Lecturecise 12
Abstract Interpretation - Solutions to exercises

2013

Exercise

Prove the following:

1. $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
2. $\sqcup A \sqsubseteq \sqcap B \;\Leftrightarrow\; \forall x \in A. \forall y \in B. x \sqsubseteq y$
3. Let $(A, \sqsubseteq)$ be a partial order such that every set $S \subseteq A$ has the greatest lower bound.
   Prove that then every set $S \subseteq A$ has the least upper bound.

Solution

1. Let $(x \sqcup y) \sqcup z = a$, then $z \sqsubseteq a$ and $x \sqcup y \sqsubseteq a$ and hence $x \sqsubseteq a$ and $y \sqsubseteq a$. From $z \sqsubseteq a$ and $y \sqsubseteq a$ it follows $z \sqcup y \sqsubseteq a$ and finally $x \sqcup (y \sqcup z) \sqsubseteq a = (x \sqcup y) \sqcup z$.
   Symmetrically, we also get that $(x \sqcup y) \sqcup z \sqsubseteq x \sqcup (y \sqcup z)$ and hence $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$.

2. Let $a = \sqcup A$ and $b = \sqcap B$. Then

   $$a = \sqcup A \Leftrightarrow \forall x \in A.x \sqsubseteq a \wedge \forall a'.\forall x \in A.x \sqsubseteq a' \rightarrow a \sqsubseteq a'$$

   and

   $$b = \sqcap B \Leftrightarrow \forall y \in B.b \sqsubseteq y \wedge \forall b'.\forall y \in B.b' \sqsubseteq y \rightarrow b' \sqsubseteq b$$

   $\Rightarrow$
   Suppose $a \sqsubseteq b$, then by transitivity it follows that $\forall x \in A.\forall y \in B.x \sqsubseteq y$.

   $\Leftarrow$
   Now suppose $\forall x \in A.\forall y \in B.x \sqsubseteq y$. This implies that the set A is a set of lower bounds to B, and B is a set of upper bounds on A. Hence $a \sqsubseteq b$ (intuitively).

Solution

3. Suppose we have some arbitrary set $S \subseteq A$, then we know that $\sqcap S$ exists.
   Let U be the set of all its upper bounds, i.e. $U = \{x | \forall y \in S . y \sqsubseteq x\}$.
   Since every subset of A has a greatest lower bound, we know that $a = \sqcap U$ exists. Now we want to show that a is an upper bound on S and that it is the least one.
   We want to show that $\forall y \in S . y \sqsubseteq a$.
   Let L be the set of lower bounds on U, i.e. $L = \{z | \forall x \in U . z \sqsubseteq x\}$.
   Clearly, $S \subseteq L$, since U are the upper bounds on S.
   Then a is the greatest element in L, and thus $\forall y \in S . y \sqsubseteq a$ and so a is an upper bound on S.
   Now take any upper bound $x \in U$. Since $a = \sqcap U$, we have $a \sqsubseteq x$ and so a is the least upper bound.