Lecturecise 3
Presburger Arithmetic and Quantifier Elimination

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Presburger Arithmetic for Verification

```
res = 0
i = x
while // invariant I(res,i): res + 2*i == 2*x && 0 <= i
(i > 0) {
    i = i - 1
    res = res + 2
}
```

Verification condition (VC) for preservation of loop invariant:

$$\left[I(res, i) \land i' = i - 1 \land res' = res + 2 \land 0 < i\right] \rightarrow I(res', i')$$

To prove that this VC is valid, we check whether its *negation*

$$I(res, i) \land i' = i - 1 \land res' = res + 2 \land 0 < i \land \neg I(res', i')$$

is *satisfiable*, i.e. whether this PA formula is true:

$$\begin{aligned} \exists x, res, i, res', i'.\ \big[& res + 2i = 2x \ \land \ 0 \leq i \ \land \ 0 < i \ \land \\ & i' = i - 1 \ \land \ res' = res + 2 \ \land \\ & \neg(res' + 2i' = 2x \ \land \ 0 \leq i')\big] \end{aligned}$$

Introducing: One-Point Rule

If $\bar{y}$ is a tuple of variables not containing x, then

$$\exists x.(x = t(\bar{y}) \wedge F(x, \bar{y})) \iff F(t(\bar{y}), \bar{y})$$

Proof:

- $\rightarrow$: Consider the values of $\bar{y}$ such that there exists x, say x_1, for which $x_1 = t(\bar{y}) \wedge F(x_1, \bar{y})$. Because $F(x_1, \bar{y})$ evaluates to true and the values of x_1 and $t(\bar{y})$ are the same, $F(t, \bar{y})$ also evaluates to true.
- $\leftarrow$: Let $\bar{y}$ be such that $F(t, \bar{y})$ holds. Let x be the value of $t(\bar{y})$. Then of course $x = t(\bar{y})$ evaluates to true and so does $F(x, \bar{y})$. So there exists x for which $x = t(\bar{y}) \wedge F(x, \bar{y})$ holds.

One point rule:
replaces left side (LHS) of equivalence by the right side (RHS).
Flattening, used when t is complex, replaces RHS by LHS.

Dual One-Point Rule for ∀

$$\forall x.(x = t(\bar{y}) \rightarrow F(x, \bar{y})) \iff F(t(\bar{y}), \bar{y})$$

To prove it, negate both sides:

$$\exists x.(x = t(\bar{y}) \land \neg F(x, \bar{y})) \iff \neg F(t(\bar{y}), \bar{y})$$

so it reduces to the rule for $\exists$.

Using One-Point Rule on Negated Verification Condition

$$\exists x, res, i, res', \underline{\mathbf{i'}}.\ \big[res + 2i = 2x \ \wedge \ 0 \leq i \ \wedge \ 0 < i \ \wedge \\ \underline{\mathbf{i' = i - 1}} \ \wedge \ res' = res + 2 \ \wedge \\ \neg(res' + 2i' = 2x \ \wedge \ 0 \leq i')\big]$$

$$\exists x, res, i, \underline{\mathbf{res'}}.\ \big[res + 2i = 2x \ \wedge \ 0 \leq i \ \wedge \ 0 < i \ \wedge \\ \underline{\mathbf{res' = res + 2}} \ \wedge \\ \neg(res' + 2(i-1) = 2x \ \wedge \ 0 \leq i - 1)\big]$$

$$\exists x, res, i.\ \big[\underline{res + 2i = 2x} \ \wedge \ 0 \leq i \ \wedge \ 0 < i \ \wedge \\ \neg(res + 2 + 2(i-1) = 2x \ \wedge \ 0 \leq i - 1)\big]$$

$$\exists x, \underline{\mathbf{res}}, i.\ \big[\underline{\mathbf{res = 2x - 2i}} \ \wedge \ 0 \leq i \ \wedge \ 0 < i \ \wedge \\ \neg(res + 2 + 2(i-1) = 2x \ \wedge \ 0 \leq i - 1)\big]$$

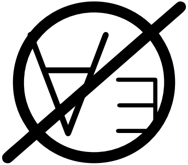
$$\exists x, i.\ \big[0 \leq i \ \wedge \ 0 < i \ \wedge \\ \neg(2x - 2i + 2 + 2(i-1) = 2x \ \wedge \ 0 \leq i - 1)\big]$$

Simplifies to $\exists x, i. 0 < i \wedge \neg(0 \leq i - 1)$ and then to false.

But there is more

One-point rule is one of the many steps used in **quantifier elimination** procedures.

