R^2 \quad \ldots \quad R^{i-1} \quad false \quad false \quad false \quad \ldots$

Periodic Relations

A relation $R(\mathbf{x}, \mathbf{x}')$ is called periodic if and only if either:

- R is $*$-inconsistent

  $R^0 \quad R^1 \quad R^2 \quad \ldots \quad R^{i-1} \quad false \quad false \quad false \quad \ldots$

  example: $1 \leq x \leq 100 \ \wedge \ x' = x + 1$

Periodic Relations

A relation $R(\mathbf{x}, \mathbf{x}')$ is called periodic if and only if either:

- R is $*$-inconsistent

  $R^0 \quad R^1 \quad R^2 \quad \ldots \quad R^{i-1} \quad false \quad false \quad false \quad \ldots$

  example: $1 \leq x \leq 100 \ \wedge \ x' = x + 1$

- R is $*$-consistent and, given mappings $\sigma : \mathcal{R} \rightarrow \mathbb{Z}_\infty^{m \times m}$ and $\rho : \mathbb{Z}_\infty^{m \times m} \rightarrow \mathcal{R}$:

$\boxed{R^0} \quad \cdots \quad \boxed{R^b} \quad \boxed{R^{b+1}} \quad \cdots \quad \boxed{R^{b+c-1}} \quad \boxed{R^{b+c}} \quad \boxed{R^{b+c+1}} \quad \cdots \quad \boxed{R^{b+2c}} \quad \boxed{R^{b+2c+1}} \quad \cdots$

Periodic Relations

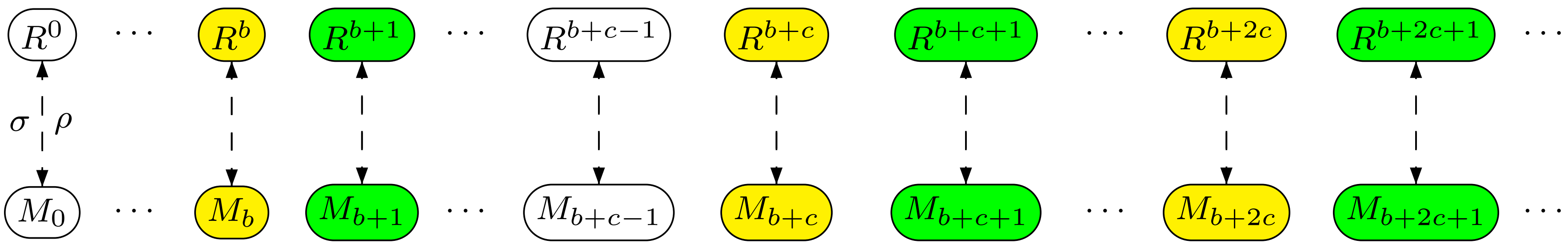
A relation $R(\mathbf{x}, \mathbf{x}')$ is called periodic if and only if either:

- R is $*$-inconsistent

  $R^0 \quad R^1 \quad R^2 \quad \ldots \quad R^{i-1} \quad false \quad false \quad false \quad \ldots$

  example: $1 \leq x \leq 100 \;\wedge\; x' = x + 1$

- R is $*$-consistent and, given mappings $\sigma : \mathcal{R} \to \mathbb{Z}_\infty^{m \times m}$ and $\rho : \mathbb{Z}_\infty^{m \times m} \to \mathcal{R}$:



Periodic Relations

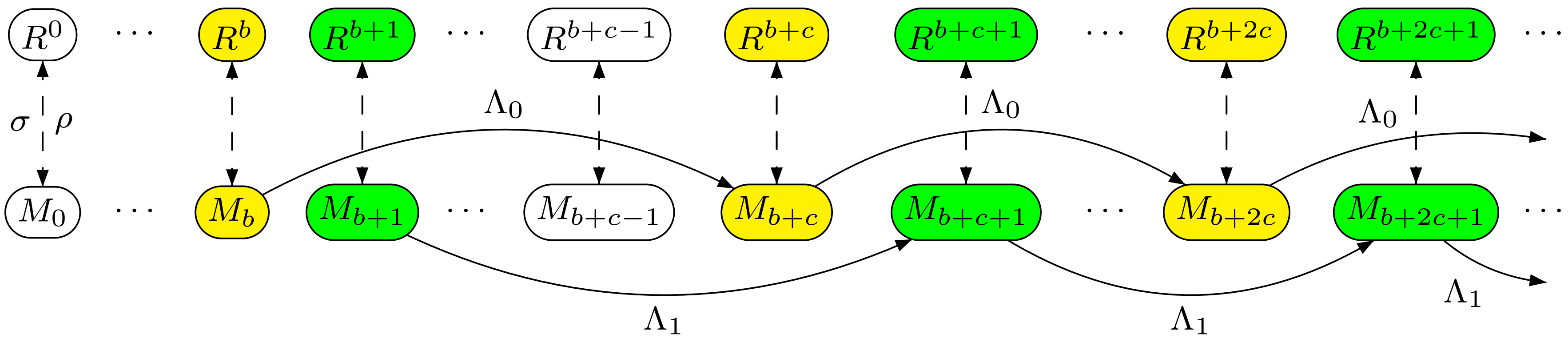
A relation $R(\mathbf{x}, \mathbf{x}')$ is called periodic if and only if either:

- R is $*$-inconsistent

  $R^0 \quad R^1 \quad R^2 \quad \ldots \quad R^{i-1} \quad false \quad false \quad false \quad \ldots$

  example: $1 \leq x \leq 100 \ \wedge \ x' = x + 1$

- R is $*$-consistent and, given mappings $\sigma : \mathcal{R} \rightarrow \mathbb{Z}_{\infty}^{m \times m}$ and $\rho : \mathbb{Z}_{\infty}^{m \times m} \rightarrow \mathcal{R}$:



