Lecture 9 Illustrations

Lattices. Fixpoints

Abstract Interpretation

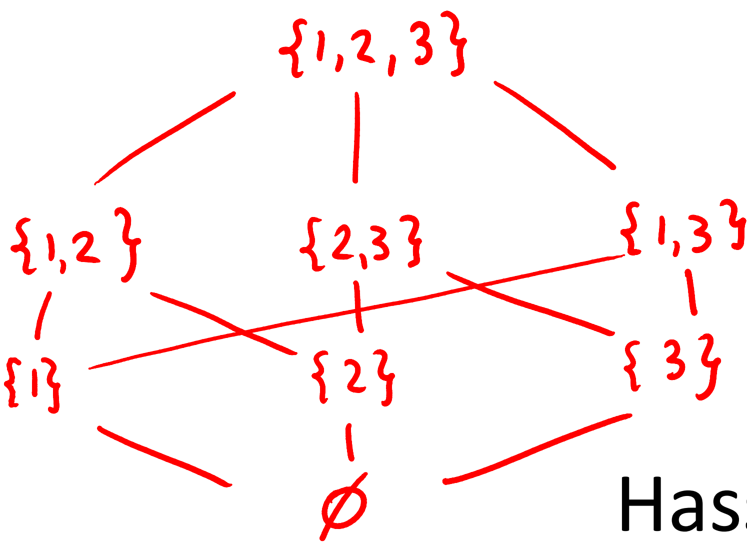
Partially Ordered Set (A, ≤)

$x \leq x$

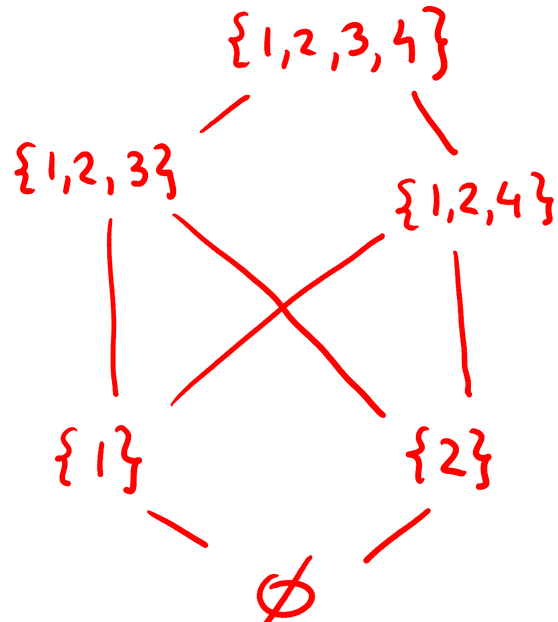
$x \leq y \wedge y \leq x \rightarrow x = y$ (else it is only pre-order)

$x \leq y \wedge y \leq z \rightarrow x \leq z$

Typical example: $(A, \subseteq)$, where $A \subseteq 2^U$ for some U



Hasse diagram

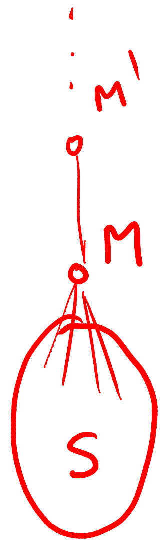


Key Terminology

Let $S \subseteq A$.

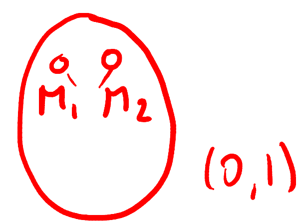
upper bound of S: **bigger than all**

dual: lower bound



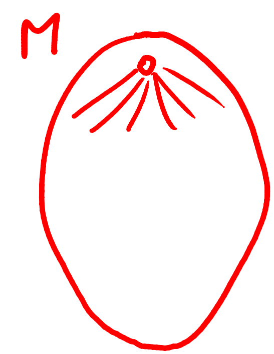
maximal element of S: **there's no bigger**

dual: minimal element



greatest element of S: upper bound on S, **in** S

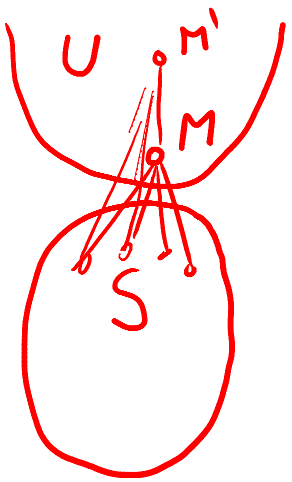
dual: least element



Least Upper Bound

Denoted lub(S), least upper bound of S is an element M, if it exists, such that M is the least element of the set

$U = \{x \mid x \text{ is upper bound on } S\}$



In other words:

- M is an upper bound on S
- For every other upper bound M' on S, we have that $M \leq M'$

Note: same definition as "inf" in real analysis

- applies not only to total orders, but any partial order

Real Analysis

Take as S the open interval of reals
$(0,1) = \{ x \mid 0 < x < 1 \}$

Then

- S has no maximal element
- S thus has no greatest element
- 2, 2.5, 3, … are all upper bounds on S
- lub(S)=1

If we had rational numbers, there would be no lub(S') in general.