Quantifier Elimination (QE)



Given a formula $F(\bar{y})$ containing quantifiers find a formula $G(\bar{y})$

- **equivalent** to $F(\bar{y})$
- that has **no quantifiers**
- and has a **subset (or equal set) of free variables** of F

Note

- Equivalence: For all $\bar{y}$, $F(\bar{y})$ and $G(\bar{y})$ have same truth value
  $\rightsquigarrow$ we can use $G(\bar{y})$ instead of $F(\bar{y})$
- No quantifiers: easier to check satisfiability of $G(\bar{y})$

$\bar{y}$ is a possibly empty tuple of variables

We are lucky when a theory has ("admits") QE

Suppose F has no free variables (all variables are quantified).
What is the result of applying QE to F?
Are there any variables in the resulting formula?

- No free variables: they are a subset of the original, empty set
- No quantified variables: because it has no quantifiers ☺

Formula without any variables! Example:

$$(2+4=7) \vee (1+1=2)$$

We check the truth value of such formula by simply evaluating it!

Using QE for Deciding Satisfiability/Validity

- To check satisfiability of $H(\bar{y})$: eliminate the quantifiers from $\exists \bar{y}.H(\bar{y})$ and evaluate.
- Validity: eliminate quantifiers from $\forall \bar{y}.H(\bar{y})$ and evaluate

We can even check formulas like this:

$$\forall x, y, r.\ \exists z.\ (5 \leq r \land x + r \leq y) \rightarrow (x < z \land z < y \land 3|z)$$

Here $3|z$ denotes that z is divisible by 3.

Does Presburger Arithmetic admit QE?

Depends on the particular set of symbols!
Given a formula $F(\bar{y})$ containing quantifiers find a formula $G(\bar{y})$

- **equivalent** to $F(\bar{y})$
- that has **no quantifiers**
- and has a **subset (or equal set) of free variables** of F

If we lack some operations that can be expressed using quantifiers, there may be no equivalent formula without quantifiers.

- $\exists y. x = y + y + y$, so we better have divisibility

Quantifier elimination says: if you can define some relationship between variables using an arbitrary, possibly quantified, formula F,

$$r \stackrel{def}{=} \{(x, y) \mid F(x, y)\}$$

then you can also define same r using another quantifier-free formula G.

Presburger Arithmetic (PA)

We look at the theory of integers with addition.

- introduce constant for each integer constant
- to be able to restrict values to natural numbers when needed, and to compare them, we introduce $<$
- introduce not only addition but also subtraction
- to conveniently express certain expressions, introduce function m_K for each $K \in \mathcal{Z}$, to be interpreted as multiplication by a constant, $m_K(x) = K \cdot x$. We write m_K as $K \cdot x$.
  Note: there is *no multiplication between variables* in PA
- to enable quantifier elimination from $\exists x. y = K \cdot x$ introduce for each K predicate $K|y$ (divisibility, $y\%K = 0$)

The resulting language has these function and relation symbols:
$\{+, -, =, <\} \cup \{K \mid K \in \mathbb{Z}\} \cup \{(K \cdot \_) \mid K \in \mathbb{Z}\} \cup \{(K|\_) \mid K \in \mathbb{Z}\}$
We also have, as usual: $\wedge, \vee, \neg, \rightarrow$ and also: $\exists, \forall$

Example

Eliminate y from this formula:

$$\exists y.\ \ 3y - 2w + 1 > -w \wedge 2y - 6 < z \wedge 4 \mid 5y + 1$$

What should we do first?
Simplify/normalize what we can using properties of integer operations:

$$\exists y.\ \ 0 < -w + 3y + 1 \quad \wedge \quad 0 < -2y + z + 6 \ \wedge \ 4 \mid 5y + 1$$

First we will consider only eliminating existential from a **conjunction of literals**.

Conjunctions of Literals

Atomic formula: a relation applied to argument.
Here, relations are: $=$, $<$, $K|\_$. So, atomic formulas are:
$t_1 = t_2, \quad t_1 < t_2, \quad K \mid t$
Literal: Atomic formula or its negation. Example: $\neg(x = y + 1)$
Conjunction of literals: $L_1 \land \ldots \land L_n$

- no disjunctions, no implications
- negation only applies to atomic formulas

We first consider the quantifier elimination problem of the form:

$$\exists y.\ L_1 \land \ldots \land L_n$$

This will prove to be sufficient to eliminate all quantifiers!