Periodic Relations

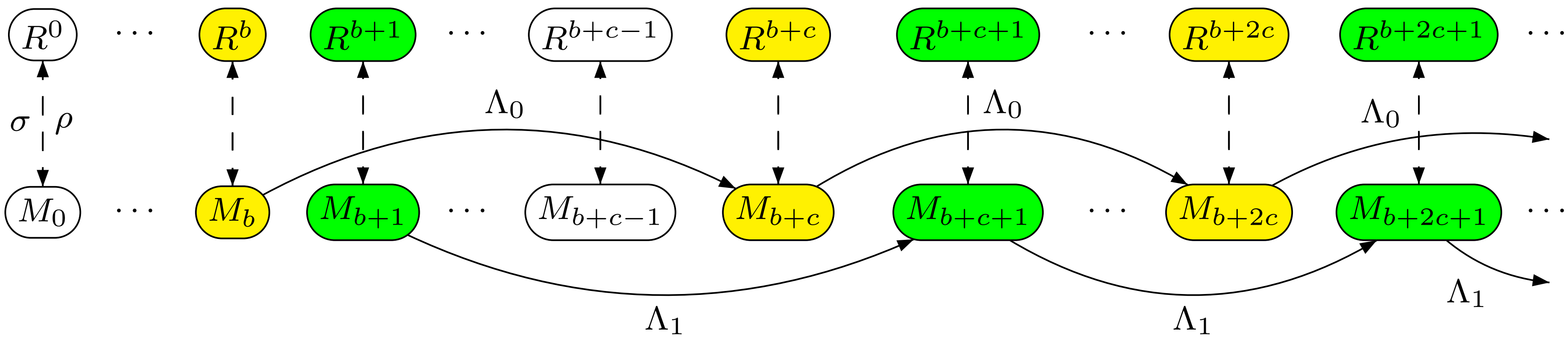
A relation $R(\mathbf{x}, \mathbf{x}')$ is called periodic if and only if either:

- R is $*$-inconsistent

  $R^0 \quad R^1 \quad R^2 \quad \ldots \quad R^{i-1} \quad false \quad false \quad false \quad \ldots$

  example: $1 \leq x \leq 100 \ \wedge \ x' = x + 1$

- R is $*$-consistent and, given mappings $\sigma : \mathcal{R} \to \mathbb{Z}_\infty^{m \times m}$ and $\rho : \mathbb{Z}_\infty^{m \times m} \to \mathcal{R}$:



$$\forall n \geq 0 \ . \ R^{b+nc} \quad \Leftrightarrow \quad \rho(\sigma(R^b) + n \cdot \Lambda_0)$$

Transitive Closure of DB Relations

```
while (x<=y)
  x = x+1
```

$$x \leq y \wedge x' = x + 1 \wedge y' = y$$

- Mapping σ is the matrix encoding of DB relations: $\sigma(R) = M_R$
- Infinite sequence $\sigma(R^0), \sigma(R^1), \sigma(R^2), \sigma(R^3), \sigma(R^4), \ldots$

$$\begin{array}{c|c} & \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{array}{c} x \\ y \\ x' \\ y' \end{array} & \begin{pmatrix} 0 & \infty & 0 & \infty \\ \infty & 0 & \infty & 0 \\ 0 & \infty & 0 & \infty \\ 0 & \infty & \infty & 0 \end{pmatrix} \end{array} \quad \begin{array}{c|c} & \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{array}{c} x \\ y \\ x' \\ y' \end{array} & \begin{pmatrix} 0 & 0 & -1 & 0 \\ \infty & 0 & \infty & 0 \\ 1 & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix} \end{array} \quad \begin{array}{c|c} & \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{array}{c} x \\ y \\ x' \\ y' \end{array} & \begin{pmatrix} 0 & -1 & -2 & 0 \\ \infty & 0 & \infty & 0 \\ 2 & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix} \end{array} \quad \begin{array}{c|c} & \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{array}{c} x \\ y \\ x' \\ y' \end{array} & \begin{pmatrix} 0 & -2 & -3 & 0 \\ \infty & 0 & \infty & 0 \\ 3 & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix} \end{array} \quad \begin{array}{c|c} & \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{array}{c} x \\ y \\ x' \\ y' \end{array} & \begin{pmatrix} 0 & -3 & -4 & 0 \\ \infty & 0 & \infty & 0 \\ 4 & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix} \end{array}$$

$$b = c = 1, \Lambda = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \sigma(R^{1+\ell}) = R^b + \ell \cdot \Lambda = \begin{pmatrix} 0 & 0-\ell & -1-\ell & 0 \\ \infty & 0 & \infty & 0 \\ 1+\ell & 1 & 0 & 1 \\ \infty & 0 & \infty & 0 \end{pmatrix}$$

$$\widehat{R}_{b,c}(\ell, \mathbf{x}, \mathbf{x}') \equiv x \leq y - \ell \wedge x' = x + 1 + \ell \wedge y' = y \wedge x' - y \leq 1 \wedge x' - y' \leq 1 \wedge y' - x \leq 0$$

$$R^* \quad \equiv \quad R^0 \quad \vee \quad \exists \ell \geq 0 \;.\; \widehat{R}_{b,c}(\ell, \mathbf{x}, \mathbf{x}')$$

Acceleration Algorithm

Accelerating Periodic Relation

- Assuming a relation is periodic, compute its transitive closure.
  - find prefix b, period c, and rate Λ

- Theorem 1. The following classes of relations are periodic
  - difference bounds relations
  - octagonal relations
  - finite monoid affine relations

- Given a relation R from one of the above classes, no precise characterization of (or an algorithm computing) b, c is known.
  - search for candidates for b, c and check if they are the right ones

Main Idea

Guess a prefix b and a period c such that:

$$\begin{tikzcd}
R^b \arrow[d, dashed, "\sigma"'] & R^{b+c} \arrow[d, dashed] & R^{b+2c} \arrow[d, dashed] \\
M_b \arrow[r, bend right, "\Lambda"'] & M_{b+c} \arrow[r, bend right, "\Lambda"'] & M_{b+2c}
\end{tikzcd}$$

