« A Tutorial on Abstract Interpretation »

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VMCAI'05 Industrial Day

Static analysis
by abstract interpretation

Example of static analysis (input)

```
n := n0;

i := n;

while (i <> 0 ) do

      j := 0;

      while (j <> i) do

            j := j + 1

      od;

      i := i - 1

od
```

Example of static analysis (output)

```
{n0>=0}
   n := n0;
{n0=n,n0>=0}
   i := n;
{n0=i,n0=n,n0>=0}
   while (i <> 0 ) do
      {n0=n,i>=1,n0>=i}
         j := 0;
      {n0=n,j=0,i>=1,n0>=i}
         while (j <> i) do
            {n0=n,j>=0,i>=j+1,n0>=i}
               j := j + 1
            {n0=n,j>=1,i>=j,n0>=i}
         od;
      {n0=n,i=j,i>=1,n0>=i}
         i := i - 1
      {i+1=j,n0=n,i>=0,n0>=i+1}
   od
{n0=n,i=0,n0>=0}
```

Example of static analysis (safety)

```
{n0>=0}
   n := n0;
{n0=n,n0>=0}
   i := n;
{n0=i,n0=n,n0>=0}
   while (i <> 0 ) do
      {n0=n,i>=1,n0>=i}
         j := 0;
      {n0=n,j=0,i>=1,n0>=i}
         while (j <> i) do
            {n0=n,j>=0,i>=j+1,n0>=i}
               j := j + 1      ⟵  j < n0 so no upper overflow
            {n0=n,j>=1,i>=j,n0>=i}
         od;
      {n0=n,i=j,i>=1,n0>=i}
         i := i - 1            ⟵  i > 0 so no lower overflow
      {i+1=j,n0=n,i>=0,n0>=i+1}
   od
{n0=n,i=0,n0>=0}
```

n0 must be initially nonnegative (otherwise the program does not terminate properly)

Static analysis by abstract interpretation

Verification: define and prove automatically a property of the possible behaviors of a complex computer program (example: program semantics);

Abstraction: the reasoning/calculus can be done on an abstraction of these behaviors dealing only with those elements of the behaviors related to the considered property;

Theory: abstract interpretation.

Example of static analysis

Verification: absence of runtime errors;

Abstraction: polyhedral abstraction (affine inequalities);

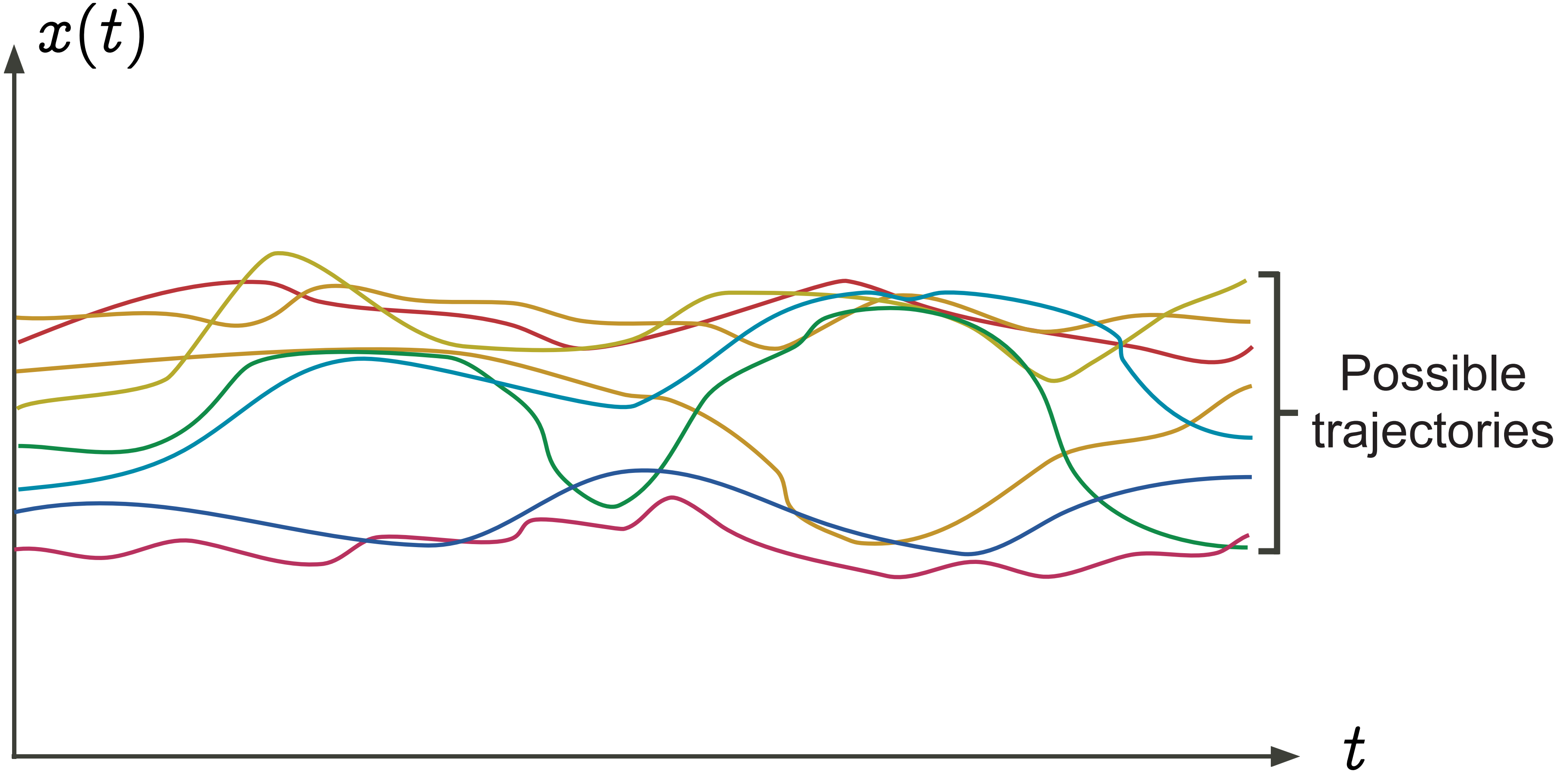
Theory: abstract interpretation.

A very informal introduction to the principles of abstract interpretation

Semantics

The *concrete semantics* of a program formalizes (is a mathematical model of) the set of all its possible executions in all possible execution environments.

Graphic example: Possible behaviors



Undecidability

- The concrete mathematical semantics of a program is an “tinfinite” mathematical object, *not computable*;
- All non trivial questions on the concrete program semantics are *undecidable*.

Example: termination

- Assume `termination(P)` would always terminates and returns `true` iff `P` always terminates on all input data;
- The following program yields a contradiction

`P ≡ while termination(P) do skip od.`

Graphic example: Safety properties

The *safety properties* of a program express that no possible execution in any possible execution environment can reach an erroneous state.

Graphic example: Safety property

Safety proofs

- A safety proof consists in proving that the intersection of the program concrete semantics and the forbidden zone is empty;
- Undecidable problem (the concrete semantics is not computable);
- Impossible to provide completely automatic answers with finite computer resources and neither human interaction nor uncertainty on the answer[1].

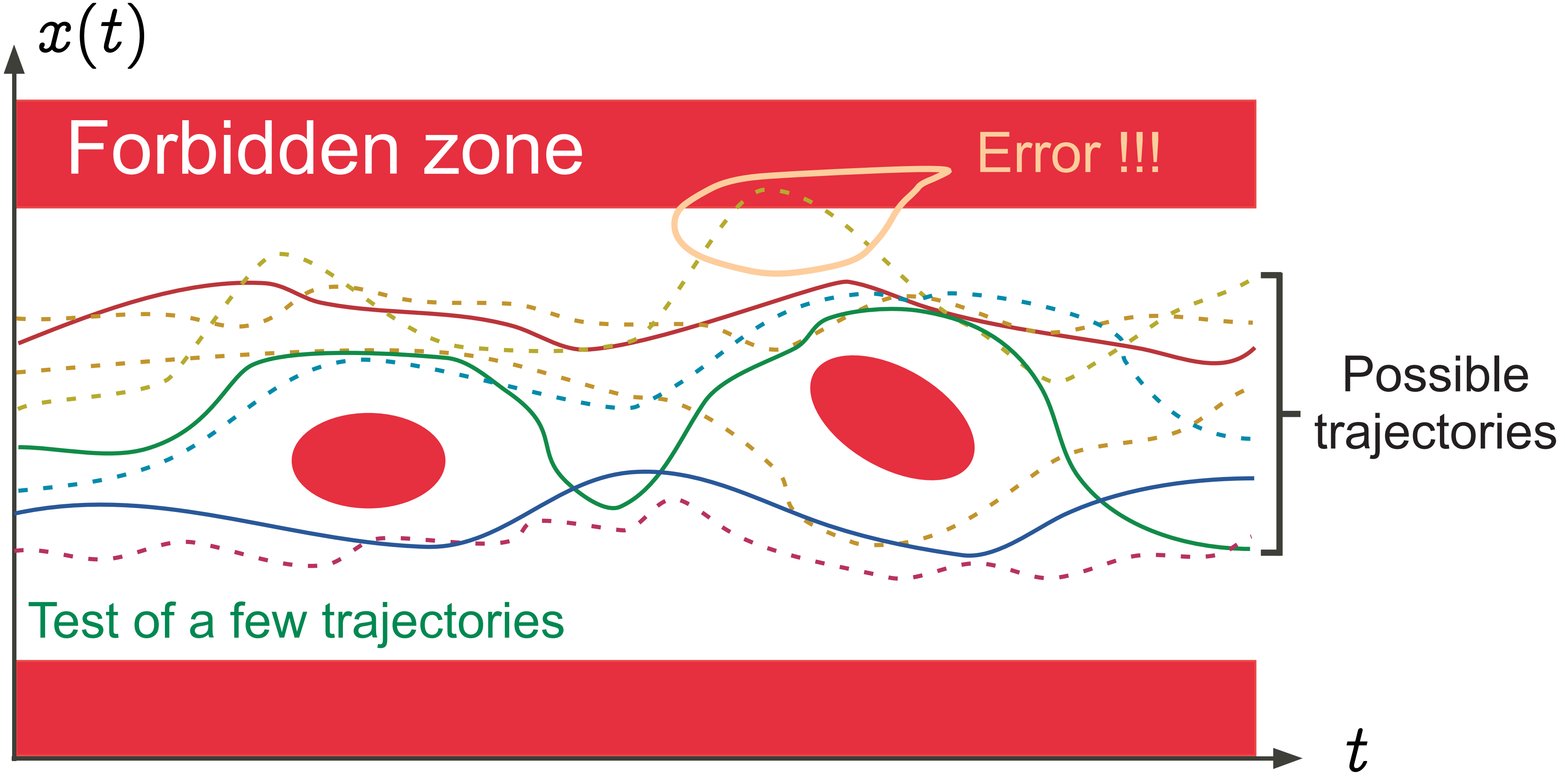
---

[1] e.g. probabilistic answer.

Test/debugging

- consists in considering a subset of the possible executions;
- not a correctness proof;
- absence of coverage is the main problem.

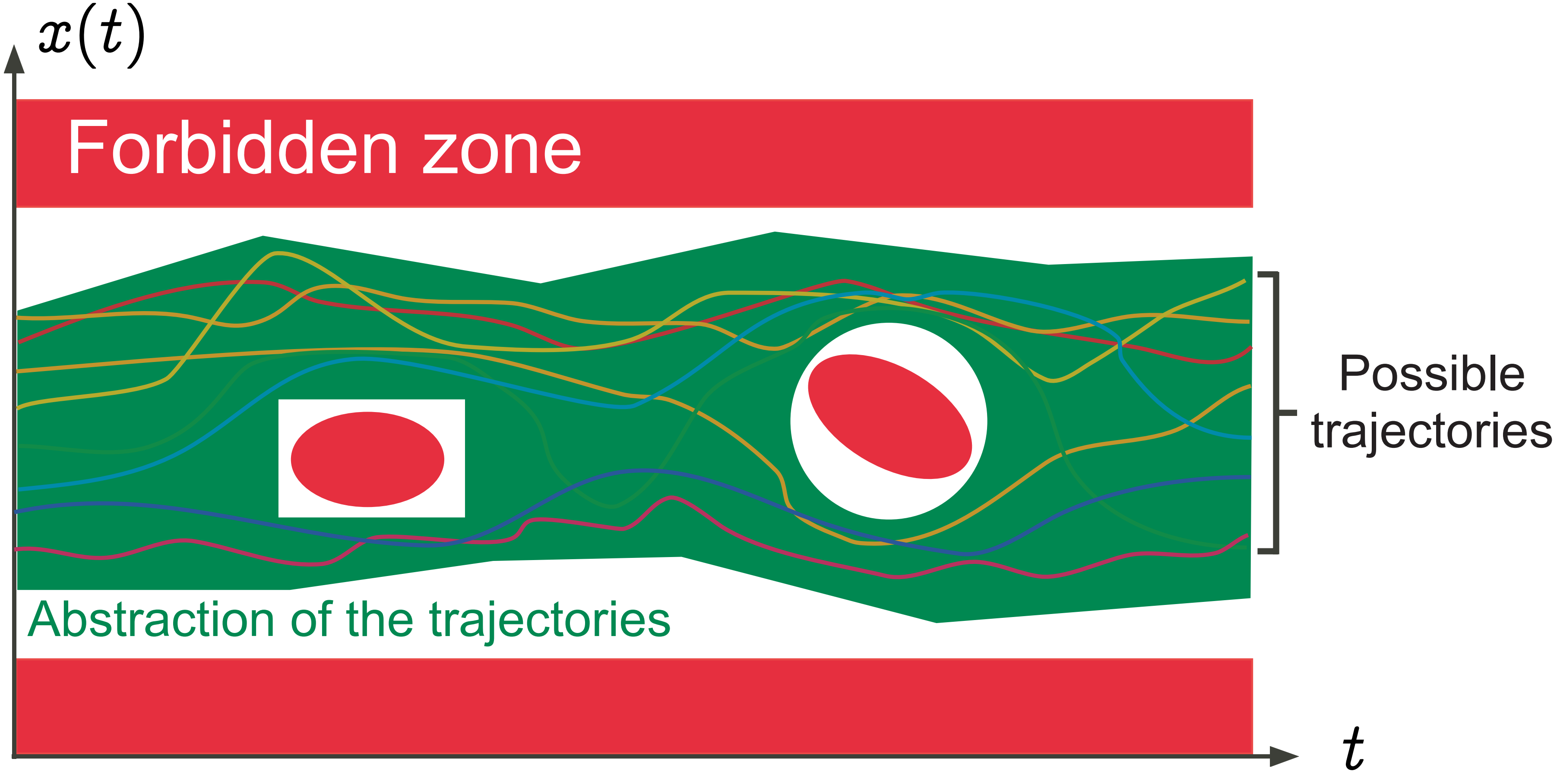
Graphic example: Property test/simulation



Abstract interpretation

- consists in considering an *abstract semantics*, that is to say a superset of the concrete semantics of the program;
- hence the abstract semantics covers all possible concrete cases;
- correct: if the abstract semantics is safe (does not intersect the forbidden zone) then so is the concrete semantics

Graphic example: Abstract interpretation



Formal methods

Formal methods are abstract interpretations, which differ in the way to obtain the abstract semantics:

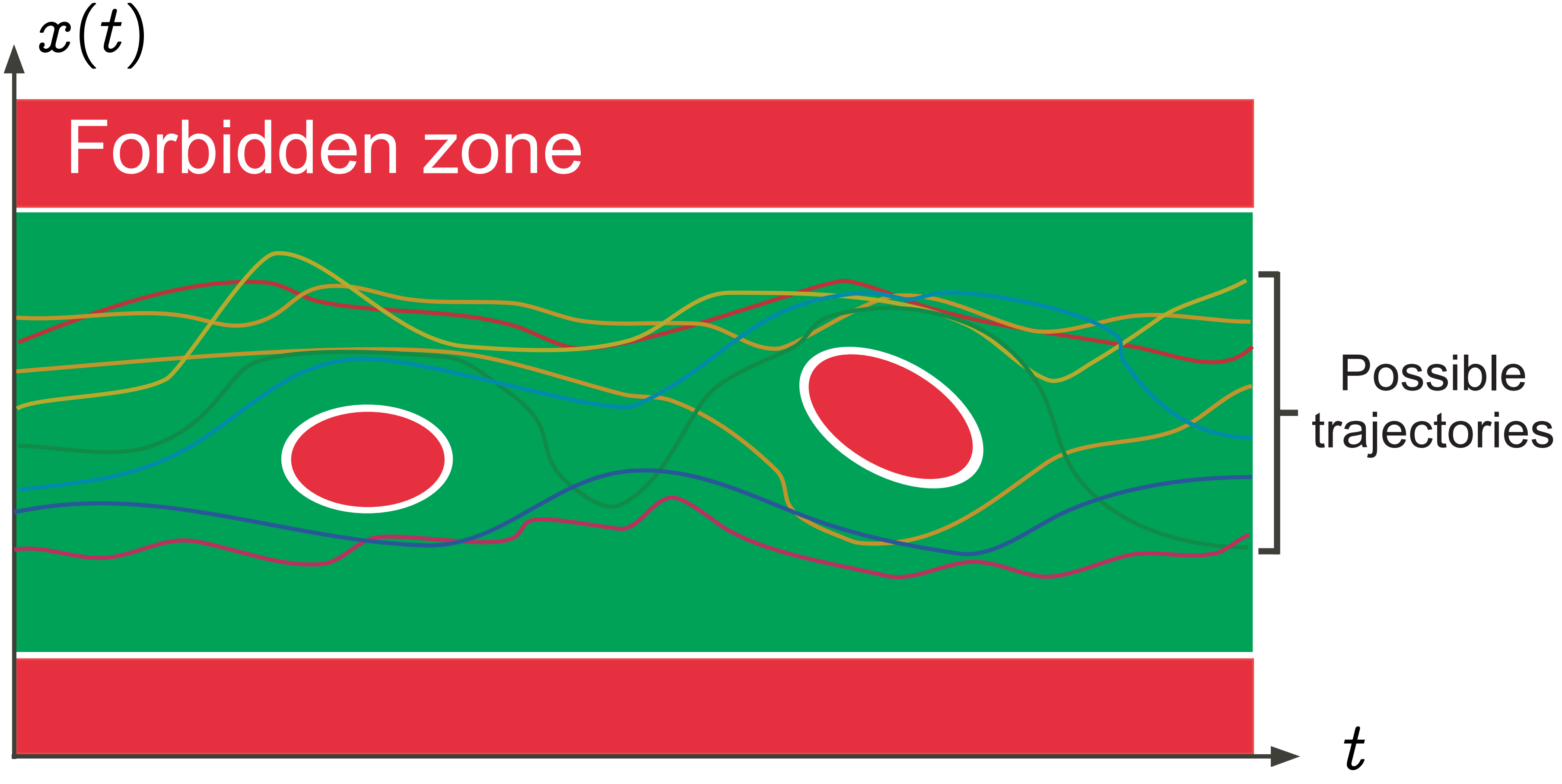
- "*model checking*":
  - the abstract semantics is given manually by the user;
  - in the form of a finitary model of the program execution;
  - can be computed automatically, by techniques relevant to static analysis.

- "*deductive methods*":
  - the abstract semantics is specified by verification conditions;
  - the user must provide the abstract semantics in the form of inductive arguments (e.g. invariants);
  - can be computed automatically by methods relevant to static analysis.
- "*static analysis*": the abstract semantics is computed automatically from the program text according to predefined abstractions (that can sometimes be tailored automatically/manually by the user).