Eliminating $\exists$ from conjunction of literals suffices

Can we eliminate $\exists$ from any **quantifier-free formula**?

$$\exists x. F(x, \bar{y})$$

where F is quantifier-free?
Formula without quantifiers has $\wedge$,$\vee$,$\neg$ applied to atomic formulas.
Convert F to **disjunctive normal form**:

$$F \iff \bigvee_{i=1}^{m} C_i$$

each C_i is a **conjunction of literals**.

$$[\exists x. \bigvee_{i=1}^{m} C_i] \iff \bigvee_{i=1}^{m} (\exists x. C_i)$$

How does disjunctive normal form (DNF) transformation work?

Which steps should we use?

Negation propagation:

$$\neg(p \land q) \;\leadsto\; (\neg p) \lor (\neg q)$$

$$\neg(p \lor q) \;\leadsto\; (\neg p) \land (\neg q)$$

$$\neg\neg p \;\leadsto\; p$$

Result is **negation-normal form**, NNF
NNF transformation is polynomial (exercise!)
Distributivity

$$a \land (b_1 \lor b_2) \;\leadsto\; (a \land b_1) \lor (a \land b_2)$$

This can lead to exponential explosion.
Can we obtain equivalent DNF formula without explosion?
No! See exercise.

Eliminating from quantifier free formulas

$$\exists x. F \iff \left[\exists x. \bigvee_{i=1}^{m} C_i\right] \iff \bigvee_{i=1}^{m} (\exists x. C_i)$$

Nested Existential Quantifiers

$$\exists x_1.\exists x_2.\exists x_3.F_0(x_1, x_2, x_3, \bar{y})$$

$$\exists x_1.\exists x_2.F_1(x_1, x_2, \bar{y})$$

$$\exists x_1.F_2(x_1, \bar{y})$$

$$F_3(\bar{y})$$

☺

Universal Quantifiers

If $F_0(x, \bar{y})$ is quantifier-free, how to eliminate

$$\forall y. F_0(x, \bar{y})$$

Equivalence (property always holds if there is no counterexample):

$$\forall y. F_0(x, \bar{y}) \iff \neg\left[\exists y. \neg F_0(x, \bar{y})\right]$$

It thus suffices to process:

$$\neg\left[\exists y. \neg F_0(x, \bar{y})\right]$$

Note that $\neg F_0(x, \bar{y})$ is quantifier-free, so we know how to handle it:

$$\exists y. \neg F_0(x, \bar{y}) \quad \rightsquigarrow \quad F_1(\bar{y})$$

Therefore, we obtain

$$\neg F_1(\bar{y})$$

Removing any **alternation** of quantifiers: illustration

Alternation: switch between existentials and universals

$$\exists x_1 . \forall x_2 . \forall x_3 . \exists x_4 . F_0(x_1, x_2, x_3, x_4, \bar{y})$$

$$\exists x_1 . \neg \exists x_2 . \exists x_3 . \neg \underline{\exists x_4 . F_0(x_1, x_2, x_3, x_4, \bar{y})}$$

$$\exists x_1 . \neg \exists x_2 . \underline{\exists x_3 . \neg F_1(x_1, x_2, x_3, \bar{y})}$$

$$\exists x_1 . \neg \underline{\exists x_2 . F_2(x_1, x_2, \bar{y})}$$

$$\exists x_1 . \neg F_3(x_1, \bar{y})$$

$$F_4(\bar{y})$$

Each quantifier alternation involves a disjunctive normal form transformation.
In practice, we do not have many alternations.

Back to Presburger Arithmetic

Consider the quantifier elimination problem of the form:

$$\exists y.\ L_1 \wedge \ldots \wedge L_n$$

where L_i are literals from PA.
Note that, for integers:

- $\neg(x < y) \iff y \leq x$
- $x < y \iff x + 1 \leq y$
- $x \leq y \iff x < y + 1$

We use these observations below.
Instead of $\leq$ we choose to use $<$ only.
We do not write $x > y$ but only $y < x$.

Normalizing Literals for PA

Normal Form of Terms: All *terms* are built from $K, +, -, K \cdot \_$, so using standard transformations they can be represented as: $K_0 + \sum_{i=1}^{n} K_i x_i$ We call such term a linear term.
Normal Form for Literals in PA:

$$\neg(t_1 < t_2) \text{ becomes } t_2 < t_1 + 1$$
$$\neg(t_1 = t_2) \text{ becomes } t_1 < t_2 \vee t_2 < t_1$$
$$t_1 = t_2 \text{ becomes } t_1 < t_2 + 1 \wedge t_2 < t_1 + 1 \quad (*)$$
$$\neg(K \mid t) \text{ becomes } \bigvee_{i=1}^{K-1} K \mid t + i$$
$$t_1 < t_2 \text{ becomes } 0 < t_2 - t_1$$

To remove disjunctions we generated, compute DNF again.
$(*)$ We transformed equalities just for simplicity. Usually we handle them directly.