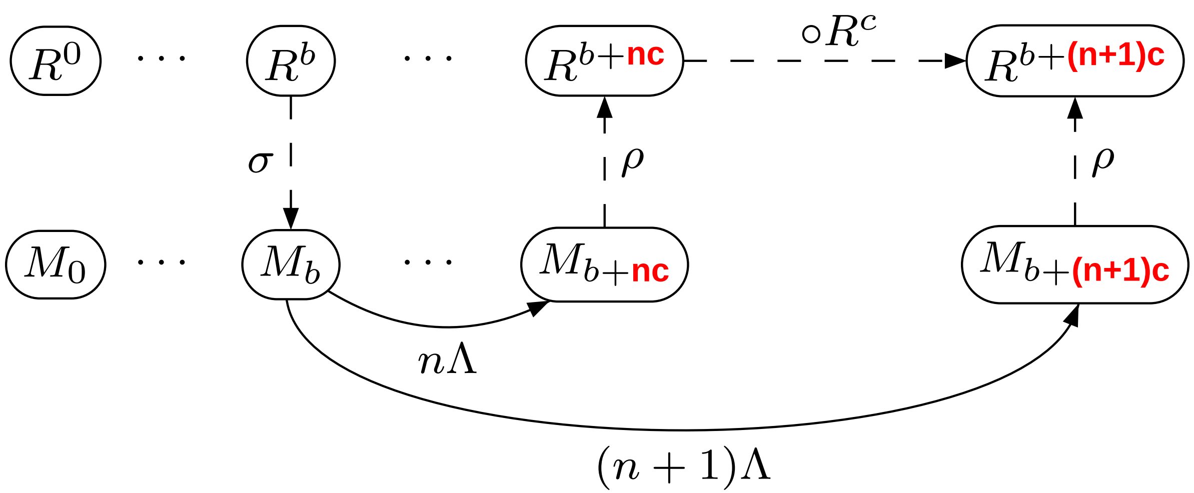
for some matrix $\Lambda \in \mathbb{Z}_{\infty}^{m \times m}$

$\sigma(R^{c+b}) = \Lambda + \sigma(R^b)$ and $\sigma(R^{2c+b}) = \Lambda + \sigma(R^{c+b})$

Main Idea

- Verify the guess. $\forall n \geq 0 \,.\, R^{b+nc} \;\Leftrightarrow\; \rho(\sigma(R^b) + n \cdot \Lambda) \qquad (\mathcal{Q}_1')$
- Validity of the above formula cannot be checked (R^{b+nc} is not known).

$$\forall n \geq 0 \;.\; \rho(\sigma(R^b) + n \cdot \Lambda) \circ R^c \;\Leftrightarrow\; \rho(\sigma(R^b) + (n+1) \cdot \Lambda) \qquad (\mathcal{Q}_1)$$



Main Idea

$$\forall n \geq 0 \;.\; R^{b+nc} \;\Leftrightarrow\; \rho(\sigma(R^b) + n \cdot \Lambda) \qquad (\mathcal{Q}_1')$$

is equivalent to

$$\forall n \geq 0 \;\;.\;\; \rho(\sigma(R^b) + n \cdot \Lambda) \circ R^c \;\Leftrightarrow\; \rho(\sigma(R^b) + (n+1) \cdot \Lambda) \qquad (\mathcal{Q}_1)$$

$(\mathcal{Q}_1') \Rightarrow (\mathcal{Q}_1)$

$$\begin{array}{ccc} R^{b+nc} \circ R^c & \Leftrightarrow & R^{b+(n+1)c} \\ \Updownarrow & & \Updownarrow \\ \rho(\sigma(R^b) + n \cdot \Lambda) \circ R^c & & \rho(\sigma(R^b) + (n+1)\Lambda) \end{array}$$

Main Idea

$$\forall n \geq 0 \ . \ R^{b+nc} \ \Leftrightarrow \ \rho(\sigma(R^b) + n \cdot \Lambda) \qquad (\mathcal{Q}_1')$$

is equivalent to

$$\forall n \geq 0 \ . \ \ \rho(\sigma(R^b) + n \cdot \Lambda) \circ R^c \ \Leftrightarrow \ \rho(\sigma(R^b) + (n+1) \cdot \Lambda) \qquad (\mathcal{Q}_1)$$

$(\mathcal{Q}_1') \Rightarrow (\mathcal{Q}_1)$

$$\begin{array}{ccc} R^{b+nc} \circ R^c & \Leftrightarrow & R^{b+(n+1)c} \\ \Updownarrow & & \Updownarrow \\ \rho(\sigma(R^b) + n \cdot \Lambda) \circ R^c & & \rho(\sigma(R^b) + (n+1)\Lambda) \end{array}$$

$(\mathcal{Q}_1) \Rightarrow (\mathcal{Q}_1')$ (by induction)

- base case – $(\mathcal{Q}_1')$ for $n = 0$ becomes $R^b \Leftrightarrow R^b$
- induction step – assuming $R^{b+nc} = \rho(\sigma(R^b) + n \cdot \Lambda)$

$$\begin{array}{rcll} R^{b+(n+1)c} & \Leftrightarrow & R^{b+nc} \circ R^c & \\ & \Leftrightarrow & \rho(\sigma(R^b) + n \cdot \Lambda) \circ R^c & \text{by ind. hypothesis} \\ & \Leftrightarrow & \rho(\sigma(R^b) + (n+1) \cdot \Lambda) & \text{by } (\mathcal{Q}_1') \end{array}$$