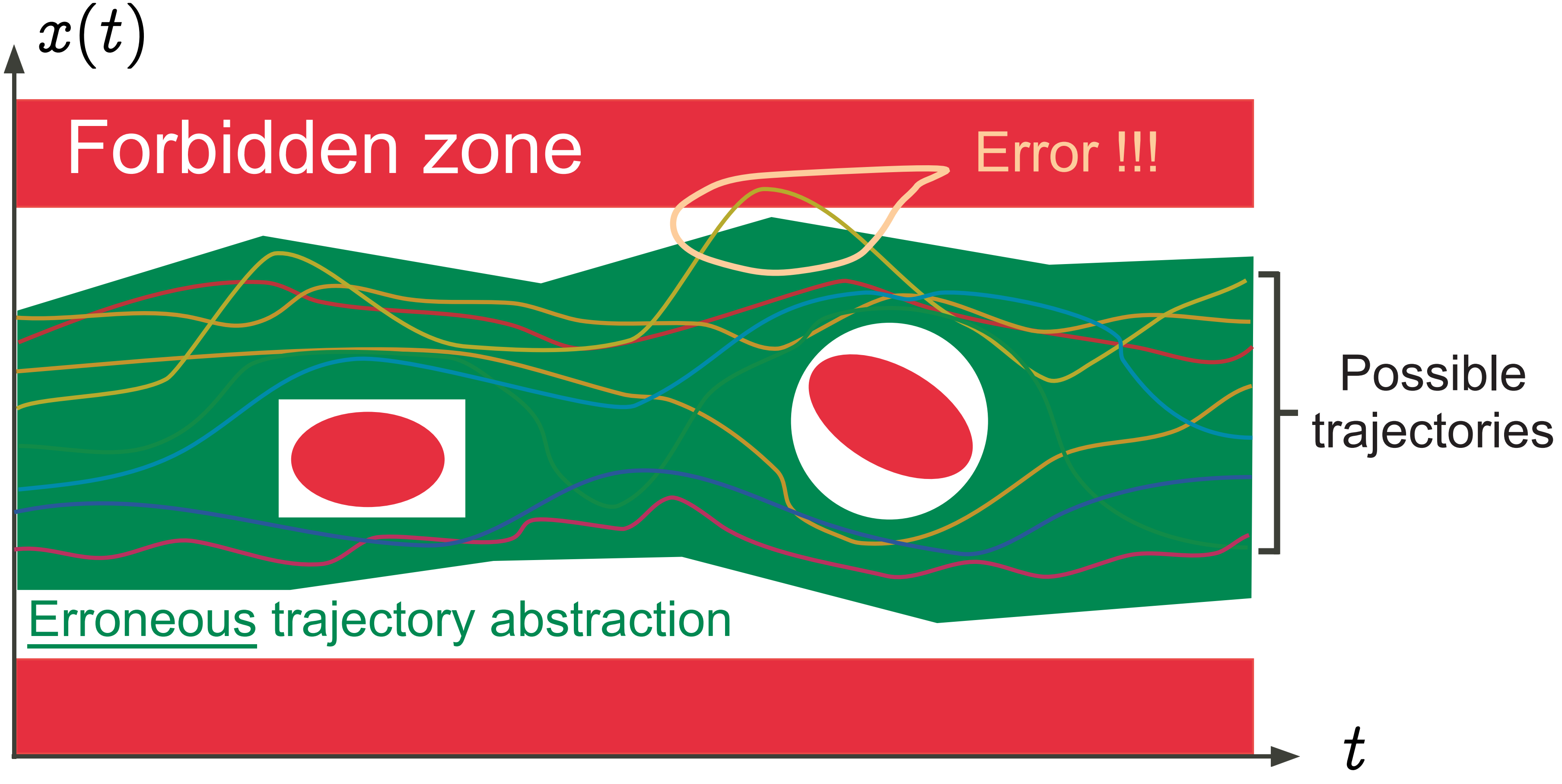
Required properties of the abstract semantics

- sound so that no possible error can be forgotten;
- precise enough (to avoid false alarms);
- as simple/abstract as possible (to avoid combinatorial explosion phenomena).

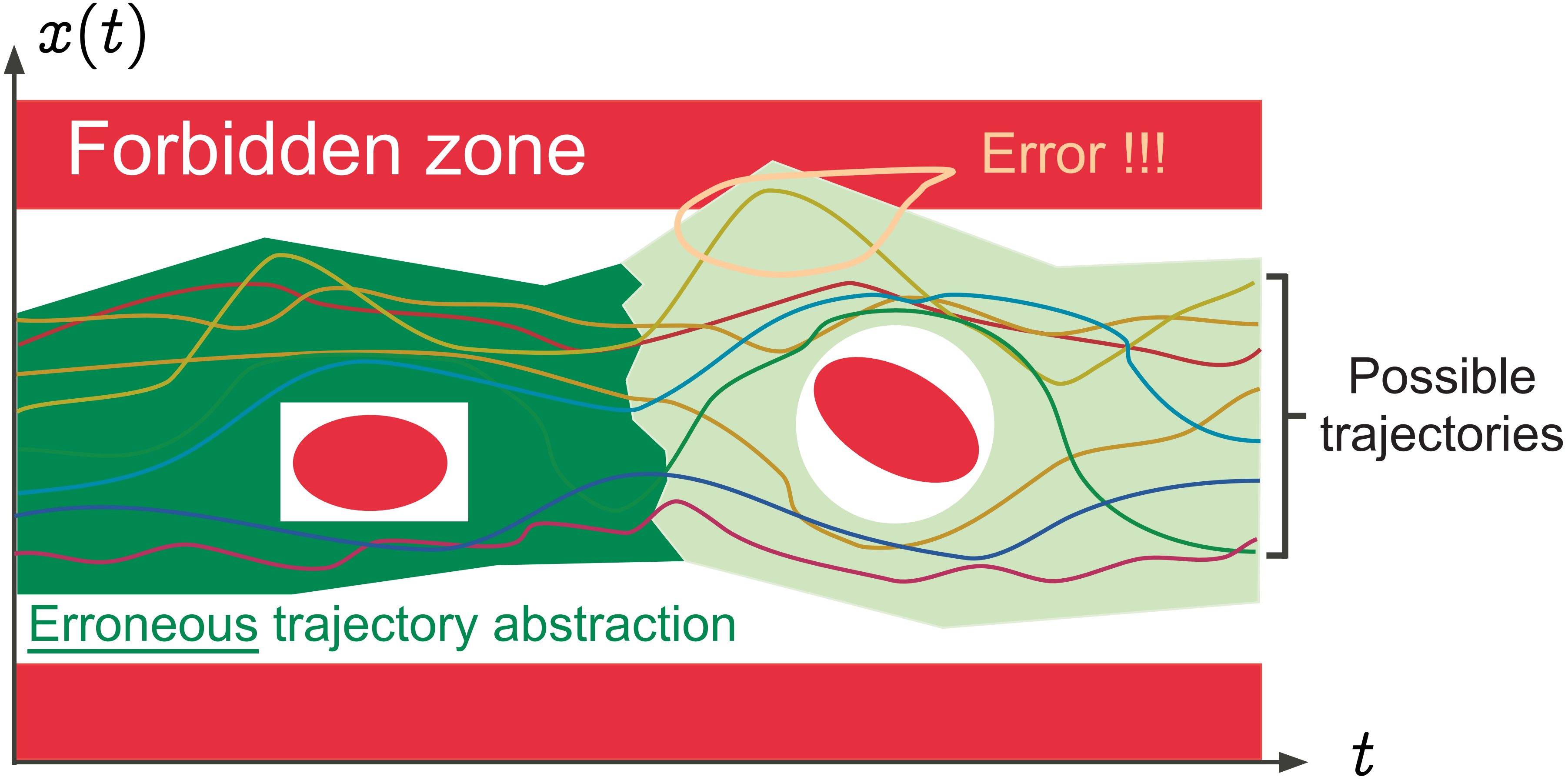
Graphic example: The most abstract correct and precise semantics



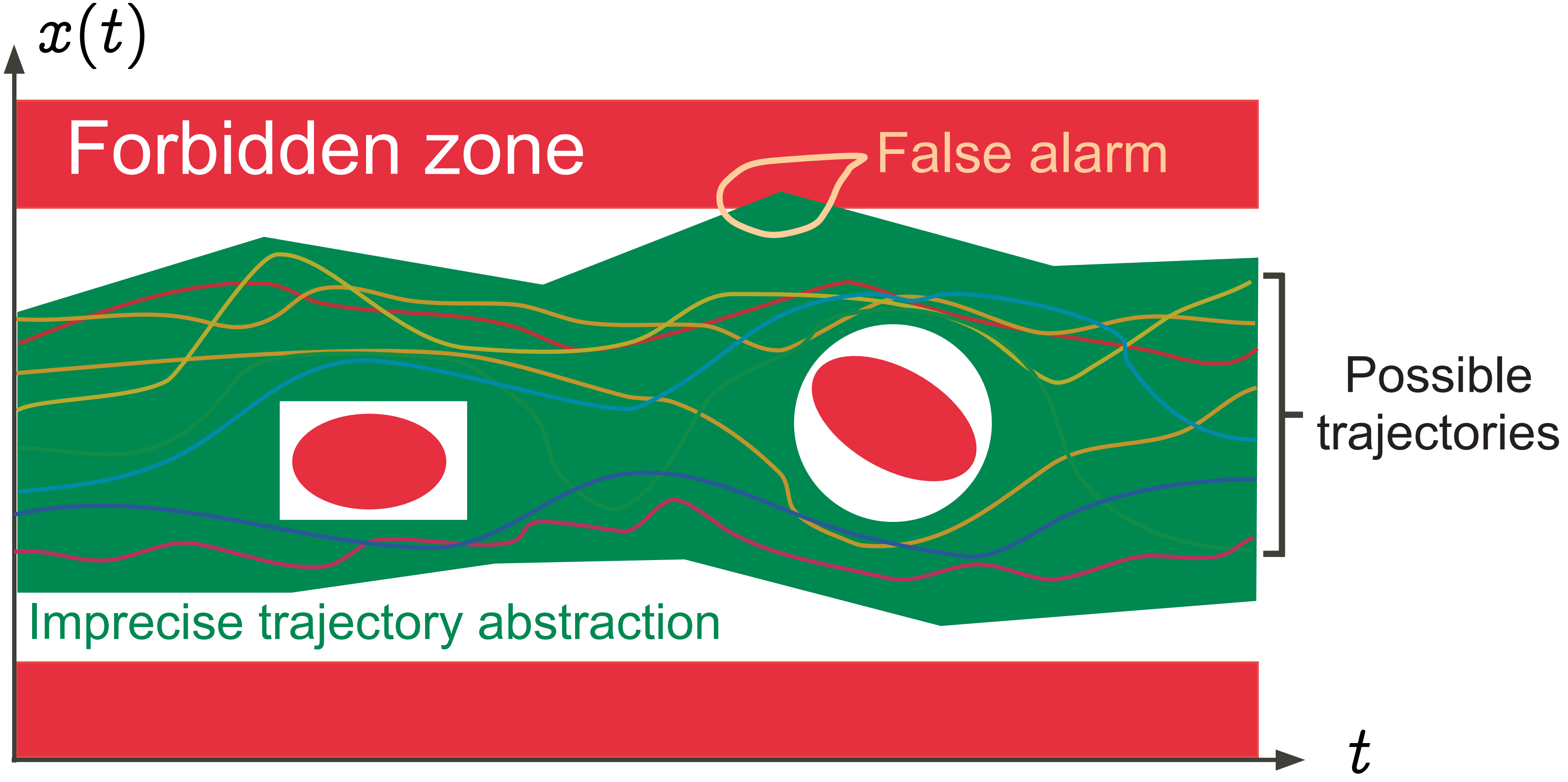
Graphic example: Erroneous abstraction — I



Graphic example: Erroneous abstraction — II



Graphic example: Imprecision ⇒ false alarms

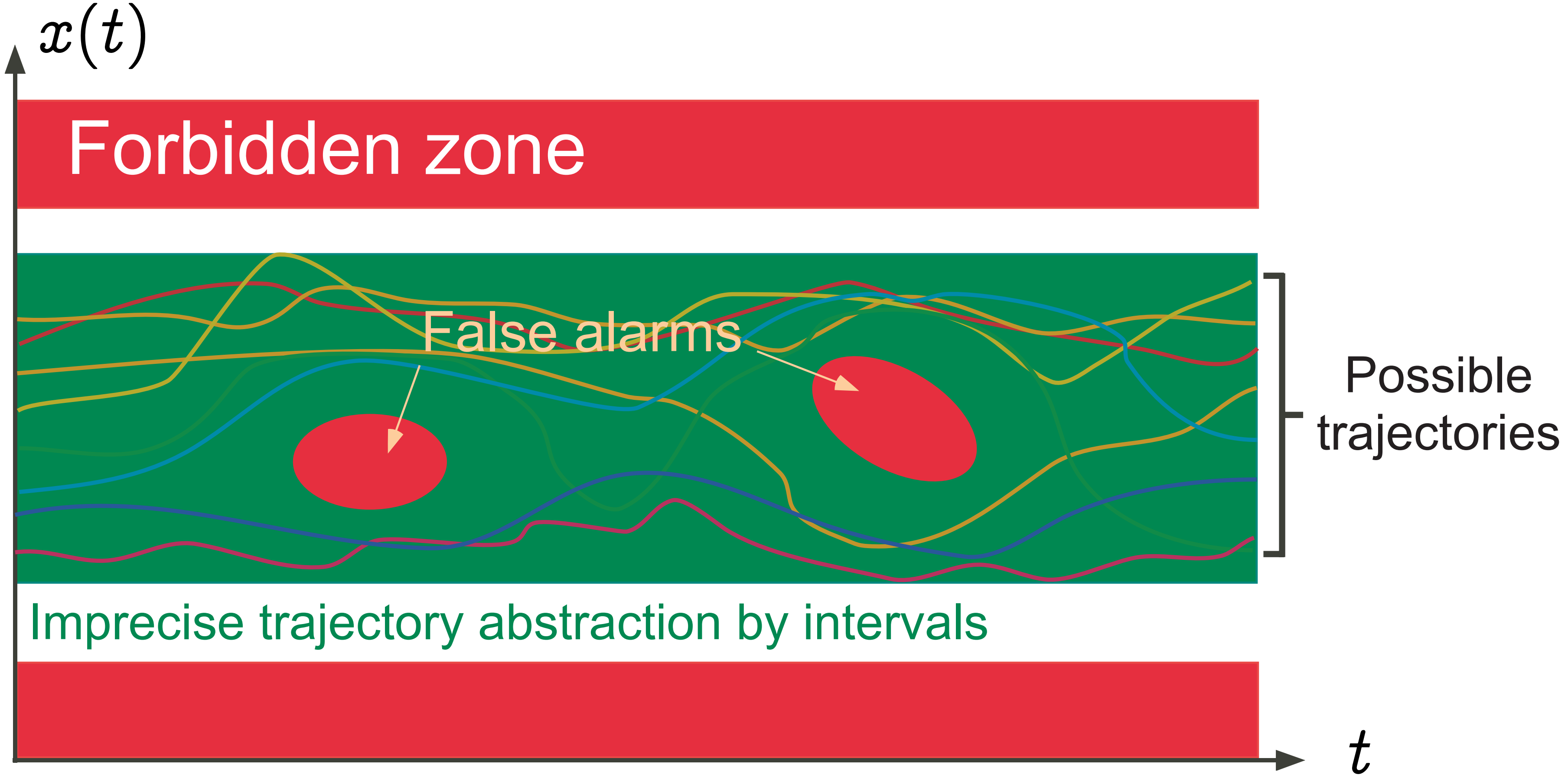


Abstract domains

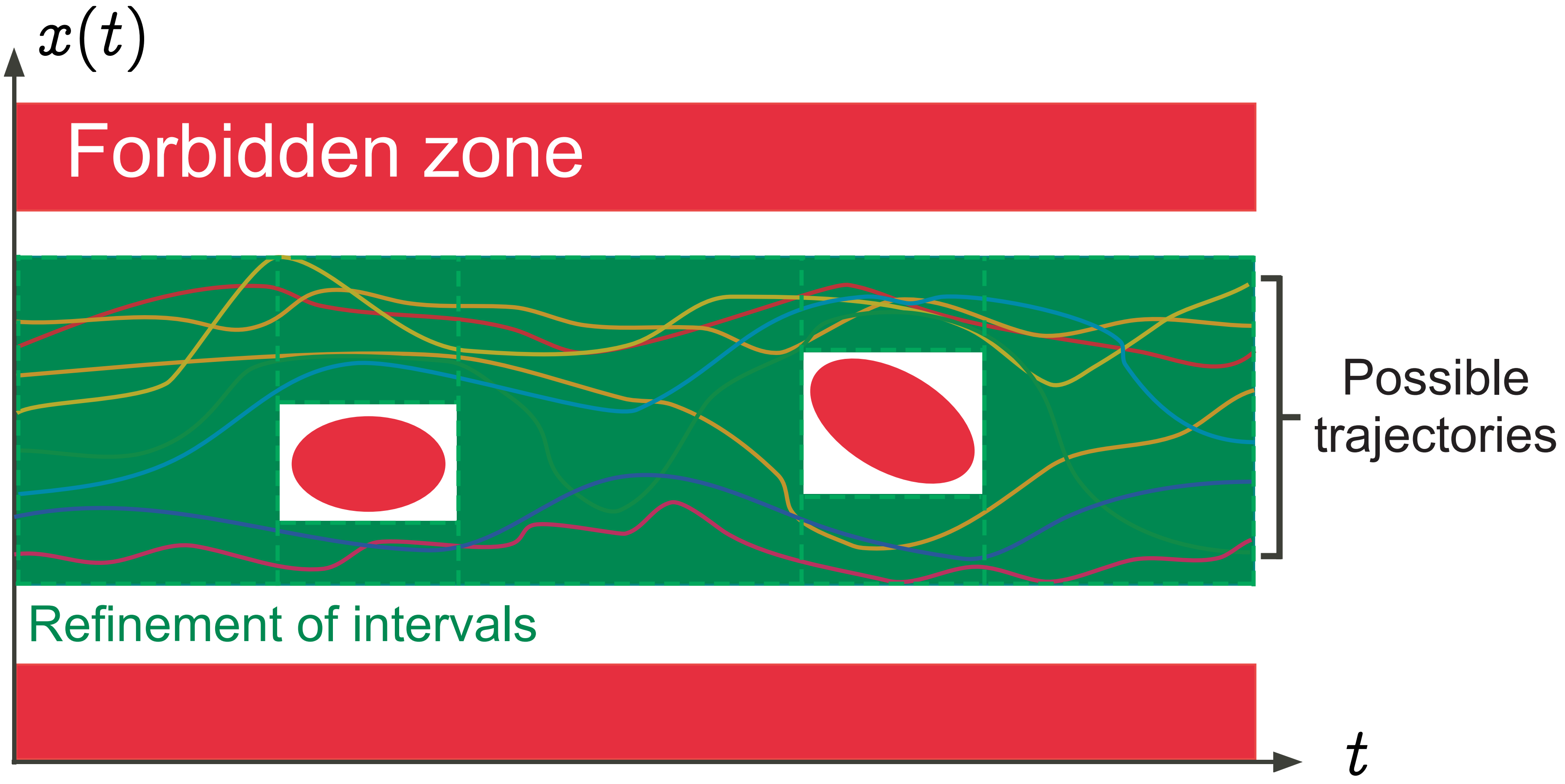
Standard abstractions

- that serve as a basis for the design of static analyzers:
  - abstract program data,
  - abstract program basic operations;
  - abstract program control (iteration, procedure, concurrency, . . .);
- can be parametrized to allow for manual adaptation to the application domains.

Graphic example: Standard abstraction by intervals



Graphic example: A more refined abstraction



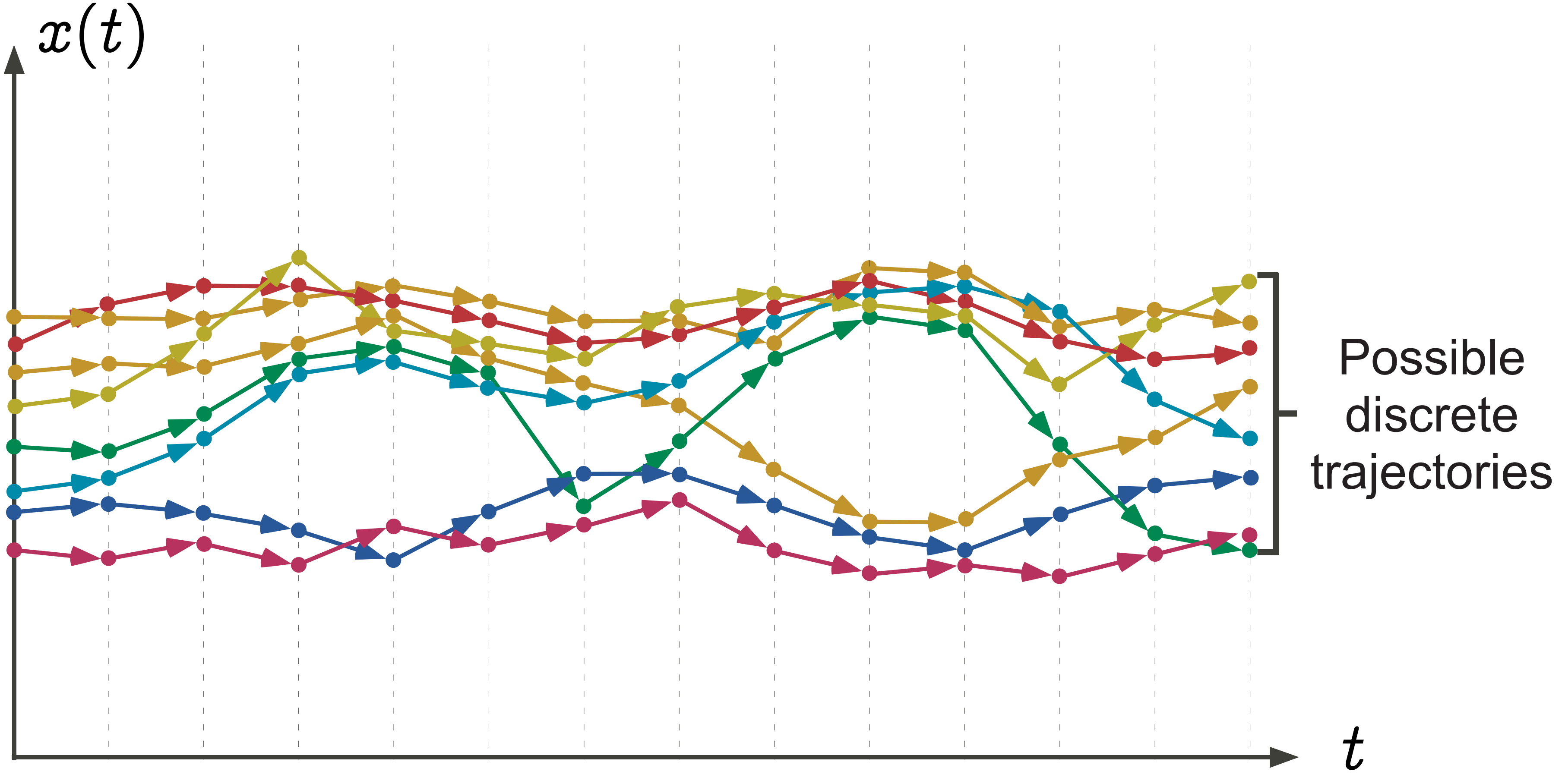
A very informal introduction to static analysis algorithms

Standard operational semantics

Standard semantics

- Start from a standard operational semantics that describes formally:
  - states that is data values of program variables,
  - transitions that is elementary computation steps;
- Consider traces that is successions of states corresponding to executions described by transitions (possibly infinite).

