The Resolution Calculus $Res$

**Definition**

- Resolution inference rule

$$\begin{align*}
C \lor A & \quad \neg A \lor D \\
\hline
C \lor D
\end{align*}$$

- (positive) factorisation

$$\begin{align*}
C \lor A \lor A & \\
\hline
C \lor A
\end{align*}$$
Refutational Completeness of Resolution

- We have to show: $N \models \bot \Rightarrow N \vdash_{\text{Res}} \bot$, or equivalently: If $N \not\vdash_{\text{Res}} \bot$, then $N$ has a model.
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or equivalently: If \( N \not\vdash_{\text{Res}} \bot \), then \( N \) has a model.

Idea: Suppose that we have computed sufficiently many inferences (and not derived \( \bot \)).

Now order the clauses in \( N \) according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
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- We have to show: $N \models \bot \Rightarrow N \vdash_{Res} \bot$, or equivalently: If $N \not\models_{Res} \bot$, then $N$ has a model.
- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in $N$ according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
- The limit interpretation can be shown to be a model of $N$. 
Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is \textit{total} and \textit{well-founded}. (There exist many such orderings, e.g., the length-based ordering on atoms when these are viewed as words over a suitable alphabet.)
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1. We assume that $\succ$ is any fixed ordering on ground atoms that is \textit{total} and \textit{well-founded}. (There exist many such orderings, e.g., the length-based ordering on atoms when these are viewed as words over a suitable alphabet.)

2. Extend $\succ$ to an ordering $\succ_L$ on ground literals:

\[
\begin{align*}
[\neg]A & \succ_L [\neg]B, \text{ if } A \succ B \\
\neg A & \succ_L A
\end{align*}
\]
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2. Extend $\succ$ to an ordering $\succ_L$ on ground literals:

   $\lnot A \succ_L \lnot B$, if $A \succ B$

   $\lnot A \succ_L A$

3. Extend $\succ_L$ to an ordering $\succ_C$ on ground clauses:

   $\succ_C = (\succ_L)_{mul}$, the multiset extension of $\succ_L$.

   \textit{Notation:} $\succ$ also for $\succ_L$ and $\succ_C$. 
Multisets

Definition

Let $E$ be a set. A multiset $M$ over $E$ is a mapping $M : E \rightarrow \mathbb{N}$. Hereby $M(e)$ specifies the number of occurrences of elements $e$ of the base set $E$ within the multiset $M$.

Let $(M, \succ)$ be a partial ordering. The multiset extension of $\succ$ to multisets over $E$ is defined by

$$M_1 \succ_{mul} M_2 \iff M_1 \neq M_2$$

$$\land \forall e \in E : [M_2(e) > M_1(e)]$$

$$\Rightarrow \exists e' \in E : (e' \succ e \land M_1(e') > M_2(e'))$$
## Clause Orderings

### Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$.

Order the following clauses:

\[
\neg A_1 \lor \neg A_4 \lor A_3 \\
\neg A_1 \lor A_2 \\
\neg A_1 \lor A_4 \lor A_3 \\
A_0 \lor A_1 \\
\neg A_5 \lor A_5 \\
A_1 \lor A_2
\]
Clause Orderings

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$.

Then:

\[
\begin{align*}
A_0 \lor A_1 \\
\lor \\
A_1 \lor A_2 \\
\lor \\
\neg A_1 \lor A_2 \\
\lor \\
\neg A_1 \lor \neg A_4 \lor A_3 \\
\lor \\
\neg A_1 \lor \neg A_4 \lor A_3 \\
\lor \\
\neg A_5 \lor A_5
\end{align*}
\]
Properties of the Clause Ordering

Theorem

1. The orderings on literals and clauses are total and well-founded.
2. Let $C$ and $D$ be clauses with $A = \max(C)$, $B = \max(D)$, where $\max(C)$ denotes the maximal atom in $C$.
   (i) If $A \succ B$ then $C \succ D$.
   (ii) If $A = B$, $A$ occurs negatively in $C$ but only positively in $D$, then $C \succ D$. 
Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:

- All $D$ where $\max(D) = B$
  - $\vdots \lor B$
  - $\vdots \lor B \lor B$
  - $\neg B \lor \ldots$

- All $C$ where $\max(C) = A$
  - $\vdots \lor A$
  - $\vdots \lor A \lor A$
  - $\neg A \lor \ldots$
Closure of Clause Sets under $Res$

**Definition**

- $Res(N) = \{ C \mid C \text{ is concl. of a rule in } Res \text{ w/ premises in } N \}$
- $Res^0(N) = N$
- $Res^{n+1}(N) = Res(Res^n(N)) \cup Res^n(N)$, for $n \geq 0$
- $Res^*(N) = \bigcup_{n \geq 0} Res^n(N)$

$N$ is called **saturated** (wrt. resolution), if $Res(N) \subseteq N$. 
Construction of Interpretations

Given:

set $\mathcal{N}$ of ground clauses, atom ordering $\succ$. 
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Wanted:
Herbrand interpretation $I$ such that
- “many” clauses from $N$ are valid in $I$;
- $I \models N$, if $N$ is saturated and $\bot \not\in N$. 
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- "many" clauses from $N$ are valid in $I$;
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Construction according to $\succ$, starting with the minimal clause.
## Construction of Interpretations