Shorthand : $\sqcup$

$a_1 \sqcup a_2$ denotes $\mathrm{lub}(\{a_1,a_2\})$

$(\ldots(a_1 \sqcup a_2) \ldots) \sqcup a_n$ is, in fact, $\mathrm{lub}(\{a_1,\ldots, a_n\})$

So the operation is ACU

- associative
- commutative
- idempotent

Consider sets of **all** subsets of U

$A = 2^U = \{ s \mid s \subseteq U \}$ $\quad (A, \subseteq)$

Do these exist, and if so, what are they?

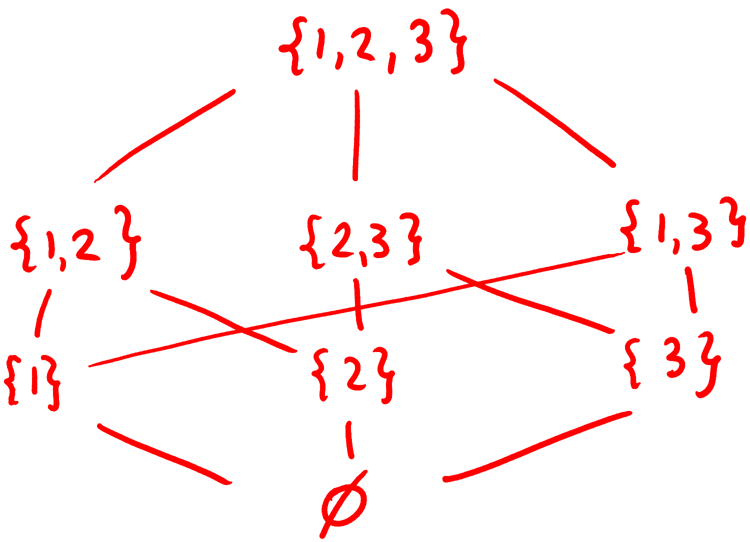
- lub($\{s_1, s_2\}$) $= s$ $\quad s_1 \subseteq s \quad s_2 \subseteq s \quad s = s_1 \cup s_2$

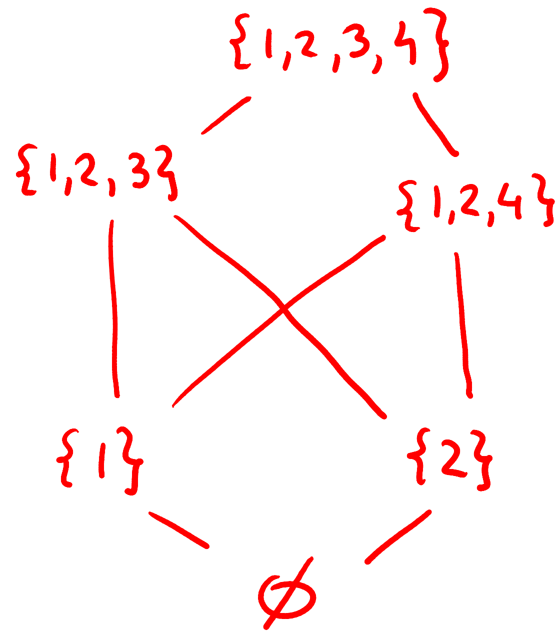
  $\forall s'. (s_1 \subseteq s' \wedge s_2 \subseteq s' \rightarrow s \subseteq s')$
- lub(S')