Why one-point rule will not be enough

Need to handle inequalities, not just equalities

If we have integers, we cannot always divide perfectly.
Variable to eliminate can occur not as y but as, e.g. $3y$

Exposing the Variable to Eliminate: Example

$$\exists y.\ \ 0 < -w + \underline{\mathbf{3y}} + 1 \ \ \wedge \ \ 0 < -\underline{\mathbf{2y}} + z + 6 \ \wedge \ 4 \mid \underline{\mathbf{5y}} + 1$$

Least common multiple of coefficients next to y,
$M = lcm(3, 2, 5) = 30$
Make all occurrences of y in the body have this coefficient:

$$\exists y.\ \ 0 < -10w + \underline{\mathbf{30y}} + 10 \wedge 0 < -\underline{\mathbf{30y}} + 15z + 90 \wedge 24 \mid \underline{\mathbf{30y}} + 6$$

Now we are quantifying over y and using $30y$ everywhere.
Let x denote $30y$.
It is **not an arbitrary** x. It is divisible by 30.

$$\exists x.\ \ 0 < -10w + x + 10 \wedge \ 0 < -x + 15z + 90 \ \wedge \ 24 \mid x + 6 \ \wedge \ 30 \mid x$$

Exposing the Variable to Eliminate in General

Eliminating y from conjunction $F(y)$ of literals:

- $0 < t$
- $K \mid t$

where t is a linear term. To eliminate $\exists y$ from such conjunction, we wish to ensure that the coefficient next to y is one or minus one.

Observation:

- $0 < t$ is equivalent to $0 < c\,t$
- $K \mid t$ is equivalent to $c\,K \mid c\,t$

for c a positive integer.

Let $K_1, \ldots, K_n$ be all coefficients next to y in the formula.

Let M be a positive integer such that $K_i \mid M$ for all i, $1 \leq i \leq n$

- for example, let M be the **least common multiple**

$$M = lcm(K_1, \ldots, K_n)$$

Ensuring Coefficient One

Multiply each literal where y occurs in subterm $K_i y$ by constant $M/|K_i|$

- the point is, M is divisible by $|K_i|$ by construction

What is the coefficient next to y in the resulting formula?

$$M \text{ or } -M$$

We obtain a formula of the form $\exists y. F(M \cdot y)$.
Letting $x = My$, we conclude the formula is equivalent to

$$\exists x.\ F(x) \wedge (M \mid x)$$

What is the coefficient next to y in the resulting formula?

$$1 \text{ or } -1$$

Lower and upper bounds:

Consider the coefficient next to x in $0 < t$. If it is -1, move the term to left side. If it is 1, move the remaining terms to the left side. We obtain formula $F_1(x)$ of the form

$$\bigwedge_{i=1}^{L} a_i < x \wedge \bigwedge_{j=1}^{U} x < b_j \wedge \bigwedge_{i=1}^{D} K_i \mid (x + t_i)$$

If there are no divisibility constraints ($D = 0$), what is the formula equivalent to?

$$\max_i a_i + 1 \leq \min_j b_j - 1 \text{ which is equivalent to } \bigwedge_{ij} a_i + 1 < b_j$$

Replacing variable by test terms

There is a an alternative way to express the above condition by replacing $F_1(x)$ with $\bigvee_k F_1(t_k)$ where t_k do not contain x. This is a common technique in quantifier elimination. Note that if $F_1(t_k)$ holds then certainly $\exists x. F_1(x)$.
What are example terms t_i when $D = 0$ and $L > 0$? Hint: ensure that at least one of them evaluates to $\max a_i + 1$.

$$\bigvee_{k=1}^{L} F_1(a_k + 1)$$

What if $D > 0$ i.e. we have additional divisibility constraints?

$$\bigvee_{k=1}^{L} \bigvee_{i=1}^{N} F_1(a_k + i)$$

What is N? least common multiple of $K_1, \ldots, K_D$
Note that if $F_1(u)$ holds then also $F_1(u - N)$ holds.

Back to Example

$$\exists x. -10 + 10w < x \wedge x < 90 + 15z \wedge 24 \mid x + 6 \wedge 30 | x$$

$$\bigvee_{i=1}^{120} 10w + i < 100 + 15z \wedge 0 < i \wedge 24 \mid 10w - 4 + i \wedge 30 | 10w - 10 + i$$

Special cases

What if $L = 0$? We first drop all constraints except divisibility, obtaining $F_2(x)$

$$\bigwedge_{i=1}^{D} K_i \mid (x + t_i)$$

and then eliminate quantifier as

$$\bigvee_{i=1}^{N} F_2(i)$$

It works

We finished describing a complete quantifier elimination algorithm for Presburger Arithmetic!