Transitive Closure Algorithm

foreach $b := 1, 2, \ldots$ do
  foreach $c := 1, 2, \ldots, b$ do
    foreach $k := 0, 1, 2$ do
      if $R^{kc+b} \Leftrightarrow$ false then return $R^* \equiv \bigvee_{i=0}^{kc+b-1} R^i$
    endfor
    if exists $\Lambda \in \mathbb{Z}_\infty^{m \times m} : \sigma(R^{b+c}) = \sigma(R^b) + \Lambda$ and $\sigma(R^{b+2c}) = \sigma(R^{b+c}) + \Lambda$ then
      if forall $n \geq 0 : \rho(\sigma(R^b) + n \cdot \Lambda) \circ R^c \Leftrightarrow \rho(\sigma(R^b) + (n+1) \cdot \Lambda)$ $(\mathcal{Q}_1)$ then
        return $R^* \equiv \bigvee_{i=0}^{b-1} R^i \ \vee \ \exists k \geq 0 \ . \ \bigvee_{i=0}^{c-1} \rho(\sigma(R^b) + k \cdot \Lambda) \circ R^i$
      endif
    endif
  endfor
endfor

Termination of the algorithm is guaranteed for periodic relations

The following universal query needs to be answered effectivelly:

$$\forall n \geq 0 \ . \quad \rho(\sigma(R^b) + n \cdot \Lambda) \circ R^c \ \Leftrightarrow \ \rho(\sigma(R^b) + (n+1) \cdot \Lambda) \qquad (\mathcal{Q}_1)$$

Illustration of the Algorithm

$$R \equiv x' = y + 1 \wedge y' = x$$

- R has no guard and thus is $*$-consistent. The test at line 4 therefore always fails.

$$\sigma(R^1) \quad \begin{array}{c} \\ x \\ y \\ x' \\ y' \end{array} \begin{array}{c} \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & 0 \\ \infty & 0 & -1 & \infty \\ \infty & 1 & 0 & \infty \\ 0 & \infty & \infty & 0 \end{pmatrix} \end{array}$$

$$\sigma(R^2) \quad \begin{array}{c} \\ x \\ y \\ x' \\ y' \end{array} \begin{array}{c} \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{pmatrix} 0 & \infty & -1 & \infty \\ \infty & 0 & \infty & -1 \\ 1 & \infty & 0 & \infty \\ \infty & 1 & \infty & 0 \end{pmatrix} \end{array}$$

$$\sigma(R^3) \quad \begin{array}{c} \\ x \\ y \\ x' \\ y' \end{array} \begin{array}{c} \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & -1 \\ \infty & 0 & -2 & \infty \\ \infty & 2 & 0 & \infty \\ 1 & \infty & \infty & 0 \end{pmatrix} \end{array}$$

$$\sigma(R^4) \quad \begin{array}{c} \\ x \\ y \\ x' \\ y' \end{array} \begin{array}{c} \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{pmatrix} 0 & \infty & -2 & \infty \\ \infty & 0 & \infty & -2 \\ 2 & \infty & 0 & \infty \\ \infty & 2 & \infty & 0 \end{pmatrix} \end{array}$$

$$\sigma(R^5) \quad \begin{array}{c} \\ x \\ y \\ x' \\ y' \end{array} \begin{array}{c} \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & -2 \\ \infty & 0 & -3 & \infty \\ \infty & 3 & 0 & \infty \\ 2 & \infty & \infty & 0 \end{pmatrix} \end{array}$$

$$\sigma(R^6) \quad \begin{array}{c} \\ x \\ y \\ x' \\ y' \end{array} \begin{array}{c} \begin{array}{cccc} x & y & x' & y' \end{array} \\ \begin{pmatrix} 0 & \infty & -3 & \infty \\ \infty & 0 & \infty & -3 \\ 3 & \infty & 0 & \infty \\ \infty & 3 & \infty & 0 \end{pmatrix} \end{array} \ \ldots$$

Let denote $\sigma(R^i)$ as M_i, for each $i \geq 0$

- $(b, c) = (1, 1)$ There is no Λ such that $M_2 = M_1 + \Lambda \wedge M_3 = M_2 + \Lambda$. Test at line 6 fails.
- $(b, c) = (2, 1)$ There is no Λ such that $M_3 = M_2 + \Lambda \wedge M_4 = M_3 + \Lambda$. Test at line 6 fails.
- $(b, c) = (2, 2)$ Test at line 6 succeeds, $M_4 = M_2 + \Lambda \wedge M_6 = M_4 + \Lambda$ for

$$\Lambda = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Illustration of the Algorithm

$$R \equiv \bigwedge \begin{array}{l} x' = y+1 \\ y' = x \end{array} \qquad \sigma(R^b) = \begin{array}{c|cccc} & x & y & x' & y' \\ \hline x & 0 & \infty & -1 & \infty \\ y & \infty & 0 & \infty & -1 \\ x' & 1 & \infty & 0 & \infty \\ y' & \infty & 1 & \infty & 0 \end{array} \qquad \Lambda = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- (line 7) The formula $(\mathcal{Q}_1)$ is constructed as follows

$$\forall n \geq 0 \ . \ \underbrace{\underbrace{\rho(\sigma(R^b) + \ell \cdot \Lambda)}_{\psi_1} \circ R^c}_{\psi_2} \ \Leftrightarrow \ \underbrace{\rho(\sigma(R^b) + (\ell+1) \cdot \Lambda)}_{\phi_3} \qquad (\mathcal{Q}_1)$$

$$\psi_1 \equiv \rho(\sigma(R^2) + \ell \cdot \Lambda) \equiv \rho(\begin{pmatrix} 0 & \infty & -1-\ell & \infty \\ \infty & 0 & \infty & -1-\ell \\ 1+\ell & \infty & 0 & \infty \\ \infty & 1+\ell & \infty & 0 \end{pmatrix}) \equiv \bigwedge \begin{array}{l} x' = x+1+\ell \\ y' = y+1+\ell \end{array}$$