$\bigsqcup S' = \bigcup_{s \in S'} s$

$\bigsqcap S' = \bigcap_{s \in S} s$

Two More Examples

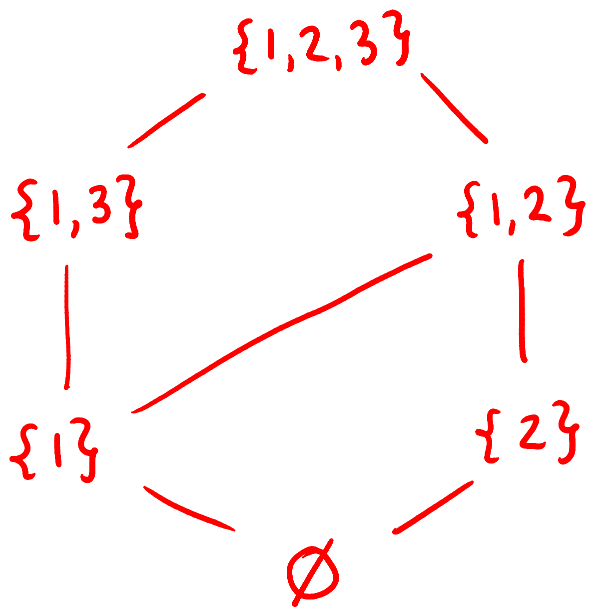




$\{1\} \sqcup \{2\} = \{1,2\}$

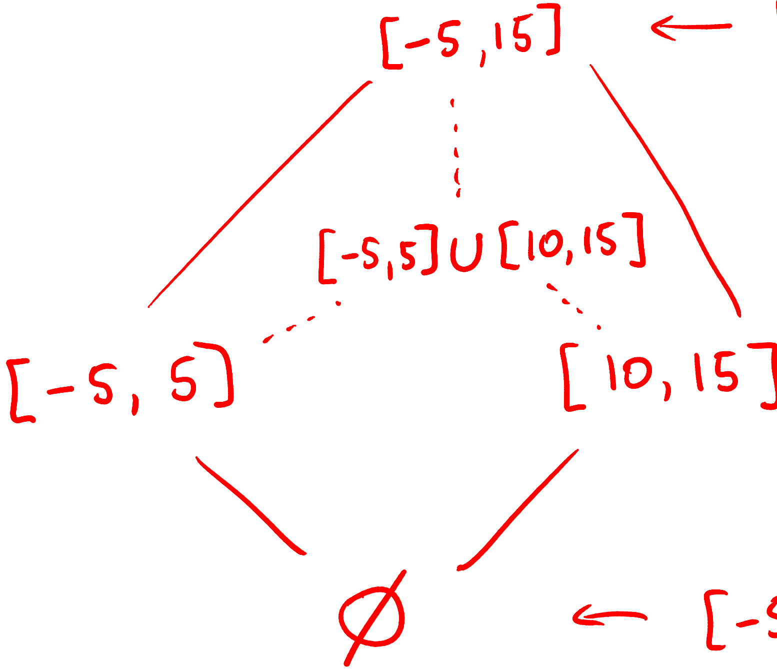
$\{1\} \sqcup \{2\} =$

Does every pair of elements in this order have least upper bound?



Dually, does it have **greatest lower bound**?

Approximation of Sets by Supersets



$[-5,5] \sqcup [10,15]$

Least Upper Bound

LUB

supremum

sup

Best approximation of union that we have in our current lattice

$[-5,5] \sqcap [10,15]$

Domain of Intervals

$D = \{\perp\} \cup \{(L,U) \mid L \in \{-\infty\} \cup Z, R \in Z \cup \{+\infty\}\}$

The domain elements, D, are

$d_1 \sqsubseteq d_2$

- pairs (L,U) where
  - L is an integer or minus infinity
  - U is an integer or plus infinity
  - if L and U are integers, then L ≤ U
- The special element ⊥ representing empty set

The associated set of elements

gamma : D → 2^Z $\gamma((L,U)) = \{x \mid L \leq x \wedge x \leq U\}$

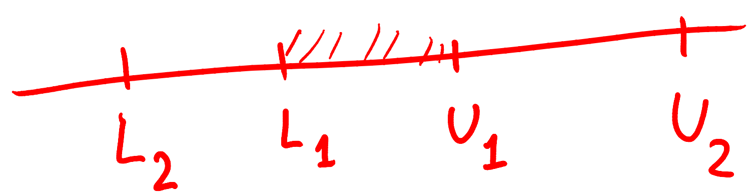
Definition of gamma, ordering, lub

$d_1, d_2 \in D$

$d_1 \sqsubseteq d_2 \quad \leftrightarrow \quad \gamma(d_1) \subseteq \gamma(d_2)$

$(L_1, U_1) \sqsubseteq (L_2, U_2) \qquad L_1, U_1, L_2, U_2 \in \mathbb{Z} \leftarrow$

$\leftrightarrow L_2 \leq L_1 \wedge U_1 \leq U_2$



$\bot \sqsubseteq d \quad \forall d \in D$

$(L_1, U_1) \sqcup (L_2, U_2) = (\min(L_1, L_2), \max(U_1, U_2))$

$[L_1, U_1]$