Graphic example: Small-steps transition semantics



Example: Small-steps transition semantics of an assignment

```
int x;
...
l:
    x := x + 1;
l':
```

$$\{\mathtt{l}:\mathtt{x}=\upsilon \rightarrow \mathtt{l}':\mathtt{x}=\upsilon+1 \mid \upsilon\in[\mathtt{min\_int},\mathtt{max\_int}-1]\}$$

$$\cup\,\{\mathtt{l}:\mathtt{x}=\mathtt{max\_int}\rightarrow \mathtt{l}':\mathtt{x}=\Omega\}\qquad\text{(runtime error)}$$

Example: Small-steps transition semantics of a loop

```
l1:
   x := 1;
l2:
   while x < 10 do
l3:
      x := x + 1
l4:
   od
l5:
```

$$\left.\begin{array}{c} \mathtt{l1} : \ldots \\ \mathtt{l1} : \mathtt{x} = -1 \\ \mathtt{l1} : \mathtt{x} = 0 \\ \mathtt{l1} : \mathtt{x} = 1 \\ \mathtt{l1} : \ldots \end{array}\right] \begin{array}{l} \searrow \\ \rightarrow \mathtt{l2} : \mathtt{x} = 1 \\ \nearrow \end{array}$$

$\mathtt{l2} : \mathtt{x} = 1 \rightarrow \mathtt{l3} : \mathtt{x} = 1$

$\mathtt{l3} : \mathtt{x} = 1 \rightarrow \mathtt{l4} : \mathtt{x} = 2$

$\mathtt{l4} : \mathtt{x} = 2 \rightarrow \mathtt{l3} : \mathtt{x} = 2$

$\mathtt{l3} : \mathtt{x} = 2 \rightarrow \mathtt{l4} : \mathtt{x} = 3$

$\ldots$

$\mathtt{l4} : \mathtt{x} = 10 \rightarrow \mathtt{l5} : \mathtt{x} = 10$

Example: Trace semantics of loop

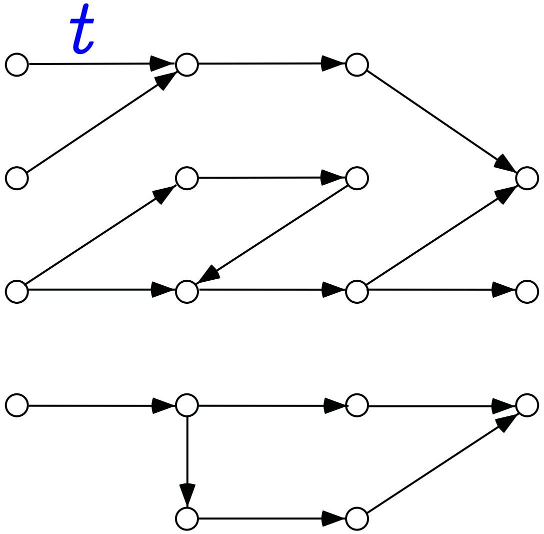
```
l1:
    x := 1;
l2:
    while x < 10 do
l3:
        x := x + 1
l4:
    od
l5:
```

$$\left.\begin{array}{c} \mathtt{l1} : \ldots \\ \mathtt{l1} : \mathtt{x} = -1 \\ \mathtt{l1} : \mathtt{x} = 0 \\ \mathtt{l1} : \mathtt{x} = 1 \\ \mathtt{l1} : \ldots \end{array}\right] \begin{array}{l} \searrow \\ \rightarrow \mathtt{l2} : \mathtt{x} = 1 \rightarrow \mathtt{l3} : \mathtt{x} = 1 \rightarrow \mathtt{l4} : \mathtt{x} = 2 \rightarrow \\ \nearrow \end{array}$$

$$\mathtt{l3} : \mathtt{x} = 2 \rightarrow \mathtt{l4} : \mathtt{x} = 3 \ldots \rightarrow \mathtt{l4} : \mathtt{x} = 10 \rightarrow \mathtt{l5} : \mathtt{x} = 10$$

Transition systems

– $\langle S, \xrightarrow{t} \rangle$ where:
  - S is a set of states/vertices/...
  - $\xrightarrow{t} \in \wp(S \times S)$ is a transition relation/set of arcs/...

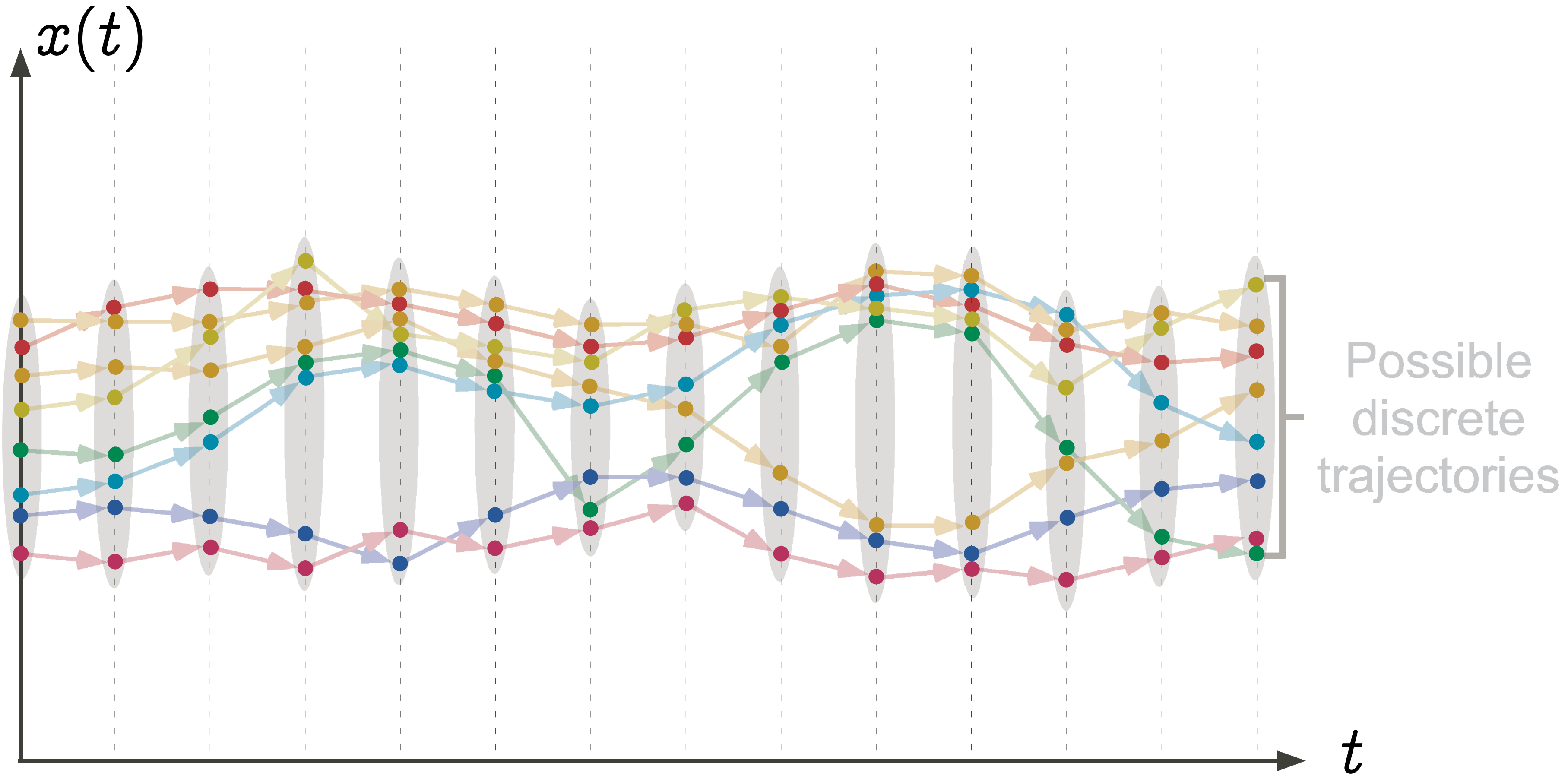


Collecting semantics in fixpoint form

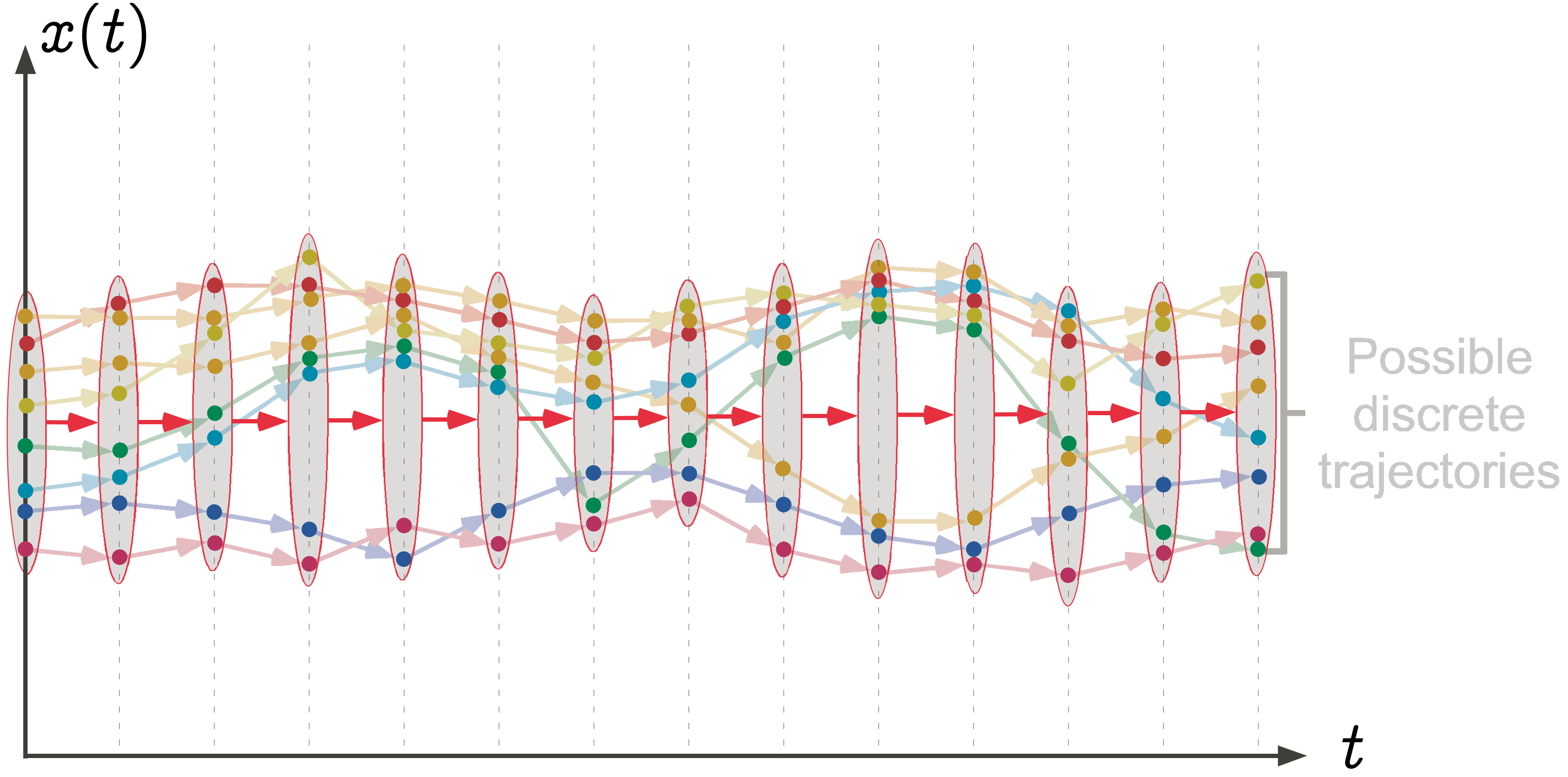
Collecting semantics

- consider all traces simultaneously;
- collecting semantics:
  - sets of states that describe data values of program variables on all possible trajectories;
  - set of states transitions that is simultaneous elementary computation steps on all possible trajectories;

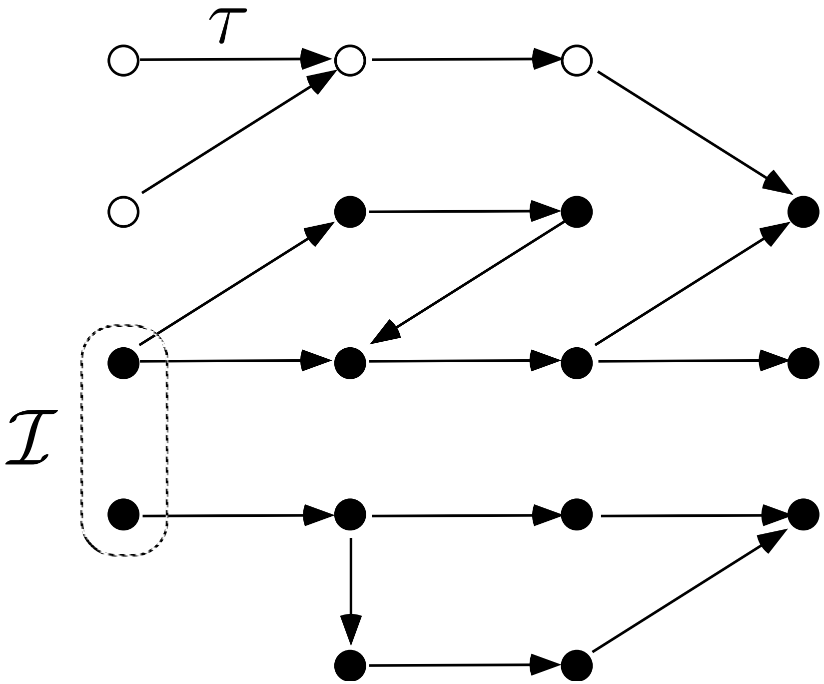
Graphic example: sets of states



Graphic example: set of states transitions



Example: Reachable states of a transition system



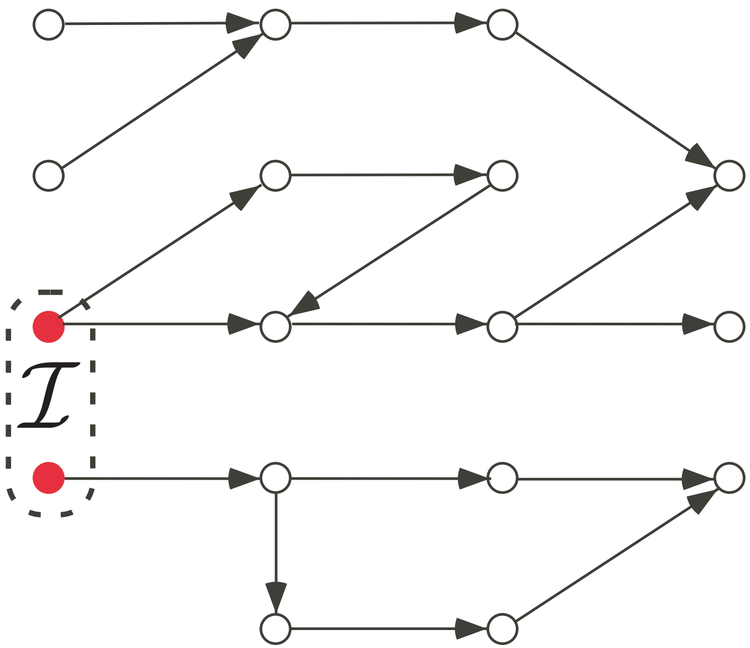
Reachable states in fixpoint form

$$F(X) = \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$$

$$\mathcal{R} = \mathsf{lfp}_{\emptyset}^{\subseteq} F$$

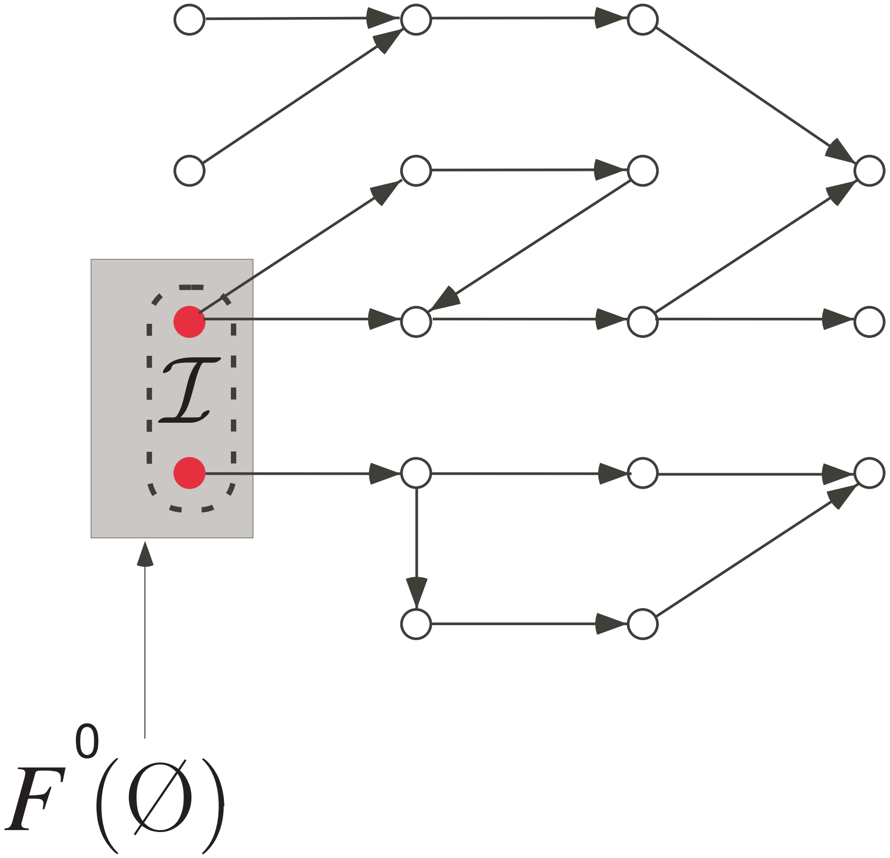
$$= \bigcup_{n=0}^{+\infty} F^n(\emptyset) \qquad \text{where} \qquad \begin{aligned} f^0(x) &= x \\ f^{n+1}(x) &= f(f^n(x)) \end{aligned}$$

Example of fixpoint iteration for reachable states $\mathsf{lfp}^{\subseteq}_{\emptyset}\ \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$