$$\psi_2 \equiv \psi_1 \circ R^c \equiv \left(\bigwedge \begin{array}{l} x' = x+1+\ell \\ y' = y+1+\ell \end{array} \right) \circ \left(\bigwedge \begin{array}{l} x' = x+1 \\ y' = y+1 \end{array} \right) \equiv \bigwedge \begin{array}{l} x' = x+2+\ell \\ y' = y+2+\ell \end{array}$$

$$\psi_3 \equiv \rho(\sigma(R^2) + \ell \cdot \Lambda) \equiv \rho(\begin{pmatrix} 0 & \infty & -2-\ell & \infty \\ \infty & 0 & \infty & -2-\ell \\ 2+\ell & \infty & 0 & \infty \\ \infty & 2+\ell & \infty & 0 \end{pmatrix}) \equiv \bigwedge \begin{array}{l} x' = x+2+\ell \\ y' = y+2+\ell \end{array}$$

$$(\mathcal{Q}_1) \equiv \forall n \geq 0 \ . \ \psi_1 \Leftrightarrow \psi_3$$

- $(\mathcal{Q}_1)$ is valid and the algorithm returns

$$R^0 \vee R^1 \vee (\exists \ell \geq 0 \ . \ x' = x+1+\ell \wedge y' = y+1+\ell) \circ (R^0 \vee R^1)$$

$*$-inconsistent Periodic Relations

$$R \ : \ x' = x + 1 \ \wedge \ 0 \leq x \leq 10^5$$

R^i is inconsistent for $i > 10^5$ and behaves periodically for $i \leq 10^5$

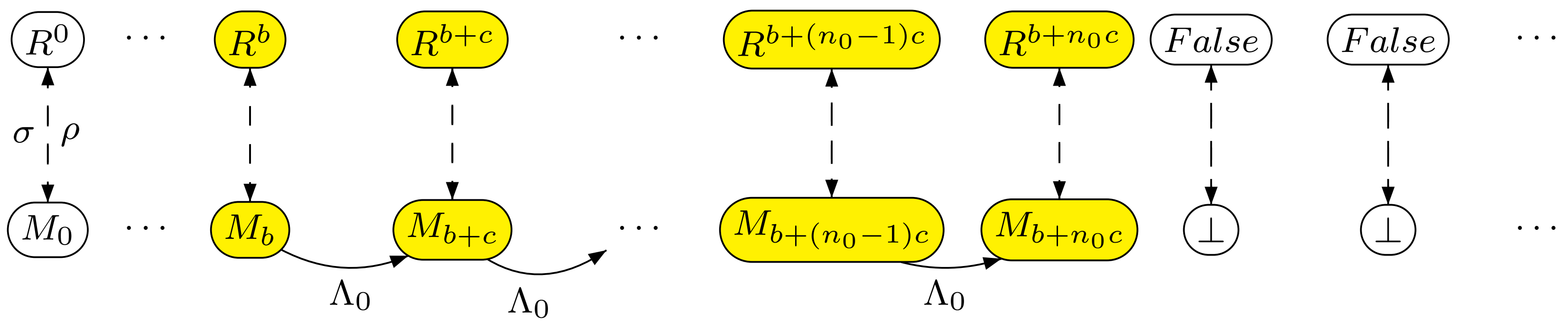
Detect consistent interval (spares time) and check for its periodicity (spares disjuncts)

$*$-inconsistent Periodic Relations

$$R \ : \ x' = x + 1 \ \wedge \ 0 \leq x \leq 10^5$$

R^i is inconsistent for $i > 10^5$ and behaves periodically for $i \leq 10^5$

Detect consistent interval (spares time) and check for its periodicity (spares disjuncts)

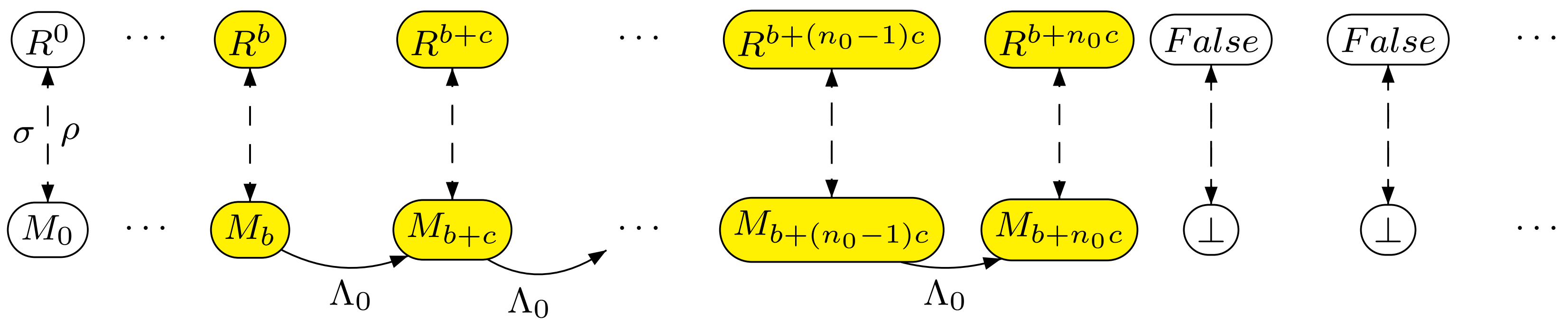


$*$-*inconsistent Periodic Relations*

$$R \; : \; x' = x + 1 \;\wedge\; 0 \le x \le 10^5$$

R^i is inconsistent for $i > 10^5$ and behaves periodically for $i \le 10^5$

Detect consistent interval (spares time) and check for its periodicity (spares disjuncts)



Termination can be accelerated by the existential query:

$\exists n \ge 0 \,.\, \rho(\sigma(R^b) + n \cdot \Lambda) \Leftrightarrow$ false $(\mathcal{Q}_2)$

Dealing with Quantifiers

For the classes of DB relations, octagonal relations and finite monoid affine transformations, the quantified queries $(\mathcal{Q}_1)$ and $(\mathcal{Q}_2)$ can be expressed in Presburger arithmetic.