This algorithm and its correctness prove that:

- PA admits quantifier elimination
- Satisfiability, validity, entailment, equivalence of PA formulas is decidable
  We can use the algorithm to prove verification conditions.
- Quantified and quantifier-free formulas have the same expressive power

Many other properties follow (e.g. interpolation).

Interpolation For Logical Theories

Interpolation can be useful in generalizing counterexamples to invariants.

Universal **Entailment**: we will write $F_1 \models F_2$ to denote that for all free variables of F_1 and F_2, if F_1 holds then F_2 holds.

Given two formulas such that

$$F_0(\bar{x}, \bar{y}) \models F_1(\bar{y}, \bar{z})$$

an interpolant for F_1, F_2 is a formula $I(\bar{y})$, which has only variables common to F_0 and F_1, such that

- $F_0(\bar{x}, \bar{y}) \models I(\bar{y})$, and
- $I(\bar{y}) \models F_1(\bar{y}, \bar{z})$

In other words, the entailment between F_0 and F_1 can be explained through $I(\bar{y})$.

Logic has **interpolation property** if, whenever $F_0 \models F_1$, then there exists an interpolant for F_0, F_1.

We often wish to have *simple* interpolants, for example ones that are quantifier free.

Quantifier Elimination Implies Interpolation

If logic has QE, it also has quantifier-free interpolants.
Consider the formula

$$\forall \bar{x}, \bar{y}, \bar{z}.\ F_0(\bar{x}, \bar{y}) \rightarrow F_1(\bar{y}, \bar{z})$$

pushing $\bar{x}$ into assumption we get

$$\forall \bar{y}, \bar{z}.\ (\exists \bar{x}. F_0(\bar{x}, \bar{y})) \rightarrow F_1(\bar{y}, \bar{z})$$

and pushing $\bar{z}$ into conclusion we get

$$\forall \bar{x}, \bar{y}.\ F_0(\bar{x}, \bar{y}) \rightarrow (\forall \bar{z}. F_1(\bar{y}, \bar{z}))$$

Given two formulas F_0 and F_1, each of the formulas satisfies properties of interpolation:

- $\exists \bar{x}. F_0(\bar{x}, \bar{y})$
- $\forall \bar{z}. F_1(\bar{y}, \bar{z})$

Applying QE to them, we obtain quantifier-free interpolants.

More on QE: One Direction to Make it More Efficient

Avoid transforming to conjunctions of literals: work directly on negation-normal form. The technique is similar to what we described for conjunctive normal form.

+ no need for DNF
- we may end up trying irrelevant bounds

This is the Cooper's algorithm:

- Reddy, Loveland: Presburger Arithmetic with Bounded Quantifier Alternation. (Gives a slight improvement of the original Cooper's algorithm.)
- Section 7.2 of the Calculus of Computation Textbook

Eliminate Quantifiers: Example

$$\exists y.\exists x.\ \ x < -2 \land 1 - 5y < x \land 1 + y < 13x$$

Check whether the formula is satisfiable

$$x < y + 2 \ \wedge \ y < x + 1 \wedge x = 3k \wedge (y = 6p + 1 \vee y = 6p - 1)$$

Apply quantifier elimination

$$\exists x.\ \ (3x + 1 < 10 \vee 7x - 6 < 7)\ \wedge\ 2 \mid x$$

Another Direction for Improvement

Handle a system of equalities more efficiently, without introducing divisibility constraints too eagerly.

Hermite normal form of an integer matrix.

Eliminate variables x and y

$$5x + 7y = a \land x \leq y \land 0 \leq x$$

Quantifier Elimination for Linear Rational Arithmetic

Consider first-order formulas with equality and $<$ relation, interpreted over rationals.
This theory is called **dense linear order without endpoints**
For example:

$$\forall\varepsilon.\exists\delta.\big(|x_1 - x_2| < \delta \wedge |y_1 - y_2| < \delta \rightarrow |3x_1 + 4y_1 - 3x_2 - 4y_2| < \varepsilon\big)$$

(i) Show that absolute value can be defined in first-order logic in terms of other linear operations and comparison.
Answer: replace $F(|t|)$ with, for example

$$(t > 0 \wedge F(t)) \vee (\neg(t > 0) \wedge F(-t))$$

Is there a way to remove $|...|$ while increasing formula size only linearly?
(ii) Give quantifier elimination algorithm for this theory.
Solution is simpler than for Presburger arithmetic—no divisibility.