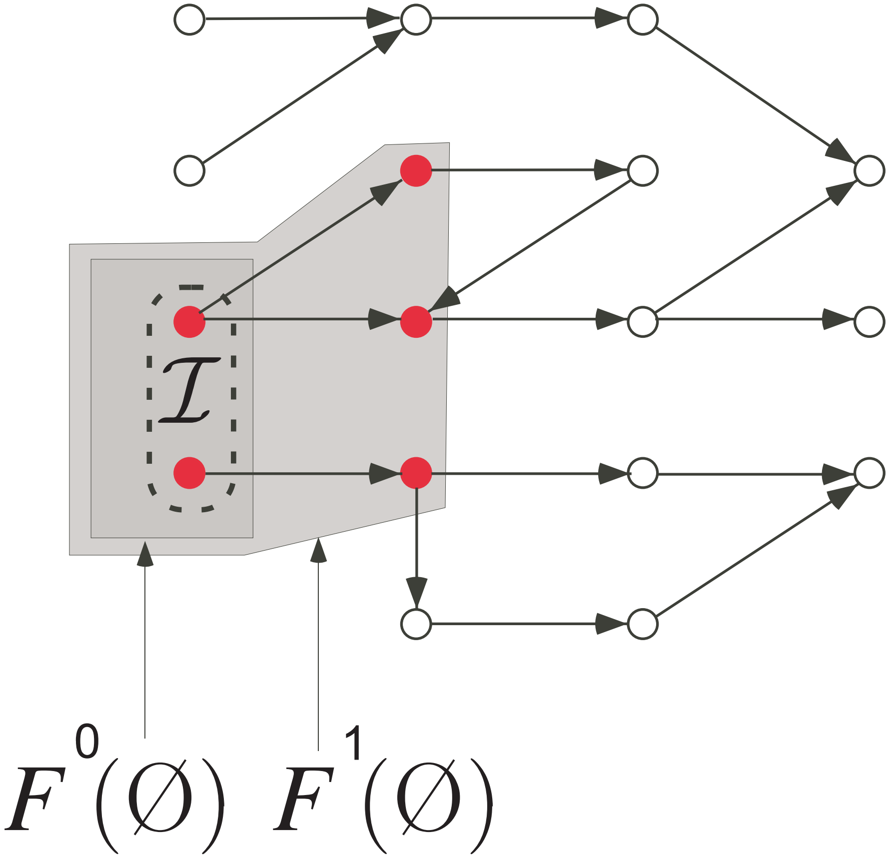


$\emptyset$

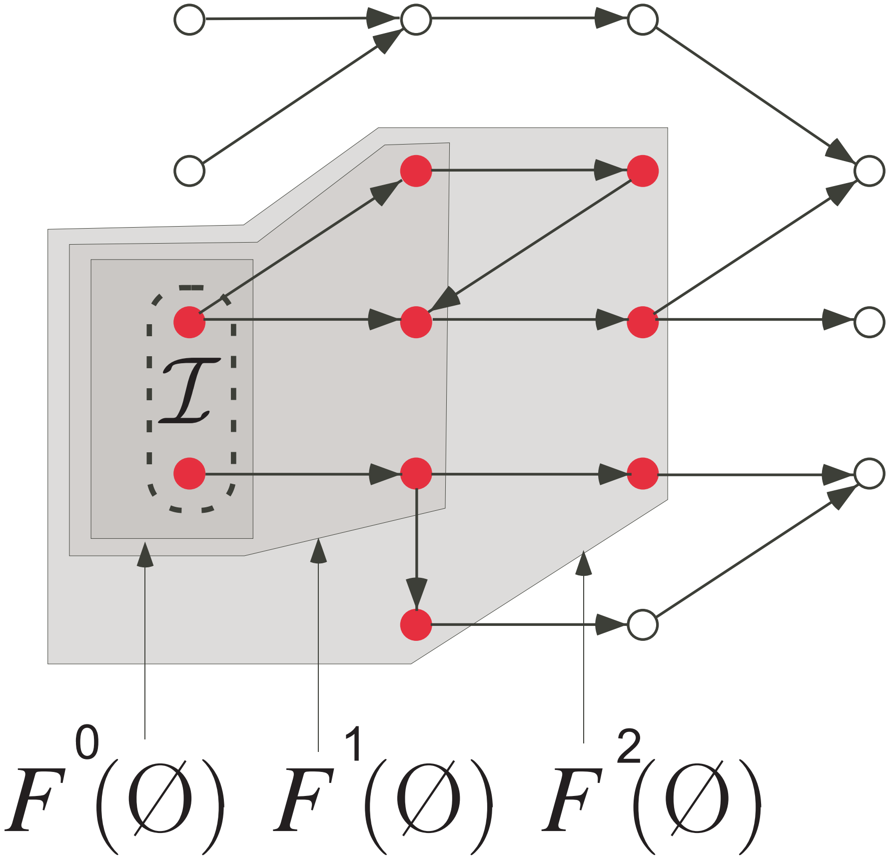
Example of fixpoint iteration for reachable states $\mathsf{lfp}_{\emptyset}^{\subseteq}\ \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$



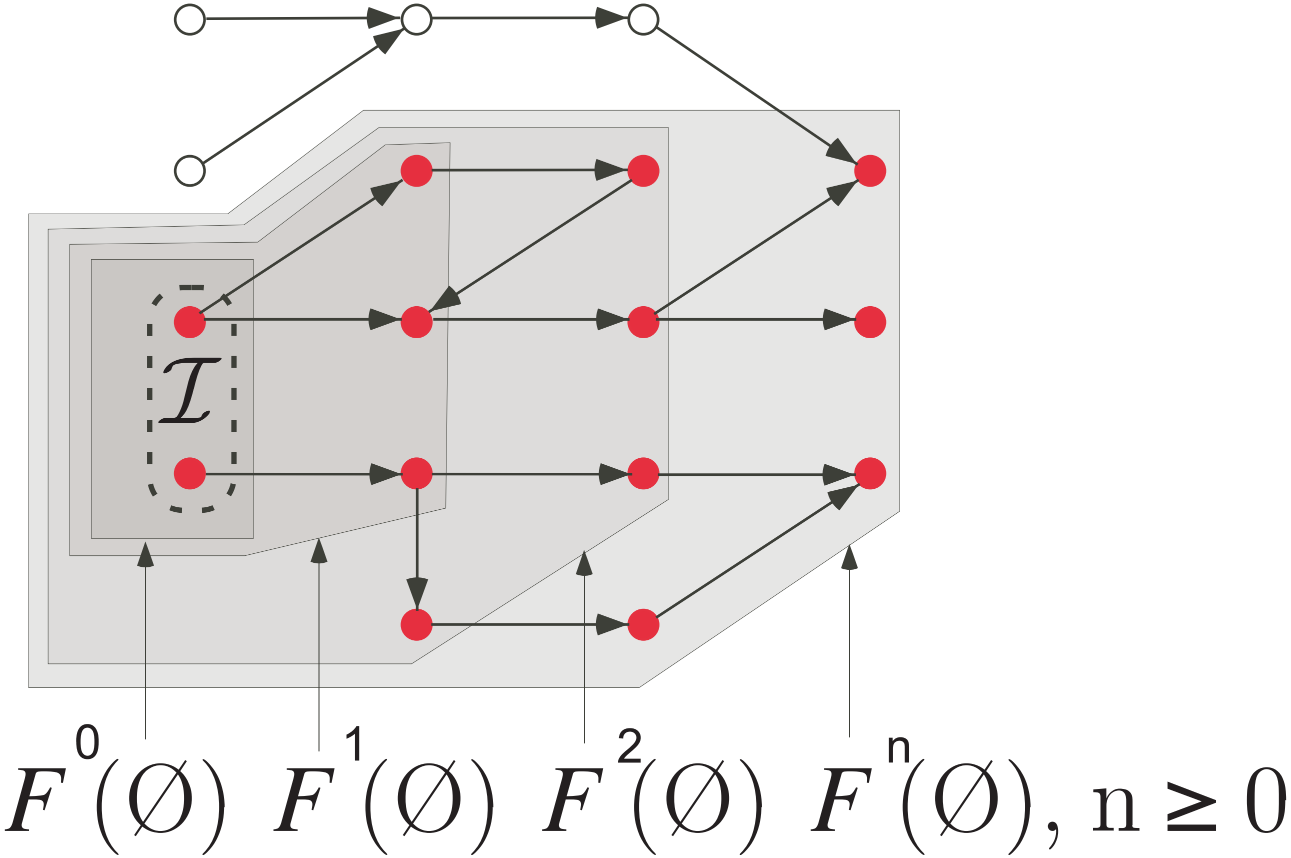
Example of fixpoint iteration for reachable states $\mathsf{lfp}_{\emptyset}^{\subseteq} \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$



Example of fixpoint iteration for reachable states $\mathsf{lfp}^{\subseteq}_{\emptyset}\ \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$



Example of fixpoint iteration for reachable states $\mathsf{lfp}^{\subseteq}_{\emptyset}\ \lambda X \cdot \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$

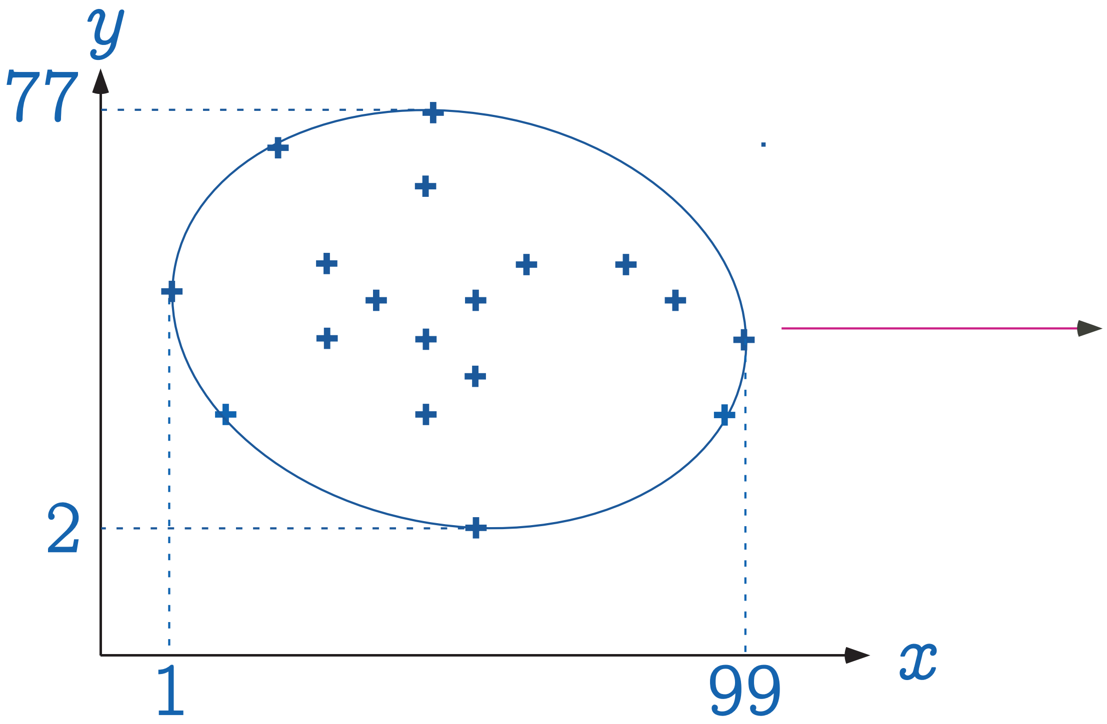


Abstraction by Galois connections

Abstracting sets (i.e. properties)

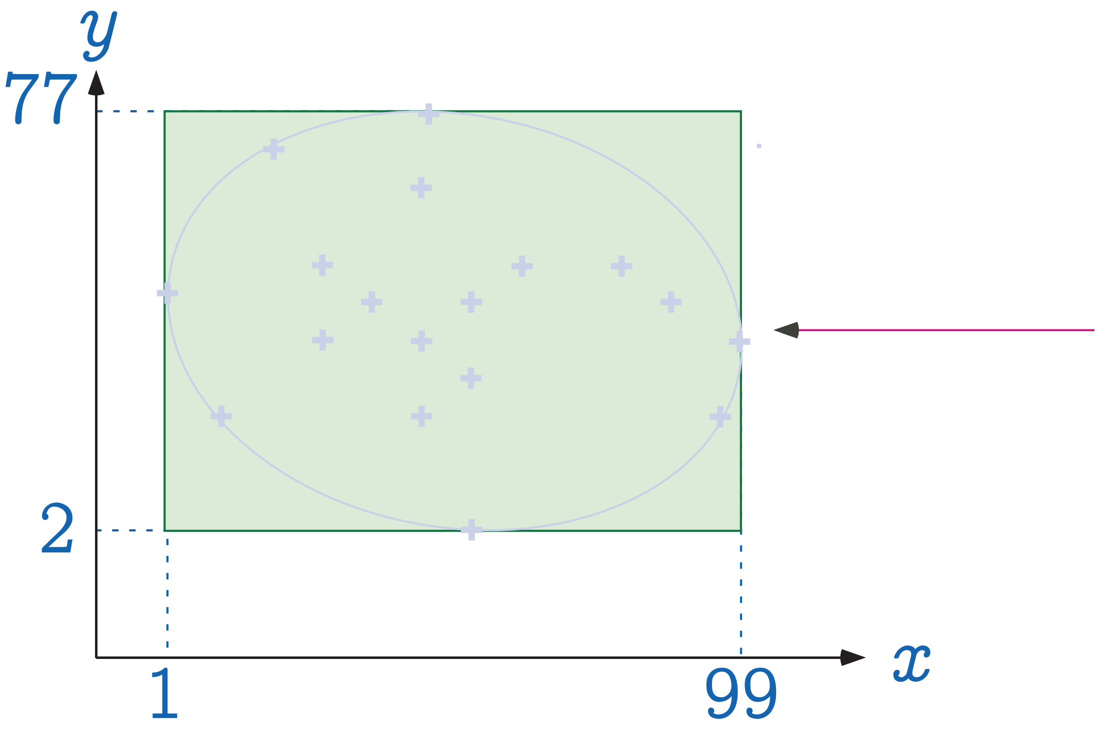
- Choose an abstract domain, replacing sets of objects (states, traces, ...) S by their abstraction $\alpha(S)$
- The abstraction function α maps a set of concrete objects to its abstract interpretation;
- The inverse concretization function γ maps an abstract set of objects to concrete ones;
- Forget no concrete objects: (abstraction from above) $S \subseteq \gamma(\alpha(S))$.

Interval abstraction α



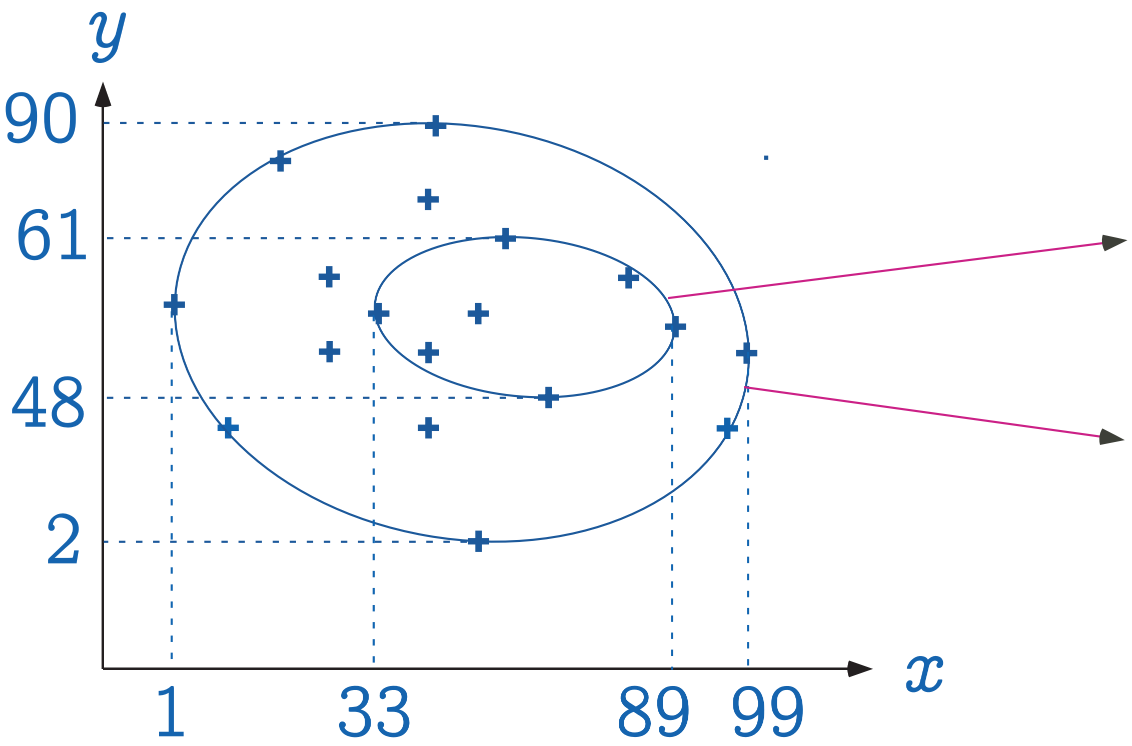
$$\{x : [1, 99], y : [2, 77]\}$$

Interval concretization γ



$\{x : [1, 99], y : [2, 77]\}$

The abstraction α is monotone



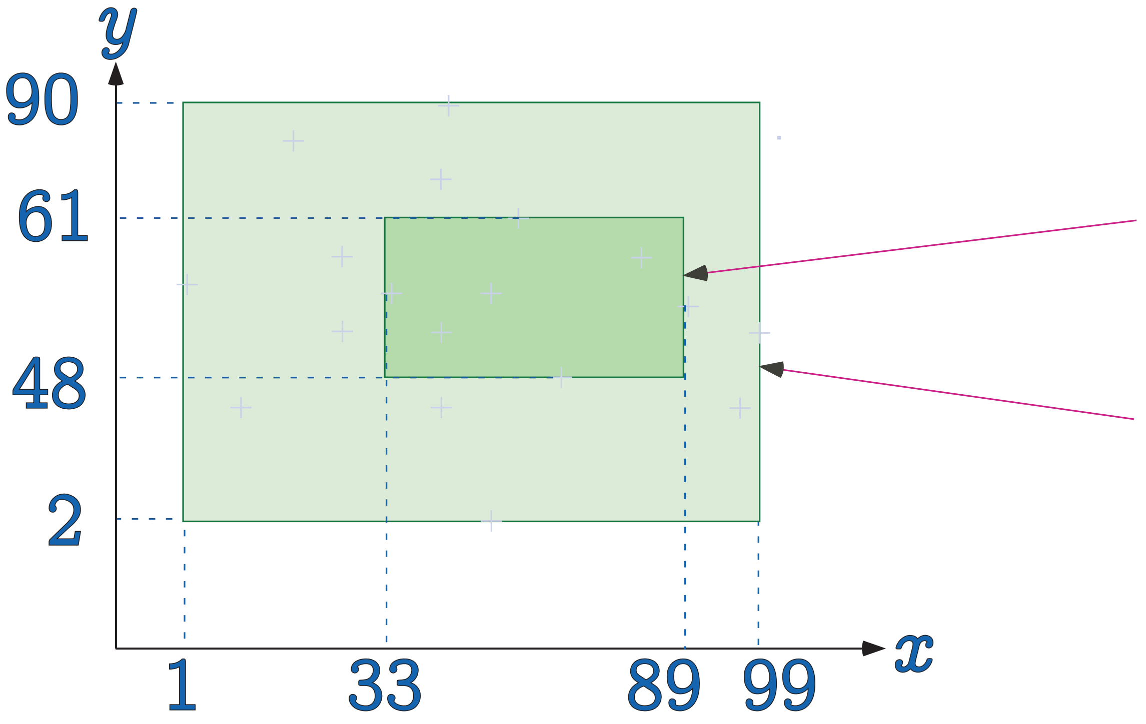
$\{x : [33, 89], y : [48, 61]\}$

$\sqsubseteq$

$\{x : [1, 99], y : [2, 90]\}$

$$X \subseteq Y \Rightarrow \alpha(X) \sqsubseteq \alpha(Y)$$

The concretization γ is monotone



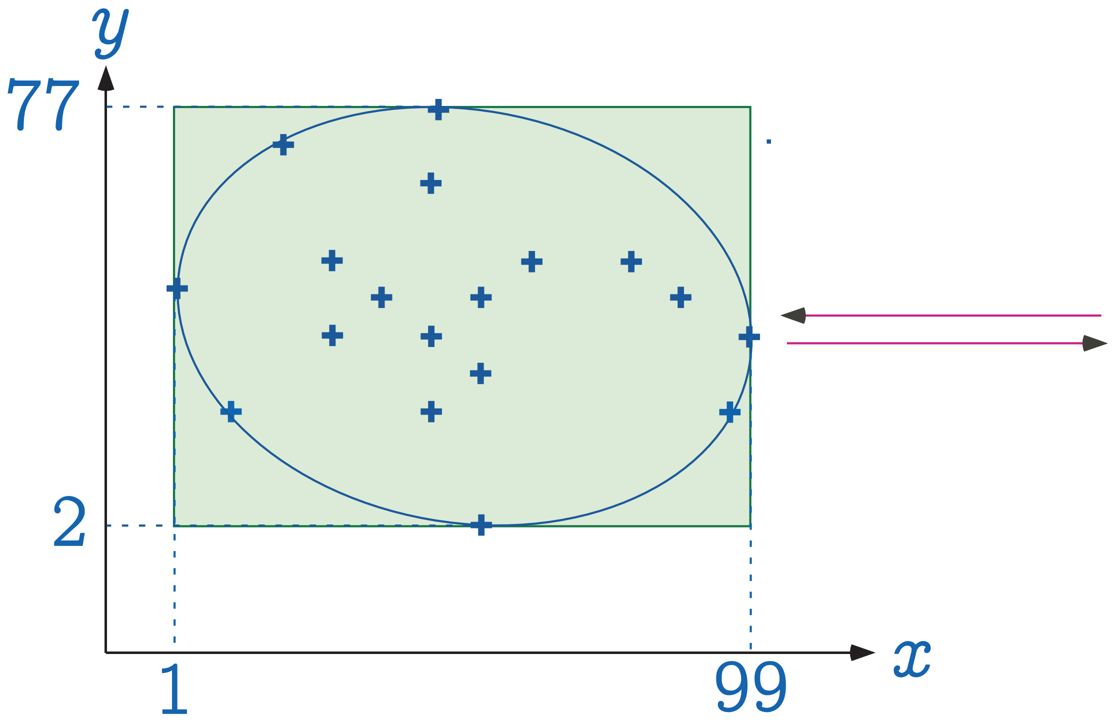
$\{x : [33, 89], y : [48, 61]\}$

$\sqsubseteq$

$\{x : [1, 99], y : [2, 90]\}$

$$X \sqsubseteq Y \Rightarrow \gamma(X) \subseteq \gamma(Y)$$

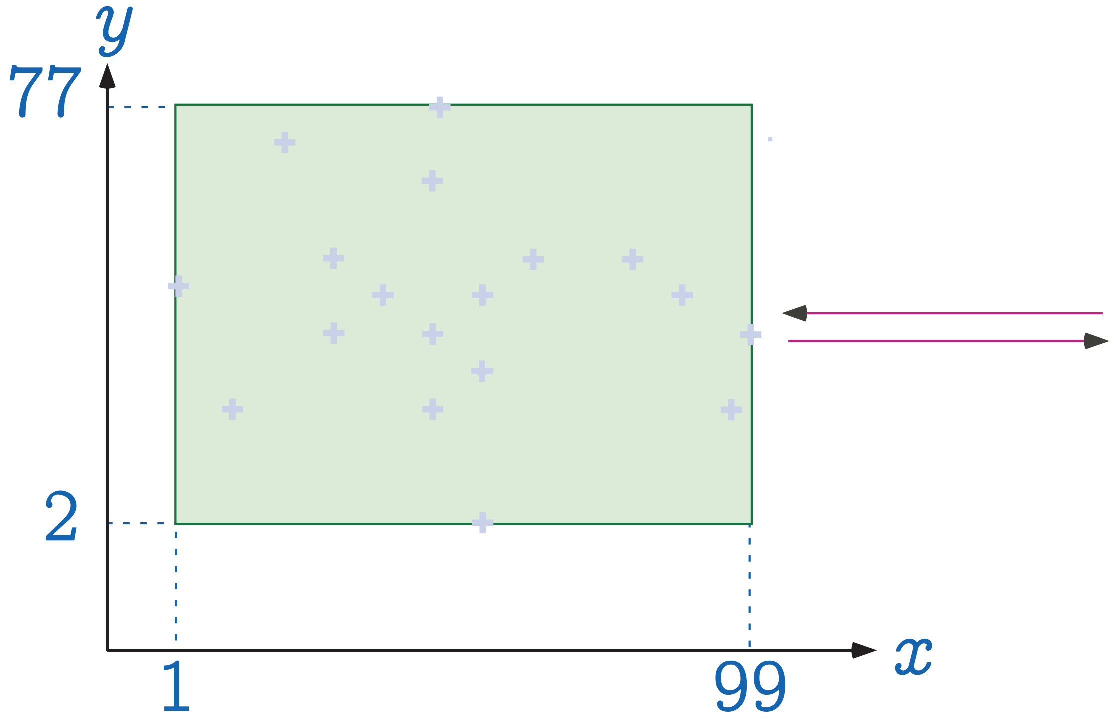
The $\gamma \circ \alpha$ composition is extensive



$\{x : [1, 99], y : [2, 77]\}$

$X \subseteq \gamma \circ \alpha(X)$

The $\alpha \circ \gamma$ composition is reductive



$$\{x : [1, 99], y : [2, 77]\}$$
$$=/\sqsubseteq$$
$$\{x : [1, 99], y : [2, 77]\}$$

$$\alpha \circ \gamma(Y) =/\sqsubseteq Y$$

Correspondance between concrete and abstract properties

- The pair $\langle \alpha, \gamma \rangle$ is a Galois connection:

$$\langle \wp(S), \subseteq \rangle \ \underset{\alpha}{\overset{\gamma}{\leftrightarrows}}\ \langle \mathcal{D}, \sqsubseteq \rangle$$

- $\langle \wp(S), \subseteq \rangle \ \mathrel{\substack{\xleftarrow{\ \gamma\ } \\ \xtwoheadrightarrow[\alpha]{}}}\ \langle \mathcal{D}, \sqsubseteq \rangle$ when α is onto (equivalently $\alpha \circ \gamma = 1$ or γ is one-to-one).