Dealing with Quantifiers

For the classes of DB relations, octagonal relations and finite monoid affine transformations, the quantified queries $(\mathcal{Q}_1)$ and $(\mathcal{Q}_2)$ can be expressed in Presburger arithmetic.

For DB relations and octagonal relations, there are efficient equivalent conditions that can

be checked without e.g., using expensive quantifier elimination

Finite Monoid Affine Relations

Definition

$\mathbf{x} = \{x_1, x_2, \ldots, x_N\}$, $\mathbf{x}' = \{x'_1, x'_2, \ldots, x'_N\}$

❖ Finite monoid linear transformation is a linear arithmetic constraint of the form

$$\mathbf{x}' = A \cdot \mathbf{x}$$

- $A \in \mathbb{Z}^{N \times N}$
- $\{A^i \mid i \geq 0\}$ is finite

❖ Finite monoid affine transformation is a linear arithmetic constraint of the form

$$\mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}$$

- $A \in \mathbb{Z}^{N \times N}$, $\mathbf{b} \in \mathbb{Z}^N$
- $\{A^i \mid i \geq 0\}$ is finite

❖ Finite monoid affine relation is a formula of the form

$$\phi(\mathbf{x}) \;\wedge\; \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}$$

- $\mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}$ is a fin. monoid aff. transformation
- $\psi(\mathbf{x})$ is a Presburger guard

Example

- A loop from Illinois cache coherence protocol modelled as an integer program

```
while (invalid>=1 && shared+exclusive>=1) {
  shared = shared + exclusive + 1
  exclusive = 0
  invalid = invalid - 1
}
```

$$\bigwedge \begin{cases} i \geq 1 \\ s + e \geq 1 \\ s' = s + e + 1 \\ e' = 0 \\ i' = i - 1 \end{cases}$$

- The loop as a fin. monoid aff. relation $\phi(\mathbf{x}) \;\wedge\; \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}$

$$\begin{array}{c} i \geq 1 \\ s + e \geq 1 \end{array} \;\wedge \begin{pmatrix} s' \\ e' \\ i' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s \\ e \\ i \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A^1$$

Fin. Monoid Lin. Transformations

$$T \equiv \mathbf{x}' = A \cdot \mathbf{x}$$

❖ Computing transitive closure of T

- consider the sequence $A^0, A^1, A^2, \ldots$
- since $\{A^i \mid i \geq 0\}$ is finite, there exist $n > 0$ and $b < n$ such that $A^n = A^b$
- consider minimal such n

$$\{A^i \mid i \geq 0\} = \{A^0, A^1, \ldots, A^{n-1}\}$$

$$T^* \equiv \bigvee_{i=0}^{b+c-1} \mathbf{x}' = A^i \cdot \mathbf{x}$$

- let $c = n - b$, we can write the sequence $A^0, A^1, A^2, \ldots$ as

$$\begin{array}{lllll} A^0, A^1, \ldots, A^{b-1}, & A^b, & A^{b+1}, & \ldots, & A^{b+c-1}, \\ & A^b, & A^{b+1}, & \ldots, & A^{b+c-1}, \\ & \ldots & & & \end{array}$$

- the sequence is periodic (prefix b, period c, rate $\lambda = \mathbf{0}$)

Fin. Monoid Aff. Transformations

- **First step**: Compute transitive closure for fin. monoid aff. transformations (ignore the Presburger guard).

$$T \equiv \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}$$

- The homogeneous form of T is:

$$T_h \equiv \underbrace{\begin{pmatrix} \mathbf{x}' \\ x'_{one} \end{pmatrix}}_{\mathbf{x}'_h} = \underbrace{\left(\begin{array}{c|c} A & \mathbf{b} \\ \hline 0 \ldots 0 & 1 \end{array}\right)}_{A_h} \cdot \underbrace{\begin{pmatrix} \mathbf{x} \\ x_{one} \end{pmatrix}}_{\mathbf{x}_h}$$

- Example.

$$T \equiv \begin{pmatrix} s' \\ e' \\ i' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s \\ e \\ i \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$T_h \equiv \begin{pmatrix} s' \\ e' \\ i' \\ x'_{one} \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} s \\ e \\ i \\ x_{one} \end{pmatrix}$$

$$\begin{aligned} T &\equiv T_h[x_{one} := 1, x'_{one} := 1] \\ T^* &\equiv T_h^*[x_{one} := 1, x'_{one} := 1] \end{aligned}$$

Periodicity of Fin. Monoid Aff. Transformations

$$T \;\equiv\; \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b} \qquad\qquad T_h \;\equiv\; \mathbf{x}_h' = A_h \cdot \mathbf{x}_h$$

- If A has a finite monoid property, then there exist integers $b \geq 0, c \geq 1$ such that $A^b = A^{b+c}$ and thus

$$\{A^i \mid i \geq 0\} = \{A^0, A^1, \ldots, A^b, \ldots, A^{b+c-1}\}$$

- How does $\{A_h^i \mid i \geq 0\}$ look like? No longer finite.
- Theorem 2. Let $b \geq 0, c \geq 1$ be integers such that

$$\{A^i \mid i \geq 0\} = \{A^0, A^1, \ldots, A^b, \ldots, A^{b+c-1}\}$$

Then, $\{A_h^i \mid i \geq 0\}$ is periodic w.r.t. prefix b and period c and moroever, the rate is of the form

$$\lambda = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{d} \\ \hline 0\ldots0 & 0 \end{array}\right) \qquad \text{for some } \mathbf{d} \in \mathbb{Z}^N$$