**Example**

Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

<table>
<thead>
<tr>
<th></th>
<th>clauses $C$</th>
<th>$I_C$</th>
<th>$\Delta_C$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\neg A_0$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>true in $I_C$</td>
</tr>
<tr>
<td>2</td>
<td>$A_0 \lor A_1$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$A_1$ maximal</td>
</tr>
<tr>
<td>3</td>
<td>$A_1 \lor A_2$</td>
<td>${A_1}$</td>
<td>$\emptyset$</td>
<td>true in $I_C$</td>
</tr>
<tr>
<td>4</td>
<td>$\neg A_1 \lor A_2$</td>
<td>${A_1}$</td>
<td>$\emptyset$</td>
<td>$A_2$ maximal</td>
</tr>
<tr>
<td>5</td>
<td>$\neg A_1 \lor A_4 \lor A_3 \lor A_0$</td>
<td>${A_1, A_2}$</td>
<td>${A_4}$</td>
<td>$A_4$ maximal</td>
</tr>
<tr>
<td>6</td>
<td>$\neg A_1 \lor \neg A_4 \lor A_3$</td>
<td>${A_1, A_2, A_4}$</td>
<td>$\emptyset$</td>
<td>$A_3$ not maximal; min. counter-ex.</td>
</tr>
<tr>
<td>7</td>
<td>$\neg A_1 \lor A_5$</td>
<td>${A_1, A_2, A_4}$</td>
<td>${A_5}$</td>
<td></td>
</tr>
</tbody>
</table>

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set
$\Rightarrow$ there exists a counterexample.
Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.
- When considering $C$, one already has a partial interpretation $I_C$ (initially $I_C = \emptyset$) available.
- If $C$ is true in the partial interpretation $I_C$, nothing is done. ($\Delta_C = \emptyset$).
- If $C$ is false, one would like to change $I_C$ such that $C$ becomes true.
Main Ideas of the Construction

- Changes should, however, be *monotone*. One never deletes anything from $I_C$ and the truth value of clauses smaller than $C$ should be maintained the way it was in $I_C$.

- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, $C$ is false in $I_C$, if $A$ occurs positively in $C$ (*adding $A$ will make $C$ become true*) and if this occurrence in $C$ is strictly maximal in the ordering on literals (*changing the truth value of $A$ has no effect on smaller clauses*).
Resolution Reduces Counterexamples

Example

\[ \neg A_1 \lor A_4 \lor A_3 \lor A_0 \]
\[ \neg A_1 \lor \neg A_4 \lor A_3 \]
\[ \neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0 \]

Construction of \( I \) for the extended clause set:

<table>
<thead>
<tr>
<th>clauses ( C )</th>
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<th>Remarks</th>
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<tbody>
<tr>
<td>( \neg A_0 )</td>
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<td>( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>( A_0 \lor A_1 )</td>
<td>( \emptyset )</td>
<td>( {A_1} )</td>
<td></td>
</tr>
<tr>
<td>( A_1 \lor A_2 )</td>
<td>{A_1}</td>
<td>( \emptyset )</td>
<td>{A_2}</td>
</tr>
<tr>
<td>( \neg A_1 \lor A_2 )</td>
<td>{A_1}</td>
<td>( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0 )</td>
<td>{A_1, A_2}</td>
<td>( \emptyset )</td>
<td>( A_3 ) occurs twice minimal counter-ex.</td>
</tr>
<tr>
<td>( \neg A_1 \lor A_4 \lor A_3 \lor A_0 )</td>
<td>{A_1, A_2}</td>
<td>( {A_4} )</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor \neg A_4 \lor A_3 )</td>
<td>{A_1, A_2, A_4}</td>
<td>( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor A_5 )</td>
<td>{A_1, A_2, A_4}</td>
<td>( {A_5} )</td>
<td></td>
</tr>
</tbody>
</table>

The same \( I \), but smaller counterexample, hence some progress was made.
Factorization Reduces Counterexamples

**Example**

\[
\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0 \\
\neg A_1 \lor \neg A_1 \lor A_3 \lor A_0
\]

**Construction of I for the extended clause set:**

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</tr>
<tr>
<td>$\neg A_3 \lor A_5$</td>
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<td></td>
</tr>
</tbody>
</table>

The resulting $I = \{A_1, A_2, A_3, A_5\}$ is a model of the clause set.
Construction of Candidate Models Formally

**Definition**

Let $N, \succ$ be given. We define sets $I_C$ and $\Delta_C$ for all ground clauses $C$ over the given signature inductively over $\succ$:

\[
I_C := \bigcup_{C \succ D} \Delta_D
\]

\[
\Delta_C := \begin{cases} 
\{A\}, & \text{if } C \in N, C = C' \lor A, A \succ C', I_C \models C \\
\emptyset, & \text{otherwise}
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We say that $C$ produces $A$, if $\Delta_C = \{A\}$. 
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We say that $C$ produces $A$, if $\Delta_C = \{A\}$.

The **candidate model** for $N$ (wrt. $\succ$) is given as $I_N^\succ := \bigcup_C \Delta_C$.

We also simply write $I_N$, or $I$, for $I_N^\succ$ if $\succ$ is either irrelevant or known from the context.
Structure of $N, \succ$

Let $A \succ B$; producing a new atom does not affect smaller clauses.

possibly productive

all $D$ with $\max(D) = B$

all $C$ with $\max(C) = A$
Model Existence Theorem

Theorem

(Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let $N$ be saturated wrt. $Res$, and suppose that $\bot \notin N$. Then $I_N \models N$. 
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Let \( \succ \) be a clause ordering, let \( N \) be saturated wrt. \( Res \), and suppose that \( \bot \notin N \). Then \( I_N^\succ \models N \).

Proof

Easy exercise! :-}