Galois connection

$$\langle \mathcal{D}, \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \overline{\mathcal{D}}, \sqsubseteq \rangle$$

iff $\forall x, y \in \mathcal{D} : x \subseteq y \Longrightarrow \alpha(x) \sqsubseteq \alpha(y)$

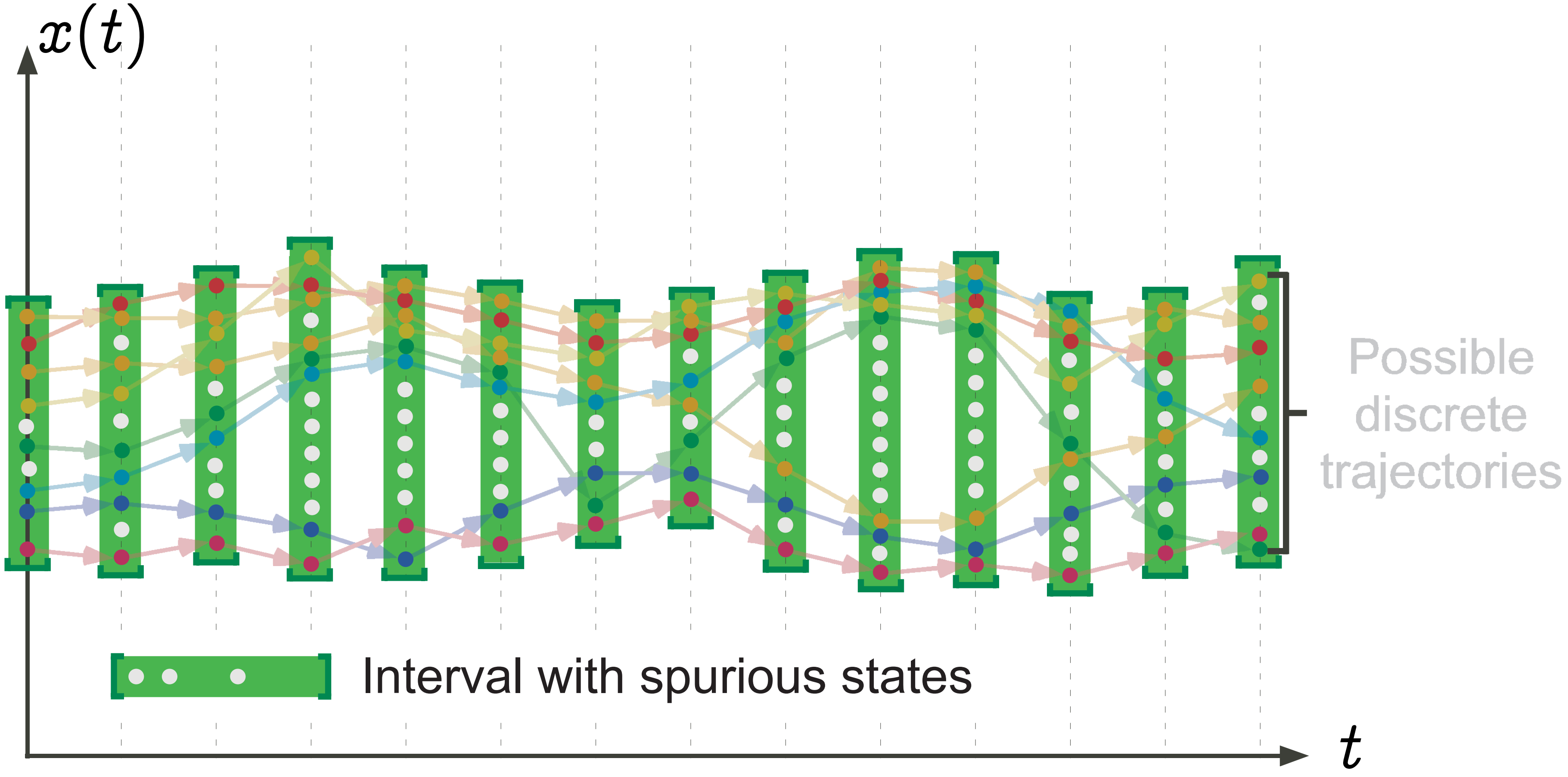
$\wedge\ \forall \overline{x}, \overline{y} \in \overline{\mathcal{D}} : \overline{x} \sqsubseteq \overline{y} \Longrightarrow \gamma(\overline{x}) \subseteq \gamma(\overline{y})$

$\wedge\ \forall x \in \mathcal{D} : x \subseteq \gamma(\alpha(x))$

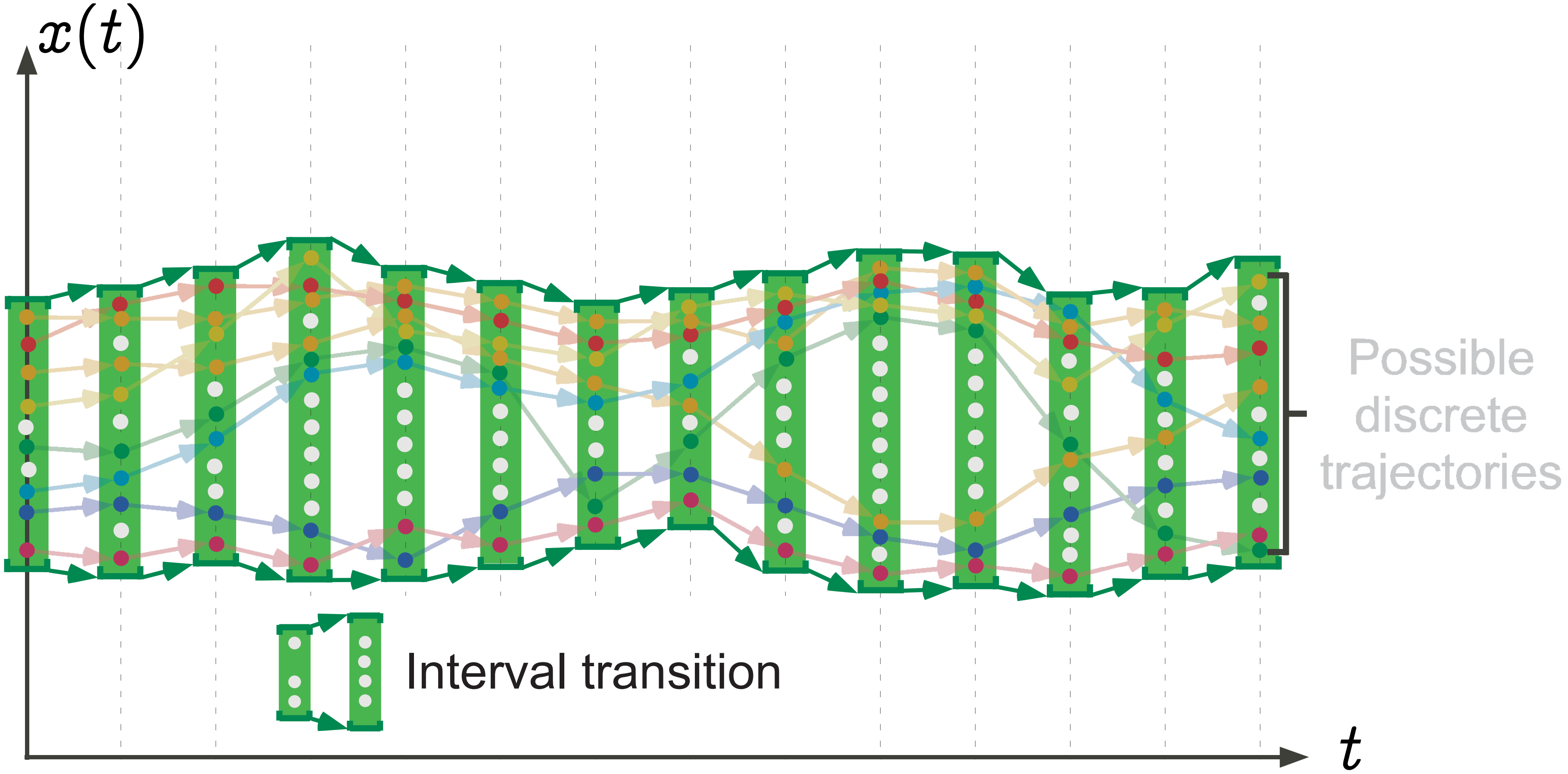
$\wedge\ \forall \overline{y} \in \overline{\mathcal{D}} : \alpha(\gamma(\overline{y})) \sqsubseteq \overline{x}$

iff $\forall x \in \mathcal{D}, \overline{y} \in \overline{\mathcal{D}} : \alpha(x) \sqsubseteq y \Longleftrightarrow x \subseteq \gamma(y)$

Graphic example: Interval abstraction



Graphic example: Abstract transitions



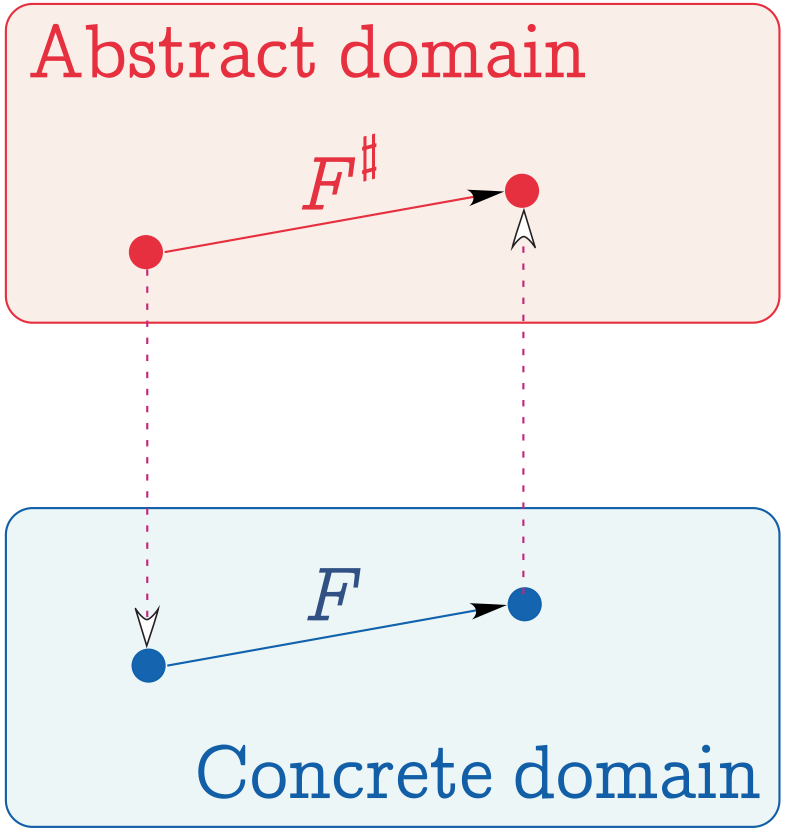
Example: Interval transition semantics of assignments

```
int x;
...
l:
    x := x + 1;
l':
```

$$\{\texttt{l} : \texttt{x} \in [\ell, h] \to \texttt{l}' : \texttt{x} \in [l + 1, \min(h + 1, \texttt{max\_int})] \cup \{\Omega \mid h = \texttt{max\_int}\} \mid \ell \leq h\}$$

where $[\ell, h] = \emptyset$ when $h < \ell$.

Function abstraction



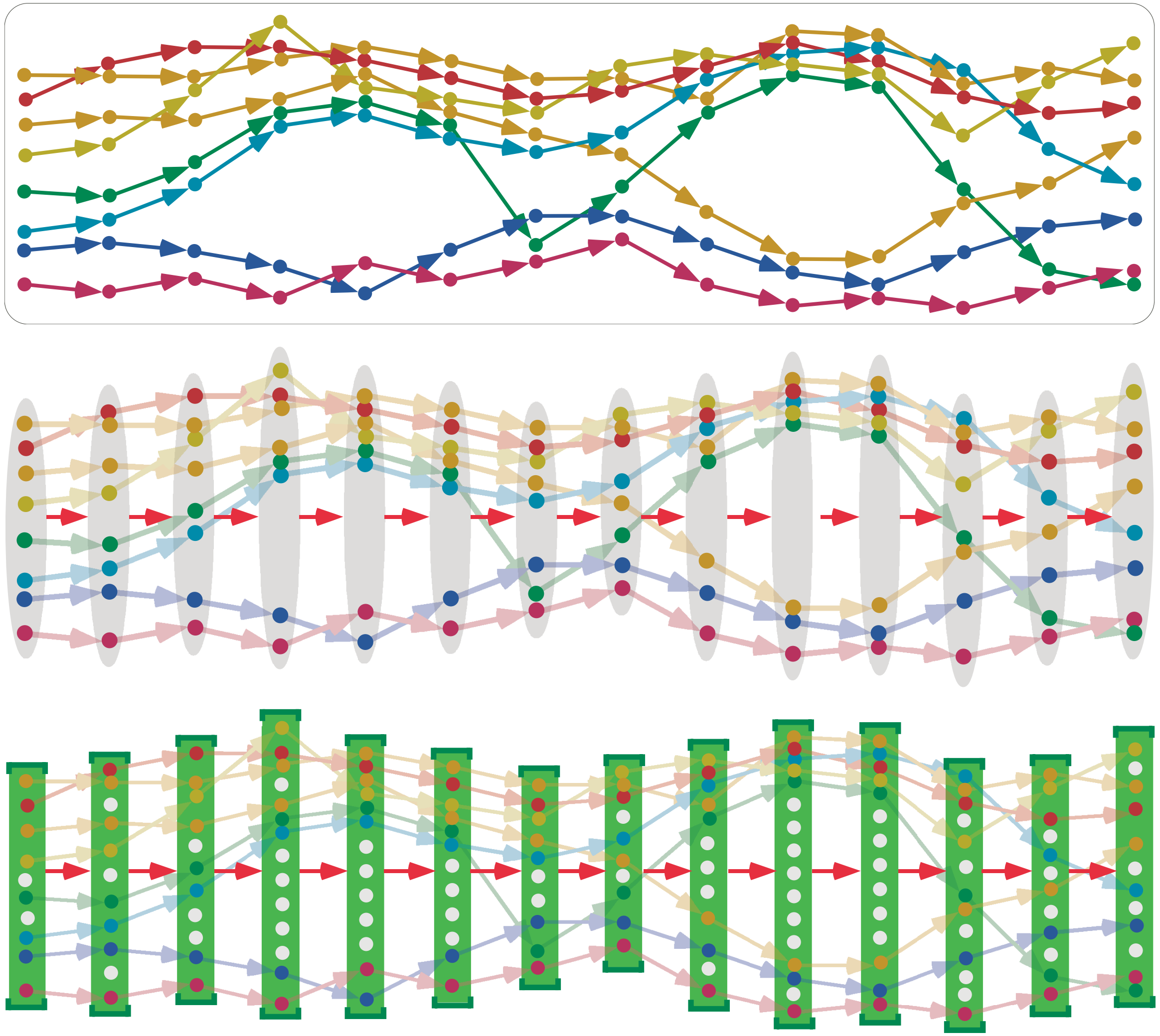
$$F^\sharp = \alpha \circ F \circ \gamma$$

i.e. $F^\sharp = \rho \circ F$

$$\langle P,\ \subseteq\rangle \xleftrightarrow[\alpha]{\gamma} \langle Q,\ \sqsubseteq\rangle \Rightarrow$$

$$\langle P \xrightarrow{\text{mon}} P,\ \dot{\subseteq}\rangle \xleftrightarrow[\lambda F\,.\,\alpha\circ F\circ\gamma]{\lambda F^\sharp\,.\,\gamma\circ F^\sharp\circ\alpha} \langle Q \xrightarrow{\text{mon}} Q,\ \dot{\sqsubseteq}\rangle$$

Example: Set of traces to trace of intervals abstraction



Set of traces:

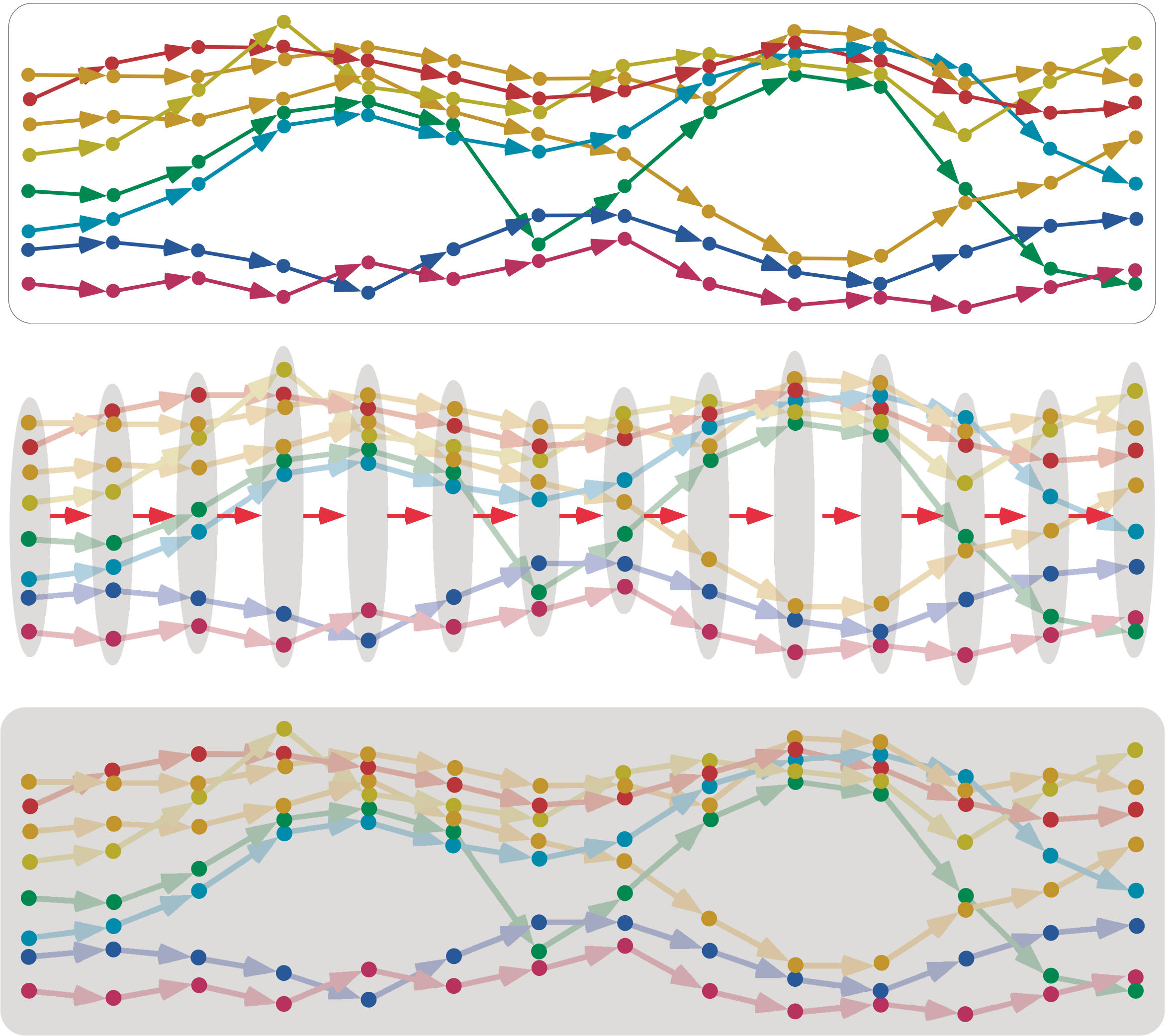
$\alpha_1 \downarrow$

Trace of sets:

$\alpha_2 \downarrow$

Trace of intervals

Example: Set of traces to reachable states abstraction



Set of traces:

$\alpha_1 \downarrow$

Trace of sets:

$\alpha_3 \downarrow$

Reachable states

Composition of Galois Connections

The composition of Galois connections:

$$\langle L,\ \leq\rangle \xleftrightarrow[\alpha_1]{\gamma_1} \langle M,\ \sqsubseteq\rangle$$

and:

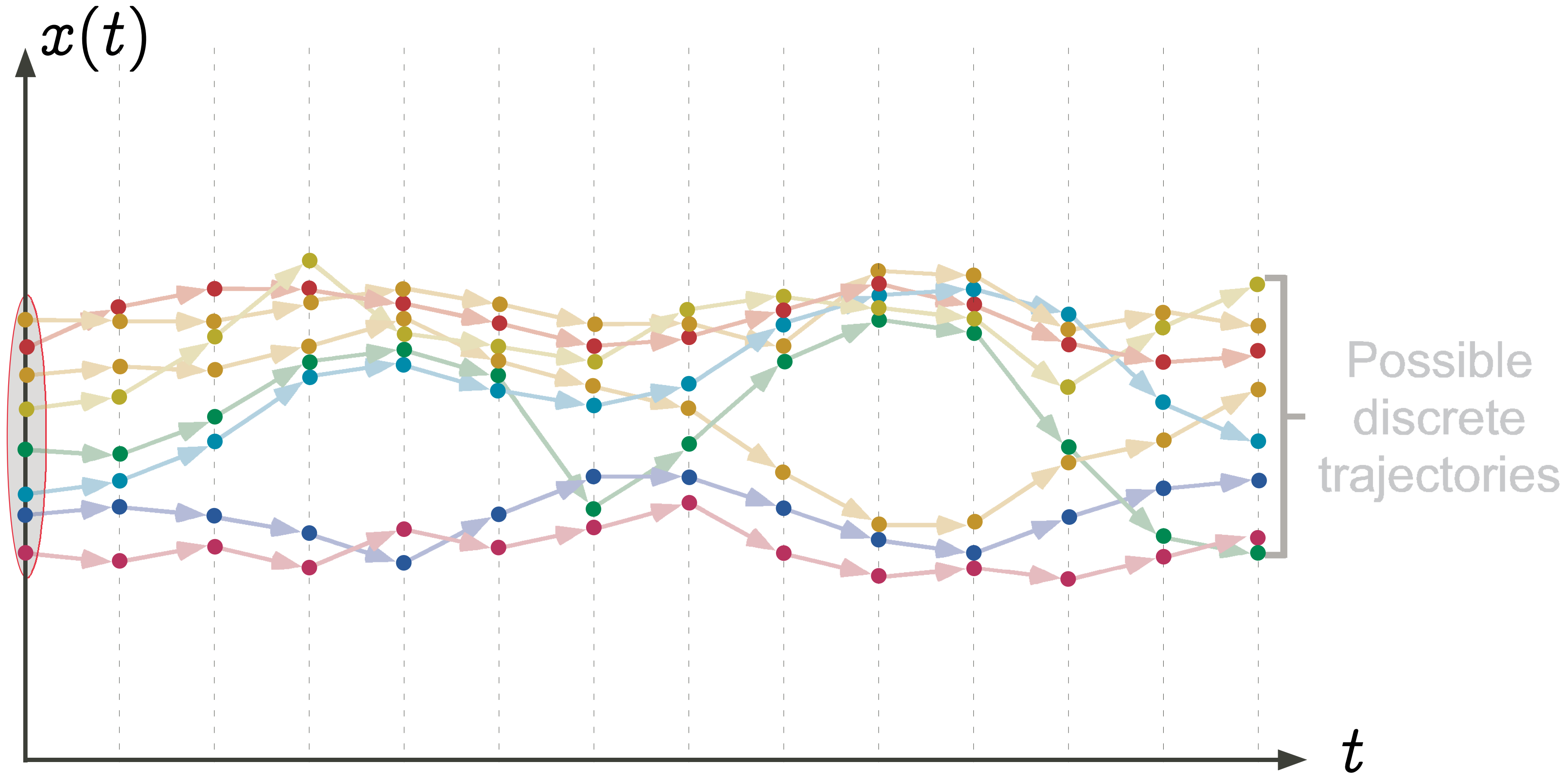
$$\langle M,\ \sqsubseteq\rangle \xleftrightarrow[\alpha_2]{\gamma_2} \langle N,\ \preceq\rangle$$

is a Galois connection:

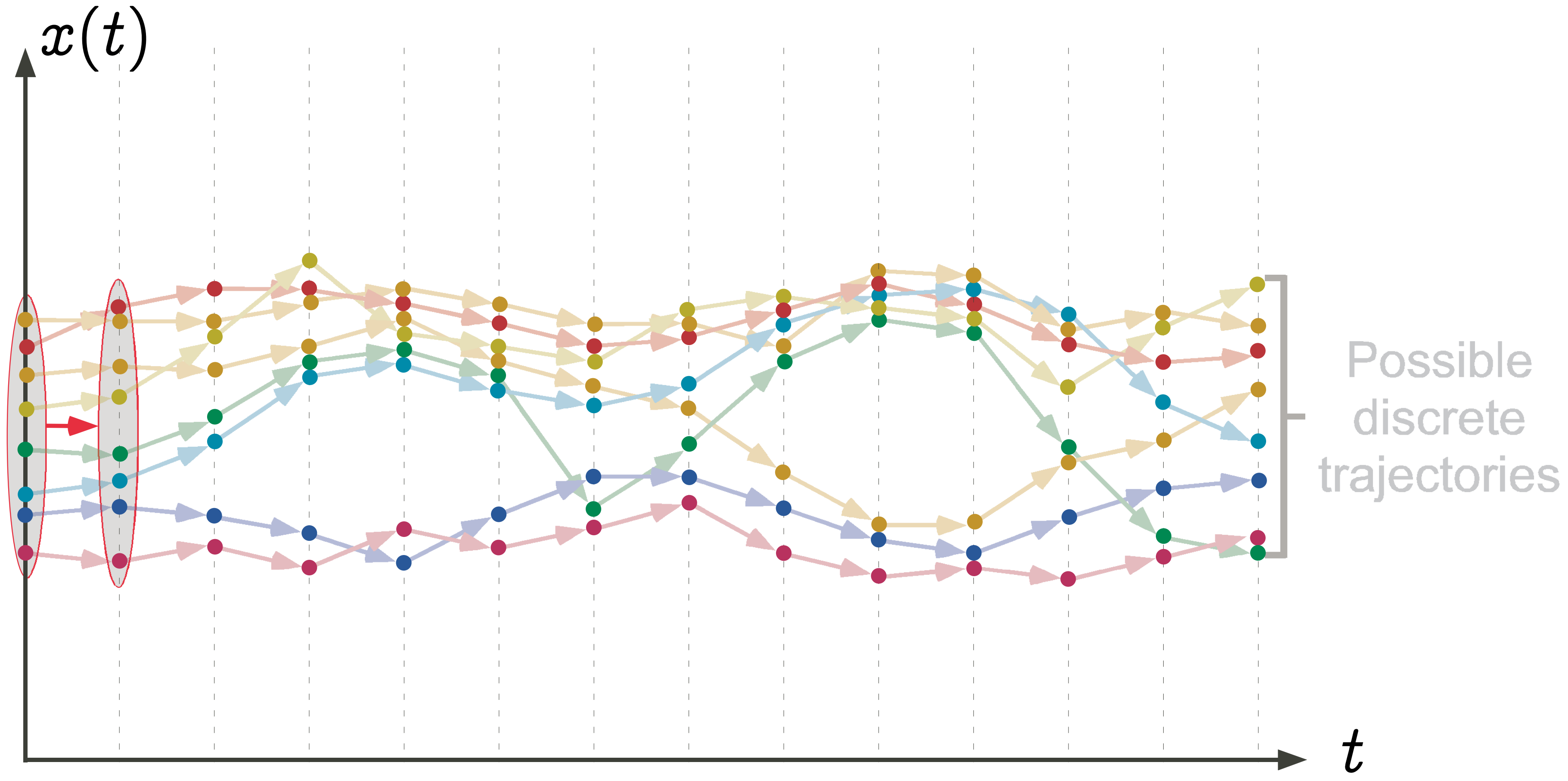
$$\langle L,\ \leq\rangle \xleftrightarrow[\alpha_2\circ\alpha_1]{\gamma_1\circ\gamma_2} \langle N,\ \preceq\rangle$$

Abstract semantics in fixpoint form

Graphic example: traces of sets of states in fixpoint form



Graphic example: traces of sets of states in fixpoint form



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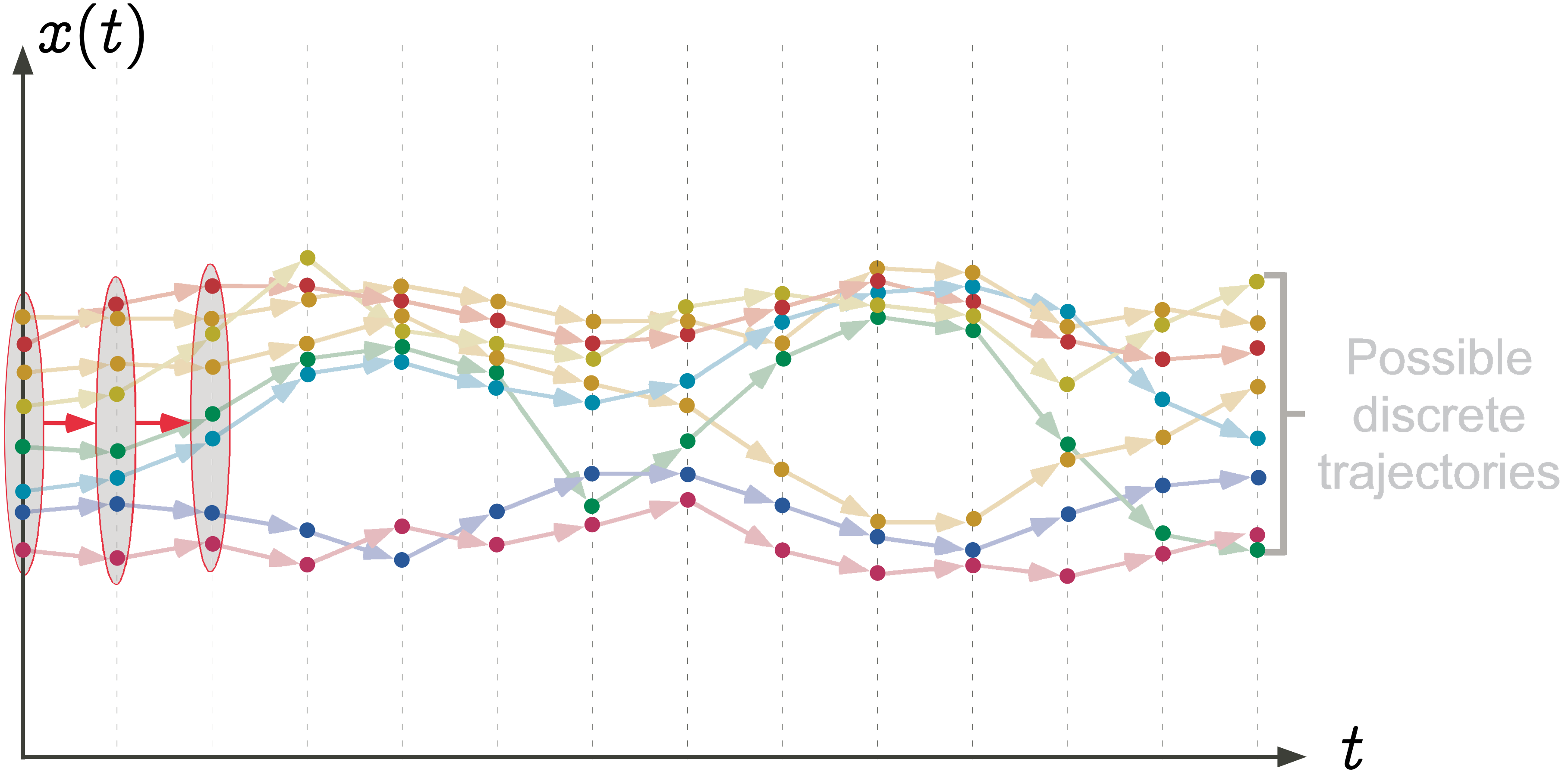
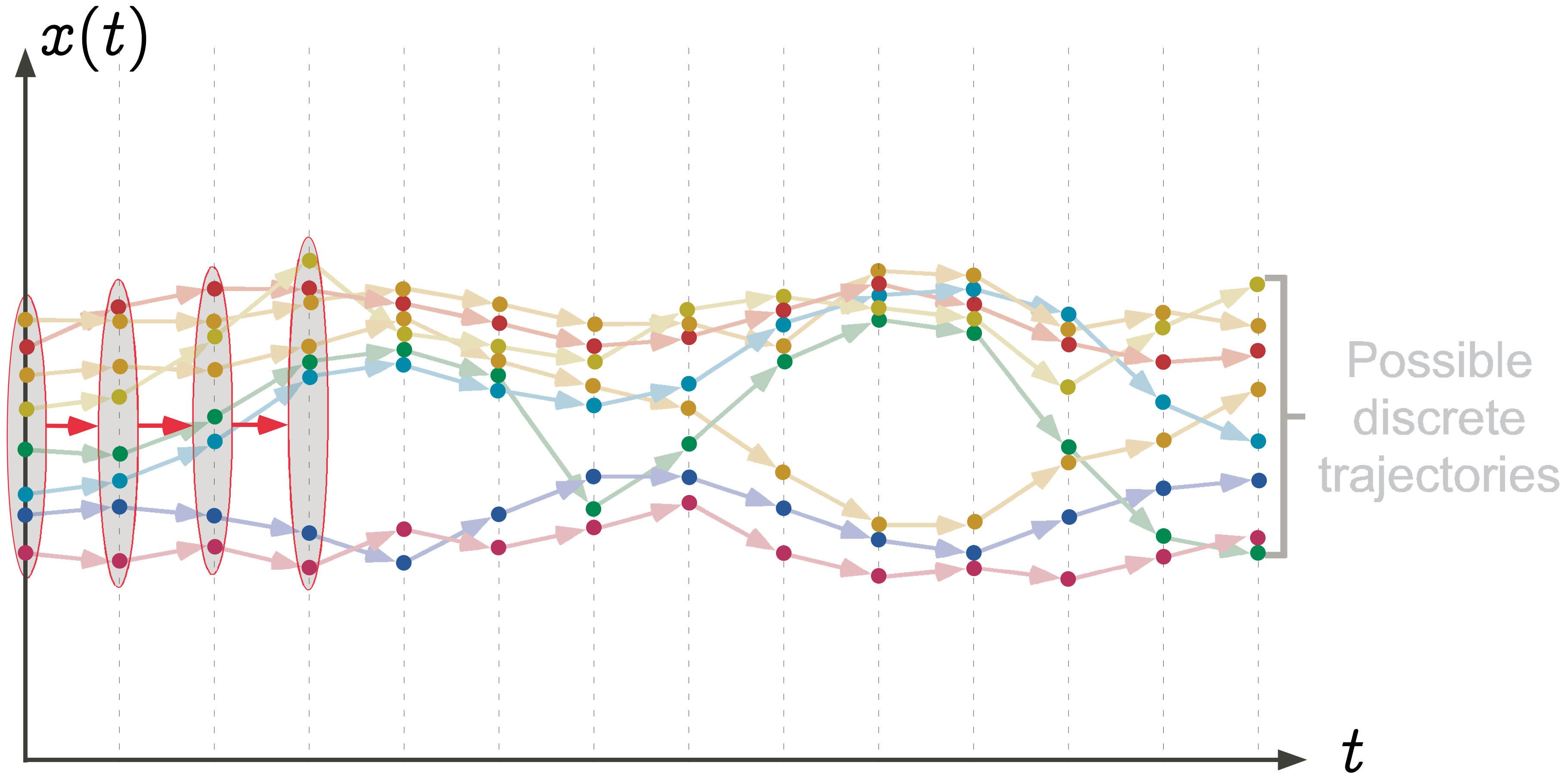
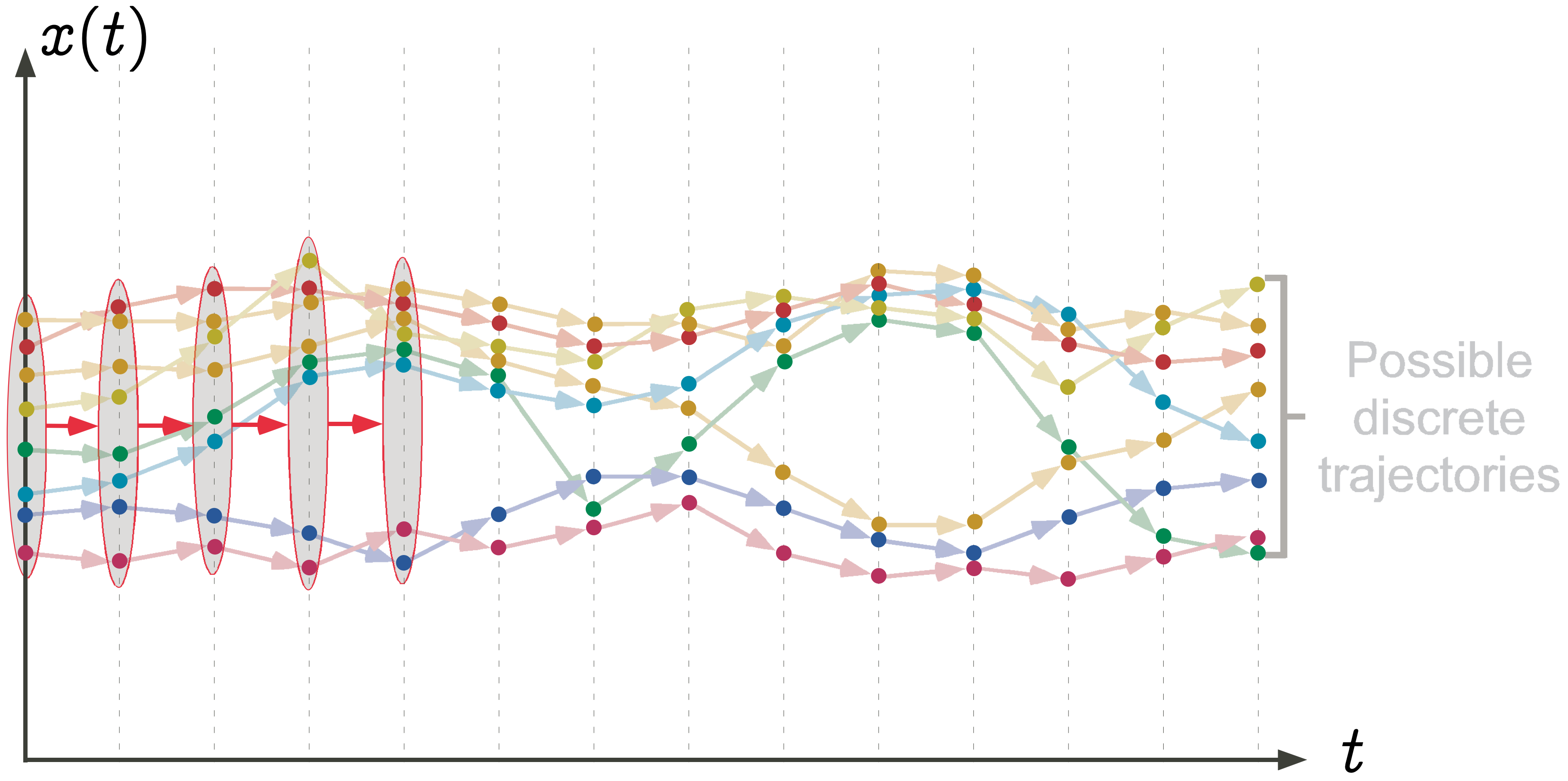