Periodicity of Fin. Monoid Aff. Transformations

$$T \;\equiv\; \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b} \qquad\qquad T_h \;\equiv\; \mathbf{x}'_h = A_h \cdot \mathbf{x}_h$$

$$T_h \;\equiv\; \begin{pmatrix} s' \\ e' \\ i' \\ x'_{one} \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} s \\ e \\ i \\ x_{one} \end{pmatrix}$$

- $\{A^i \mid i \geq 0\} = \{A^0, A^1, \ldots, A^b, \ldots, A^{b+c-1}\}$ for $b = 1, c = 1$

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^2 = A^1$$

- By Theorem 2, $\{A_h^i \mid i \geq 0\}$ is periodic w.r.t. $b = 1, c = 1$ with $\lambda = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{d} \\ \hline 0\ldots0 & 0 \end{array}\right)$

$$\overset{A_h^0}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{A_h^1}{\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{A_h^2}{\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{A_h^3}{\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \ldots \quad \Bigg| \quad \overset{\lambda}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

Accelerating Fin. Monoid Aff. Transformations

$$\overset{A_h^0}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{A_h^1}{\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{A_h^2}{\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \overset{A_h^3}{\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \cdots \quad \Bigg| \quad \overset{\lambda}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

- Defining $A_h^{1+\ell}$ for each $\ell \geq 0$ parametrically

$$A_h^{1+\ell} = A^1 + \ell \cdot \lambda = \begin{pmatrix} 1 & 1 & 0 & 1+\ell \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1+\ell \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Defining closed form of $\{T_h^\ell\}_{\ell=1}^\infty$

$$\widehat{T_h}_{\ b,c}(\ell) \equiv \mathbf{x}'_h = A_h^{1+\ell} \cdot \mathbf{x}_h = \begin{pmatrix} 1 & 1 & 0 & 1+\ell \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1+\ell \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s \\ e \\ i \\ x_{one} \end{pmatrix} \equiv \bigwedge \begin{cases} s' = s + e + (1+\ell)x_{one} \\ e' = 0 \\ i' = i + (-1+\ell)x_{one} \\ x'_{one} = x_{one} \end{cases}$$

Accelerating Fin. Monoid Aff. Transformations

$$\widehat{T_h}\ _{b,c}(\ell) \equiv \bigwedge \begin{cases} s' = s + e + (1+\ell)x_{one} \\ e' = 0 \\ i' = i + (-1+\ell)x_{one} \\ x'_{one} = x_{one} \end{cases}$$

❖ From the closed form of $\{T_h^i\}_{i=1}^{\infty}$ to the closed form of $\{T^i\}_{i=1}^{\infty}$

$$\widehat{T}_{b,c}(\ell) \equiv \widehat{T_h}_{b,c}(\ell)[x_{one} := 1, x'_{one} := 1] \equiv \bigwedge \begin{cases} s' = s + e + 1 + \ell \\ e' = 0 \\ i' = i - 1 + \ell \end{cases}$$

What would happen if λ wasn't of the form $\lambda = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{d} \\ \hline 0 \ldots 0 & 0 \end{array} \right)$

- multiplicative terms of the form $c \cdot \ell \cdot x$, $c \in \mathbb{Z}$

Deterministic Relations

❖ Second step. Consider the general case with guard (finite monoid affine relations).

$$\phi(\mathbf{x}) \ \wedge \ \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}$$

❖ A relation $R(\mathbf{x}, \mathbf{x}')$ is deterministic **iff** for each $\mathbf{v} \in \mathbb{Z}^{\mathbf{x}}$, the set

$$\{\mathbf{v}' \in \mathbb{Z}^{\mathbf{x}'} \mid \ \models R(\mathbf{v}, \mathbf{v}')\}$$

has cardinality 0 or 1.

❖ Example.

- $2|x \wedge x' = x + 2 \wedge y' = y$ is determinisitc
- $x \leq 1 \wedge x' \geq x + 1$ is not deterministic
- $x \leq y \wedge x' = x + 1$ is not deterministic

❖ A closed form of a relation R is denoted $\widehat{R}$ and defined as a closed form of $\{R^i \mid i \geq 0\}$

Acceleration of Deterministic Relations

❖ Theorem 3. Let $T(\mathbf{x}, \mathbf{x}')$ be a relation of the form $T(\mathbf{x}, \mathbf{x}') \Leftrightarrow \phi(\mathbf{x}) \wedge R(\mathbf{x}, \mathbf{x}')$ where $R(\mathbf{x}, \mathbf{x}')$ is deterministic. Then, T^+ can be defined as

$$T^+(\mathbf{x}, \mathbf{x}') \quad \Leftrightarrow \quad \exists k > 0 \;.\; \widehat{R}(k, \mathbf{x}, \mathbf{x}') \wedge \forall 0 \leq \ell < k \;.\; \exists \mathbf{y} \;.\; \widehat{R}(\ell, \mathbf{x}, \mathbf{y}) \wedge \phi(\mathbf{y})$$

where $\widehat{R}$ defines the closed form of R.

Acceleration of Deterministic Relations

❖ Theorem 3. Let $T(\mathbf{x},\mathbf{x}')$ be a relation of the form $T(\mathbf{x},\mathbf{x}') \Leftrightarrow \phi(\mathbf{x}) \wedge R(\mathbf{x},\mathbf{x}')$ where $R(\mathbf{x},\mathbf{x}')$ is deterministic. Then, T^+ can be defined as

$$T^+(\mathbf{x},\mathbf{x}') \quad \Leftrightarrow \quad \exists k > 0 \;.\; \widehat{R}(k,\mathbf{x},\mathbf{x}') \wedge \forall 0 \leq \ell < k \;.\; \exists \mathbf{y} \;.\; \widehat{R}(\ell,\mathbf{x},\mathbf{y}) \wedge \phi(\mathbf{y})$$

where $\widehat{R}$ defines the closed form of R.

❖ Proof of $(\Rightarrow)$ Let $\mathbf{v},\mathbf{v}'$ be valuations such that $\models T^+(\mathbf{v},\mathbf{v}')$

- there exist integer $n > 0$ such that $\models T^n(\mathbf{v},\mathbf{v}')$, $\models \widehat{T}(n,\mathbf{v},\mathbf{v}')$ and valuations $\mathbf{v} = \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n = \mathbf{v}'$ such that
  - for each $i = 0 \ldots n-1 \;.\; \models T(\mathbf{v}_i,\mathbf{v}_{i+1})$ (1)
  - for each $i = 0 \ldots n \;.\; \models \widehat{T}(i,\mathbf{v},\mathbf{v}_i)$ (2)
- since $T(\mathbf{x},\mathbf{x}') \Rightarrow R(\mathbf{x},\mathbf{x}')$, it follows from (1), (2) that
  - for each $i = 0 \ldots n-1 \;.\; \models R(\mathbf{v}_i,\mathbf{v}_{i+1})$ (3)
  - for each $i = 0 \ldots n \;.\; \models \widehat{R}(i,\mathbf{v},\mathbf{v}_i)$ (4)
- since $T(\mathbf{x},\mathbf{x}') \Rightarrow \phi(\mathbf{x})$, it follows from (1) that
  - for each $i = 0 \ldots n-1 \;.\; \models \phi(\mathbf{v}_i)$ (5)
- (4), (5) imply that for each $i = 0 \ldots n-1 \;.\; \models \widehat{R}(i,\mathbf{v},\mathbf{v}_i) \wedge \phi(\mathbf{v}_i)$ (6)
- (6) implies $\forall 0 \leq \ell < n \;.\; \exists \mathbf{y} \;.\; \widehat{R}(\ell,\mathbf{x},\mathbf{y}) \wedge \phi(\mathbf{y})$

Acceleration of Deterministic Relations

❖ Theorem 3. Let $T(\mathbf{x}, \mathbf{x}')$ be a relation of the form $T(\mathbf{x}, \mathbf{x}') \Leftrightarrow \phi(\mathbf{x}) \wedge R(\mathbf{x}, \mathbf{x}')$ where $R(\mathbf{x}, \mathbf{x}')$ is deterministic. Then, T^+ can be defined as

$$T^+(\mathbf{x}, \mathbf{x}') \quad \Leftrightarrow \quad \exists k > 0 \, . \, \widehat{R}(k, \mathbf{x}, \mathbf{x}') \wedge \forall 0 \leq \ell < k \, . \, \exists \mathbf{y} \, . \, \widehat{R}(\ell, \mathbf{x}, \mathbf{y}) \wedge \phi(\mathbf{y})$$

where $\widehat{R}$ defines the closed form of R.

❖ Proof of $(\Leftarrow)$

- let $\mathbf{v}, \mathbf{v}'$ be valuations and $n > 0$ an integer such that $\models \widehat{R}(n, \mathbf{v}, \mathbf{v}')$ (1)
- for each $i = 0 \ldots n-1$, let $\mathbf{v}_i$ be valuation such that $\models \widehat{R}(i, \mathbf{v}, \mathbf{v}_i) \wedge \phi(\mathbf{v}_i)$ (2)
- (1) implies that there exist valuations $\mathbf{v} = \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n = \mathbf{v}'$ such that
  - for each $i = 0 \ldots n-1 \, . \, \models R(\mathbf{w}_i, \mathbf{w}_{i+1})$ (3)
  - for each $i = 0 \ldots n \, . \, \models \widehat{R}(i, \mathbf{v}, \mathbf{w}_i)$ (4)
- since R is deterministic, (2) and (4) imply that $\begin{array}{l} \models \widehat{R}(i, \mathbf{v}, \mathbf{v}_i) \\ \models \widehat{R}(i, \mathbf{v}, \mathbf{w}_i) \end{array} \Rightarrow \mathbf{v}_i = \mathbf{w}_i$ (5)
- (3) and (5) imply that for each $i = 0 \ldots n-1 \, . \, \models R(\mathbf{v}_i, \mathbf{v}_{i+1})$ (6)
- (2) and (6) imply that for each $i = 0 \ldots n-1 \, . \, \models R(\mathbf{v}_i, \mathbf{v}_{i+1}) \wedge \phi(\mathbf{v}_i)$ (7)
- we infer from (7) that $\models T^n(\mathbf{v}, \mathbf{v}')$ and $\models T^+(\mathbf{v}, \mathbf{v}')$

Accelerating Fin. Monoid Aff. Relations

- Since a fin. monoid aff. relation

$$T(\mathbf{x}, \mathbf{x}') \equiv \phi(\mathbf{x}) \ \wedge \ \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{b}$$

is deterministic, we apply Theorem 3 and define T^+ as a Presburger formula.

- Theorem 3 can be generalized to situations

$$T(\mathbf{x}, \mathbf{x}') \equiv \phi(\mathbf{x}) \wedge R(\mathbf{z}, \mathbf{z}') \wedge \psi(\mathbf{x}')$$

where $\mathbf{z} \subseteq \mathbf{x}$ and $R(\mathbf{z}, \mathbf{z}')$ is deterministic.

Applications

Applications

- Precise reachability analysis
  - finite monoid affine relations
    - tool FAST (`www.lsv.ens-cachan.fr/Software/fast/`)
    - reachability of Petri nets and broadcast protocols
  - difference bounds and octagonal relations
    - tool FLATA (`nts.imag.fr/index.php/Flata`)
    - reachability of programs with lists, VHDL designs
    - satisfiability of formulas from an array logic SIL
    - summaries of recursive procedures (McCarthy 91 function)
- Reachability analysis by predicate abstraction
  - acceleration increases the likelihood of convergence of the reachability algorithm
- Termination analysis
  - Presburger definability of $\widehat{R}$ is used to decide the conditional termination problem for DB, octagonal, and fin. monoid affine relations
  - adaptation of summary computation to transition invariant computation (useful to check termination)