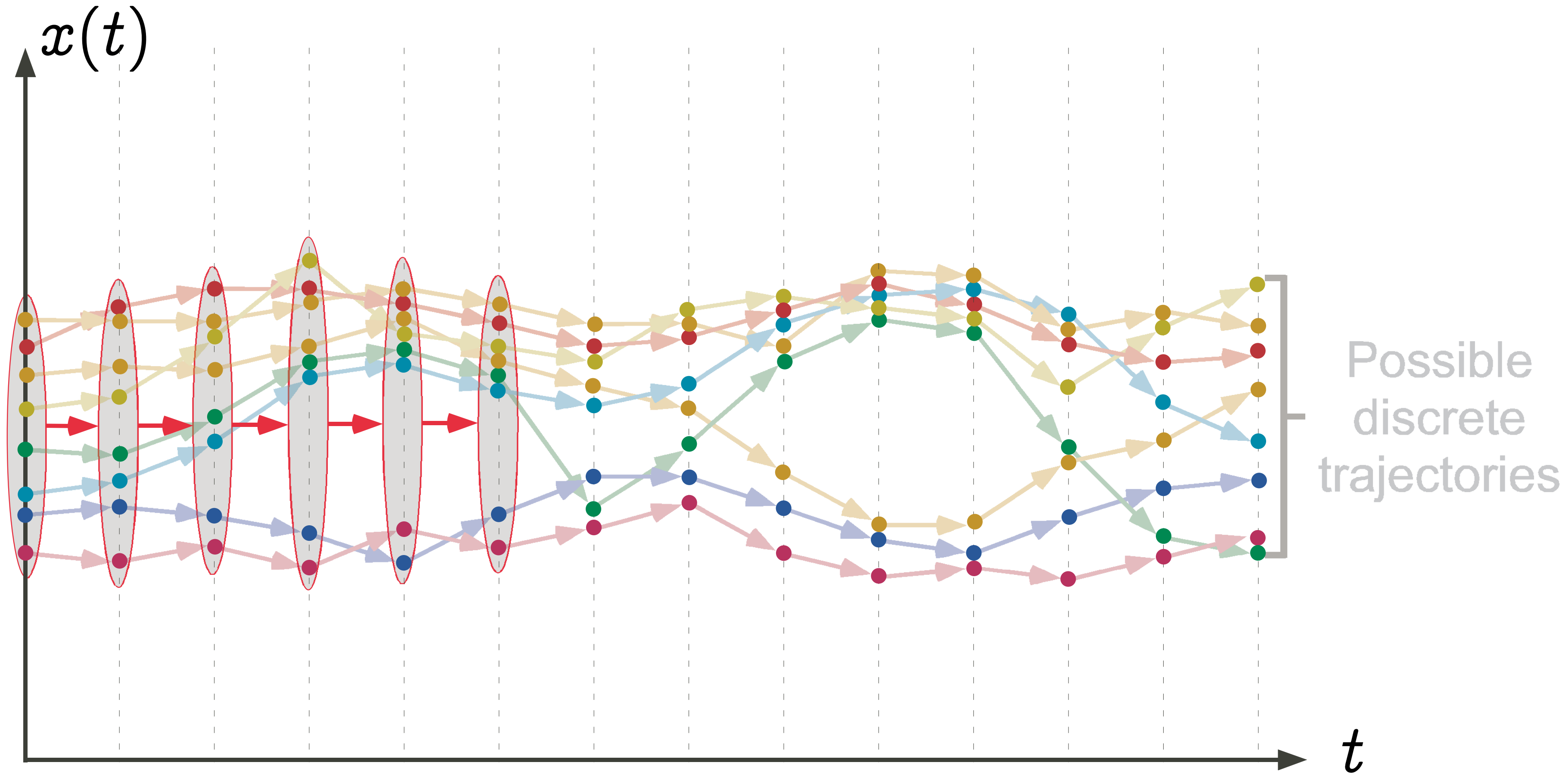
Graphic example: traces of sets of states in fixpoint form



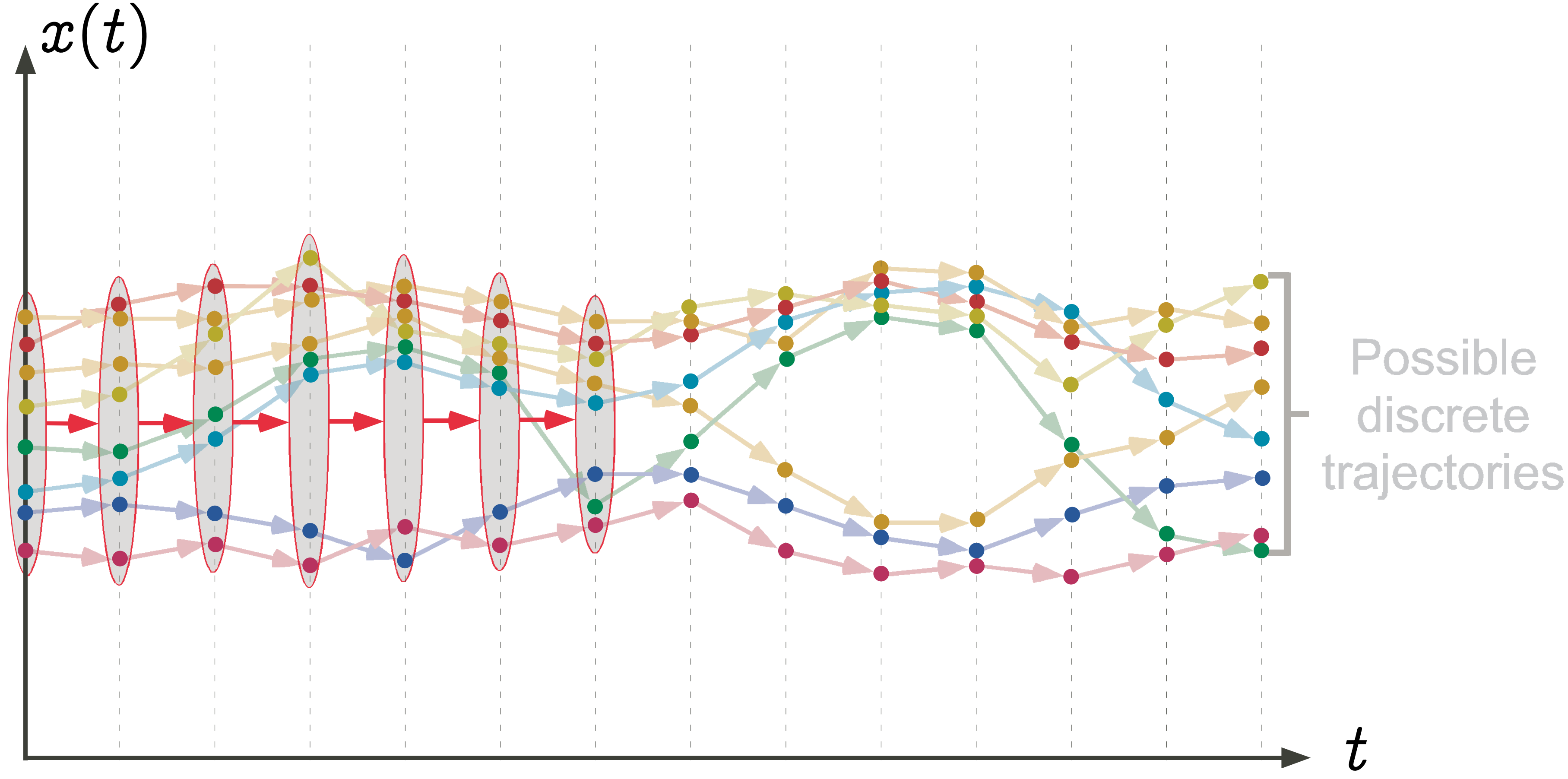
Graphic example: traces of sets of states in fixpoint form



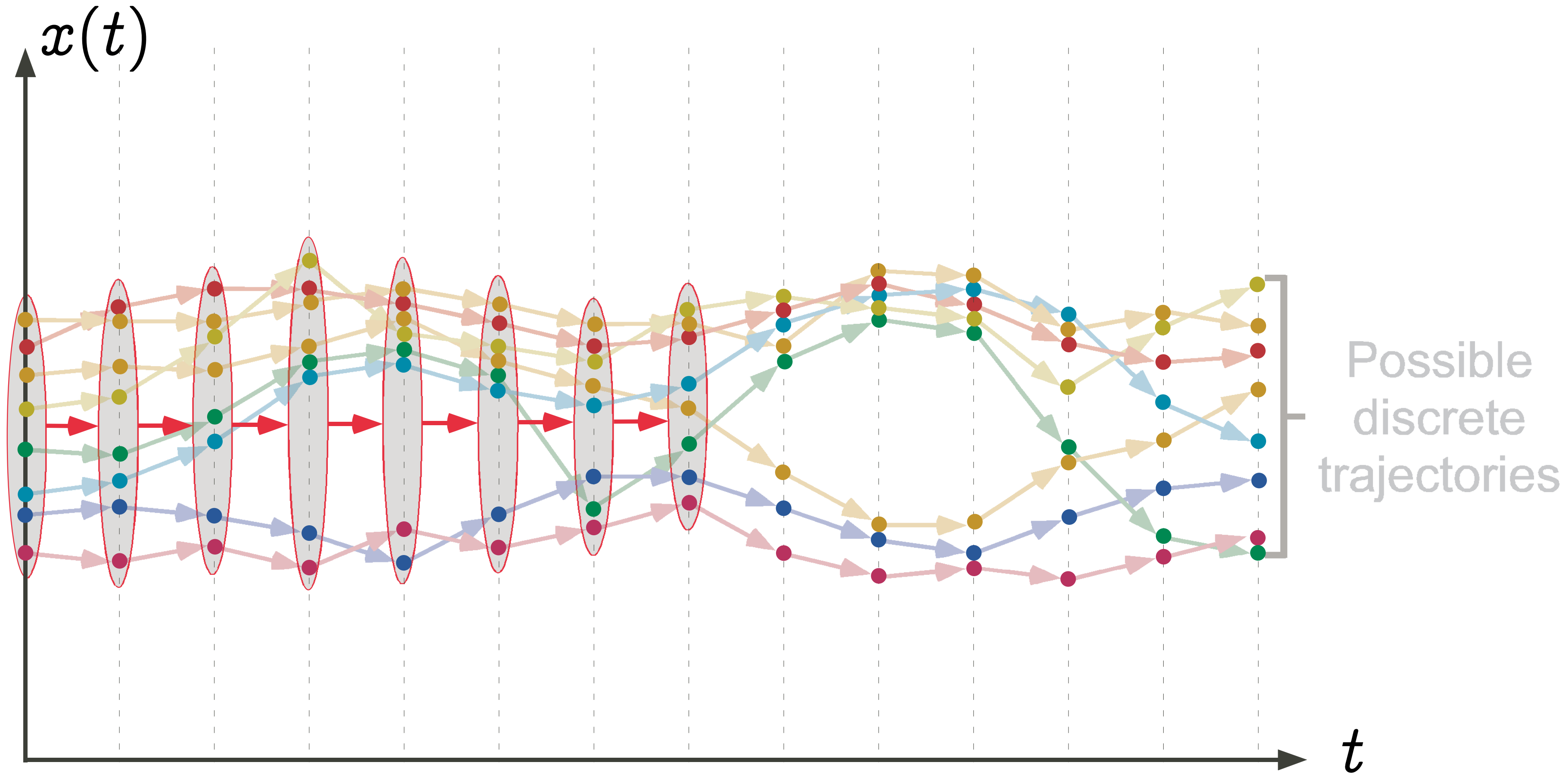
Graphic example: traces of sets of states in fixpoint form



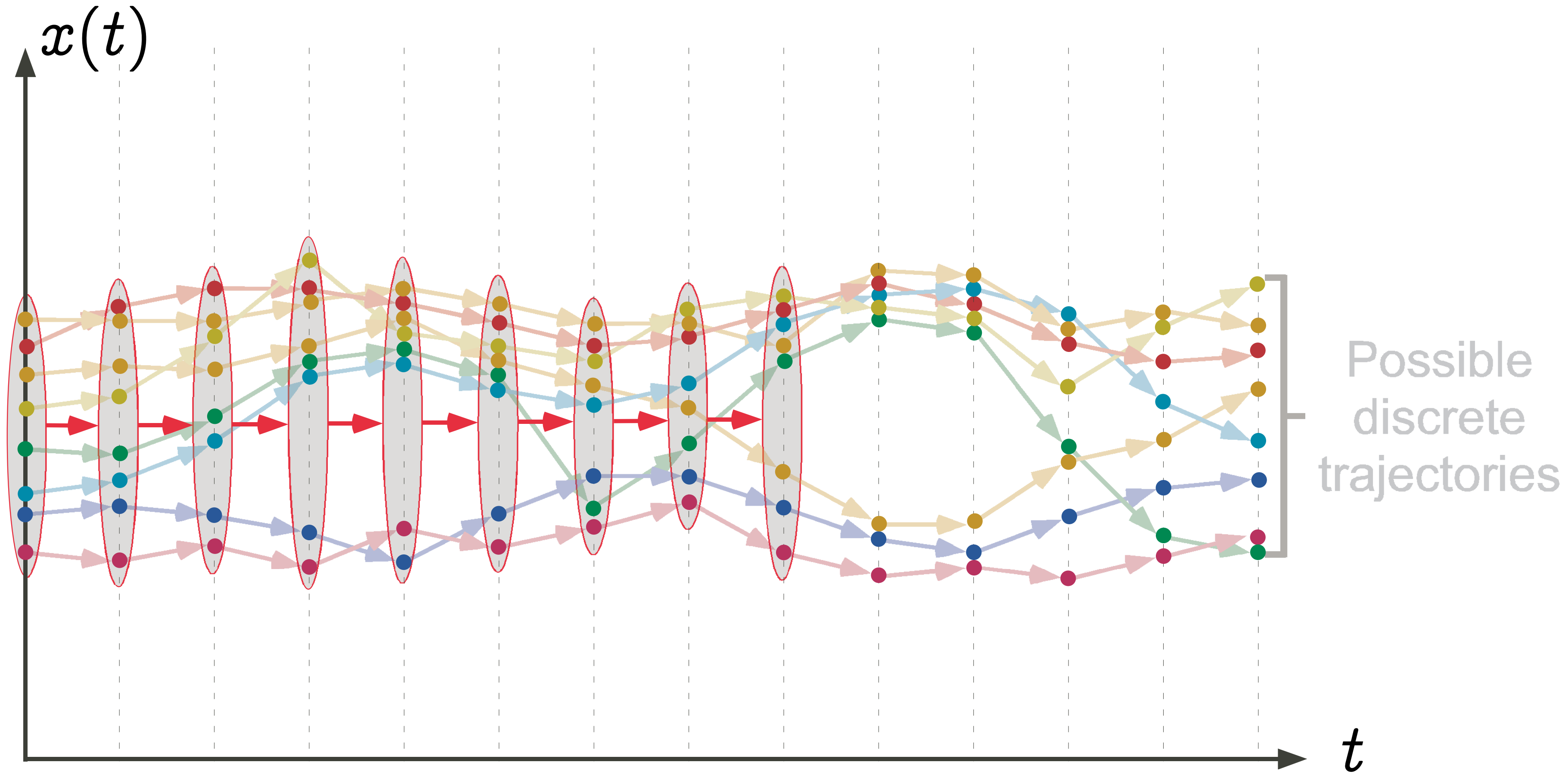
Graphic example: traces of sets of states in fixpoint form



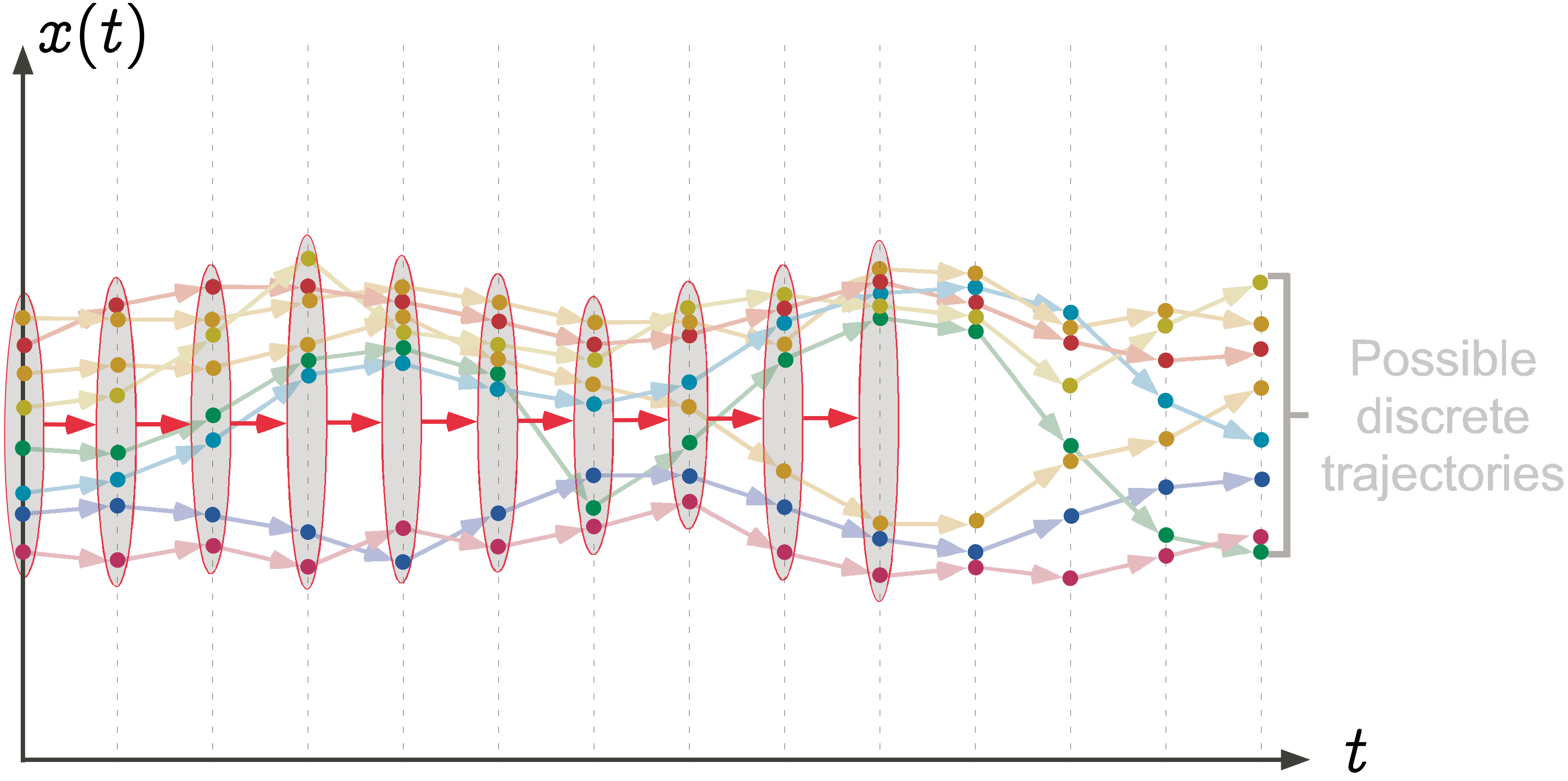
Graphic example: traces of sets of states in fixpoint form



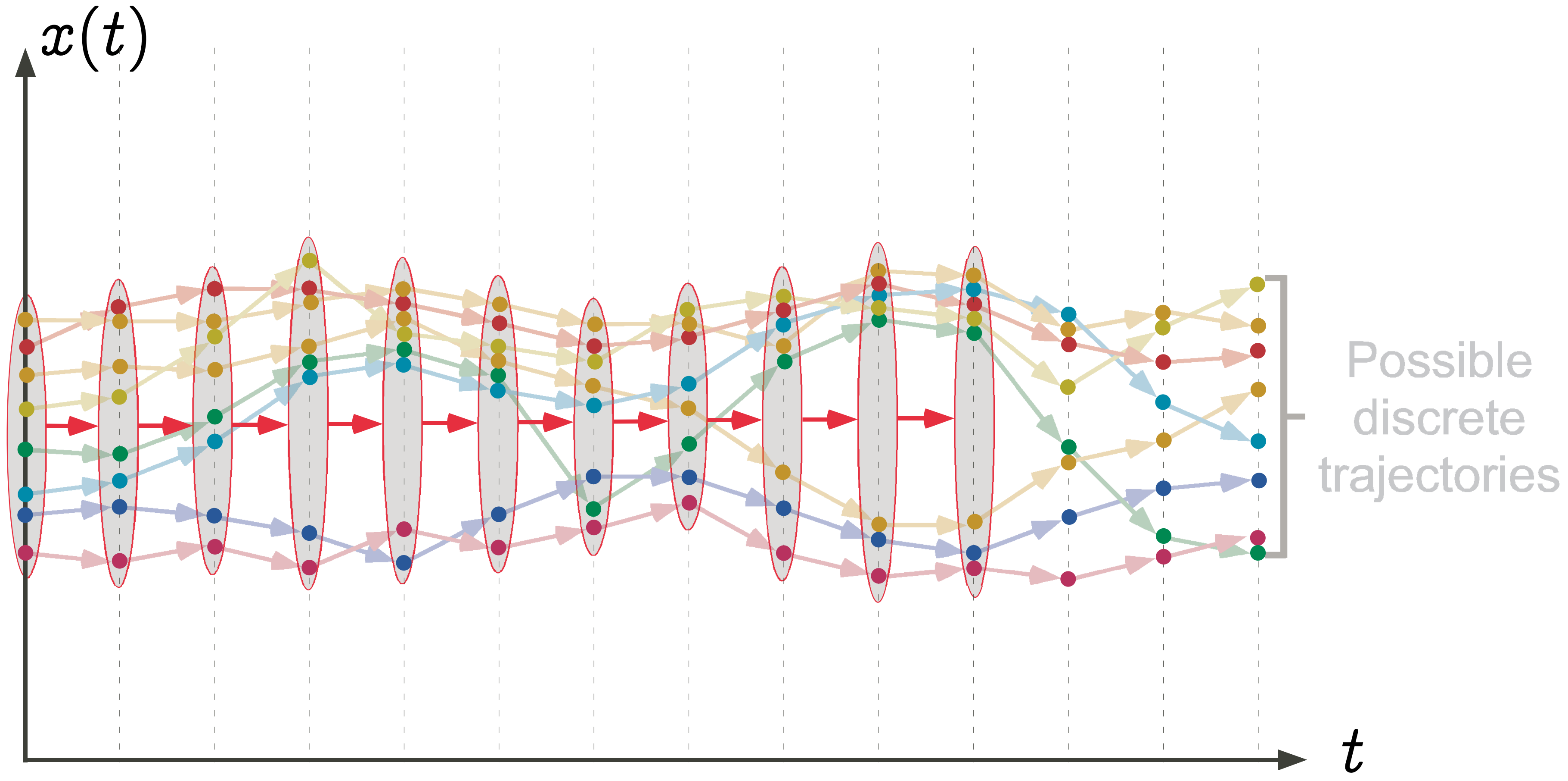
Graphic example: traces of sets of states in fixpoint form



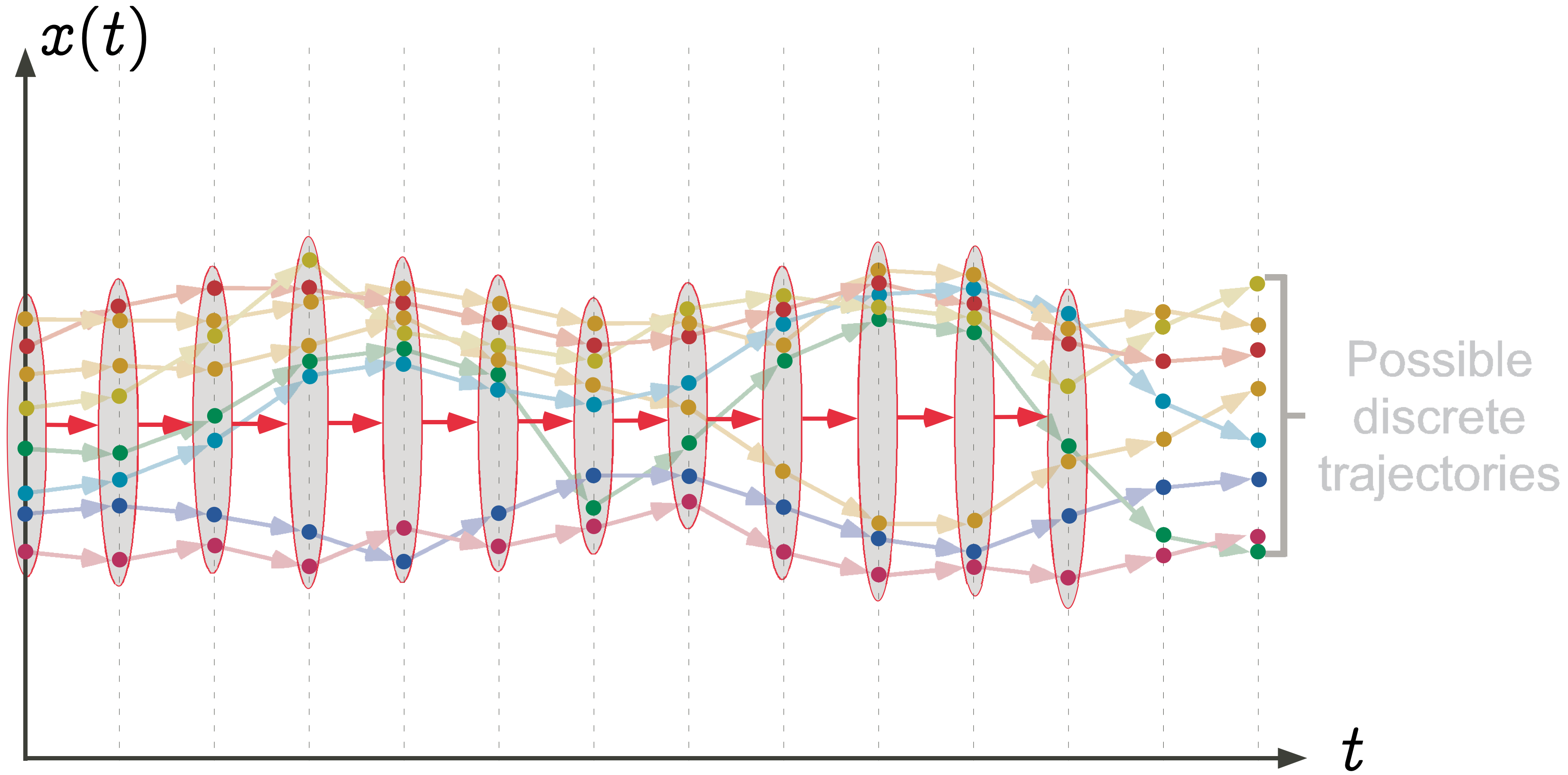
Graphic example: traces of sets of states in fixpoint form



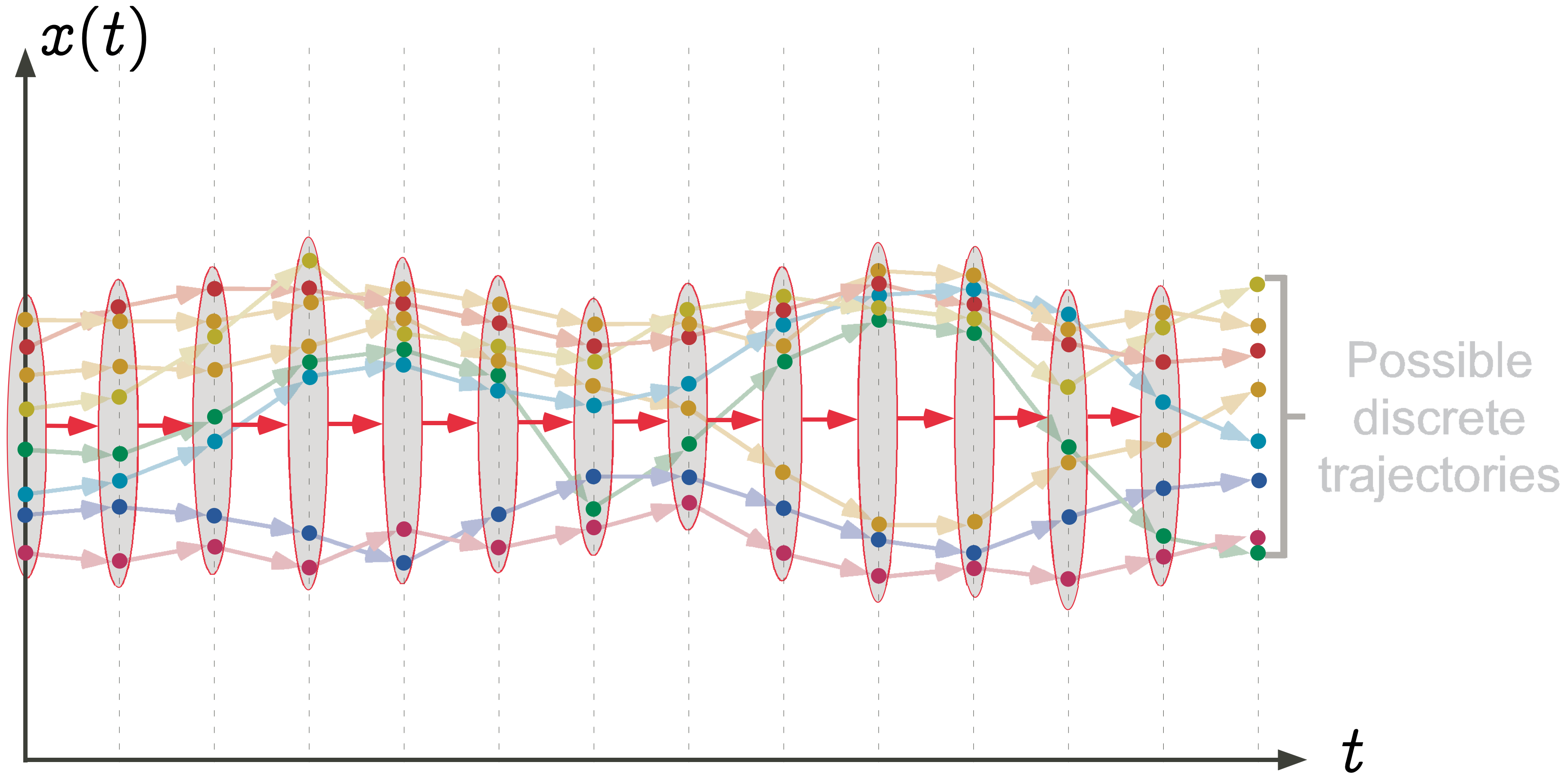
Graphic example: traces of sets of states in fixpoint form



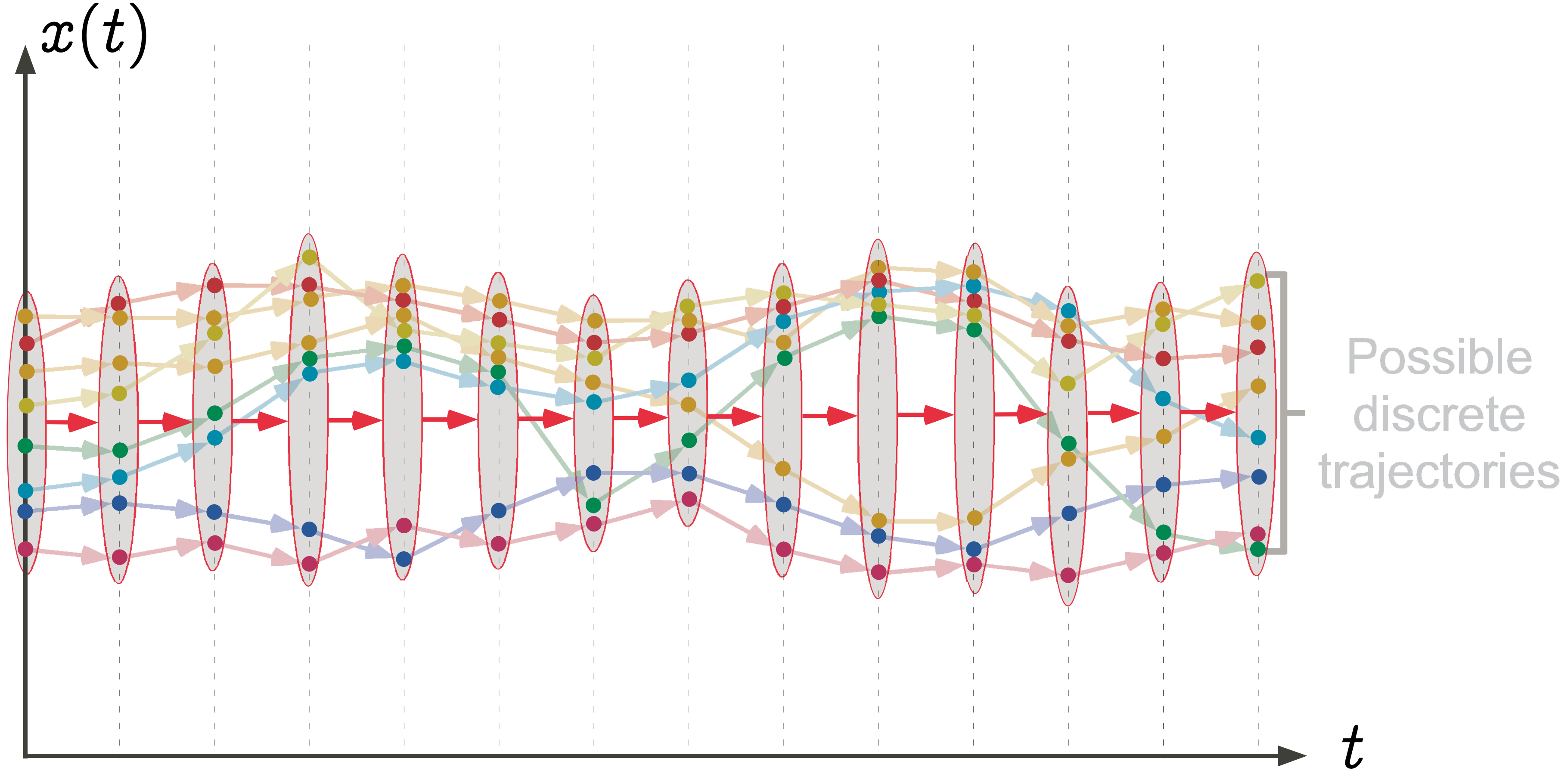
Graphic example: traces of sets of states in fixpoint form



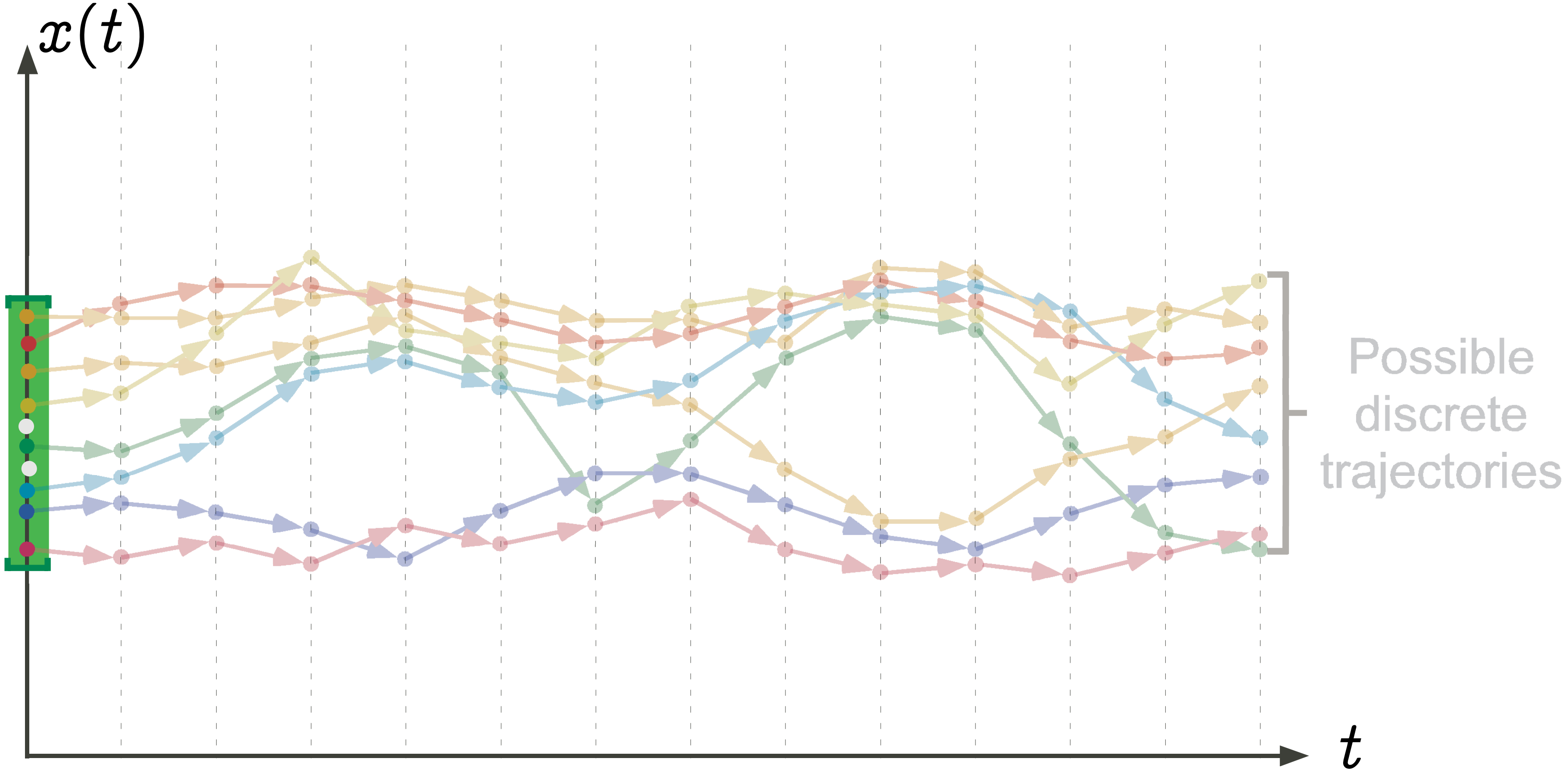
Graphic example: traces of sets of states in fixpoint form



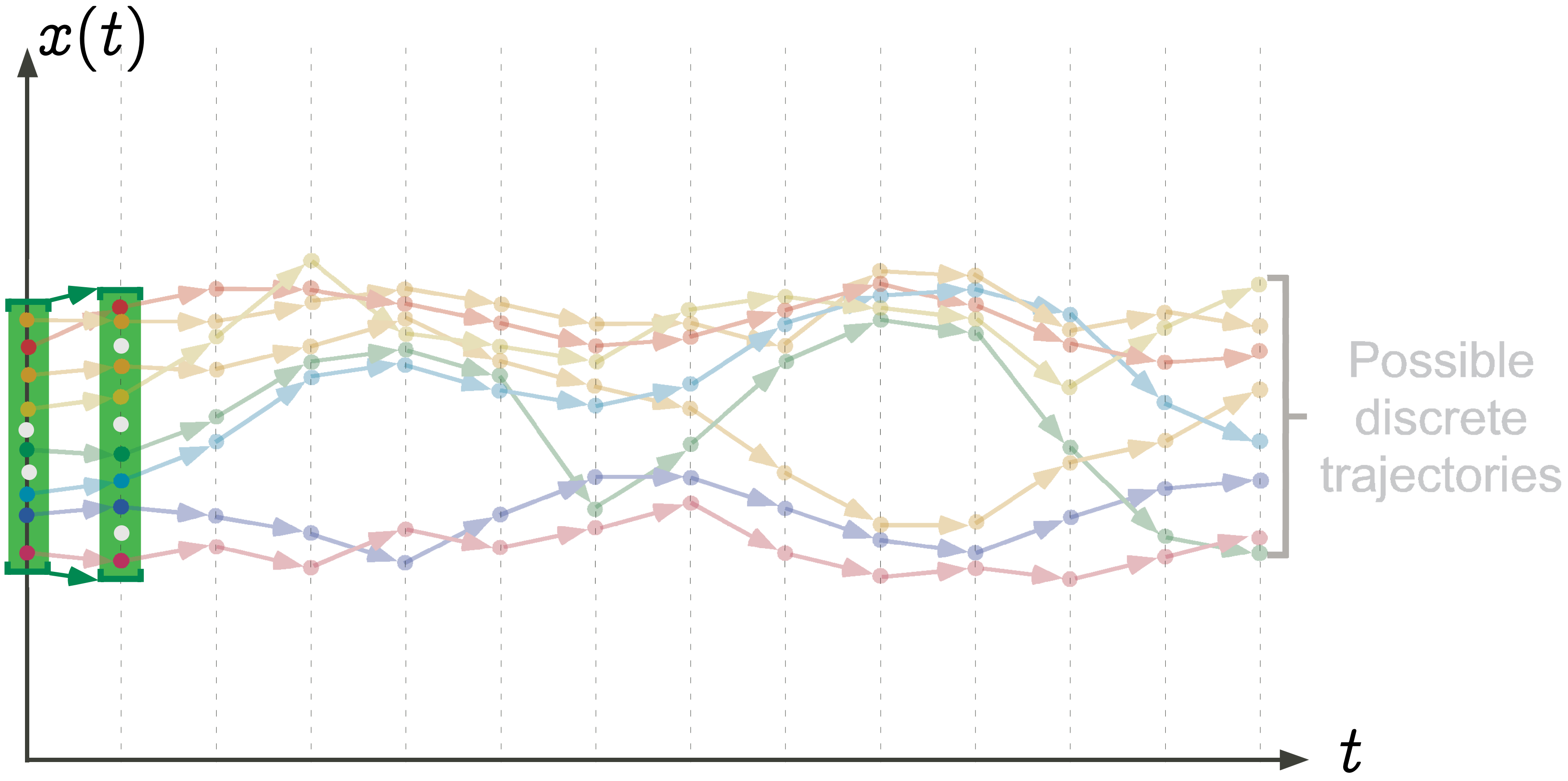
Graphic example: traces of sets of states in fixpoint form



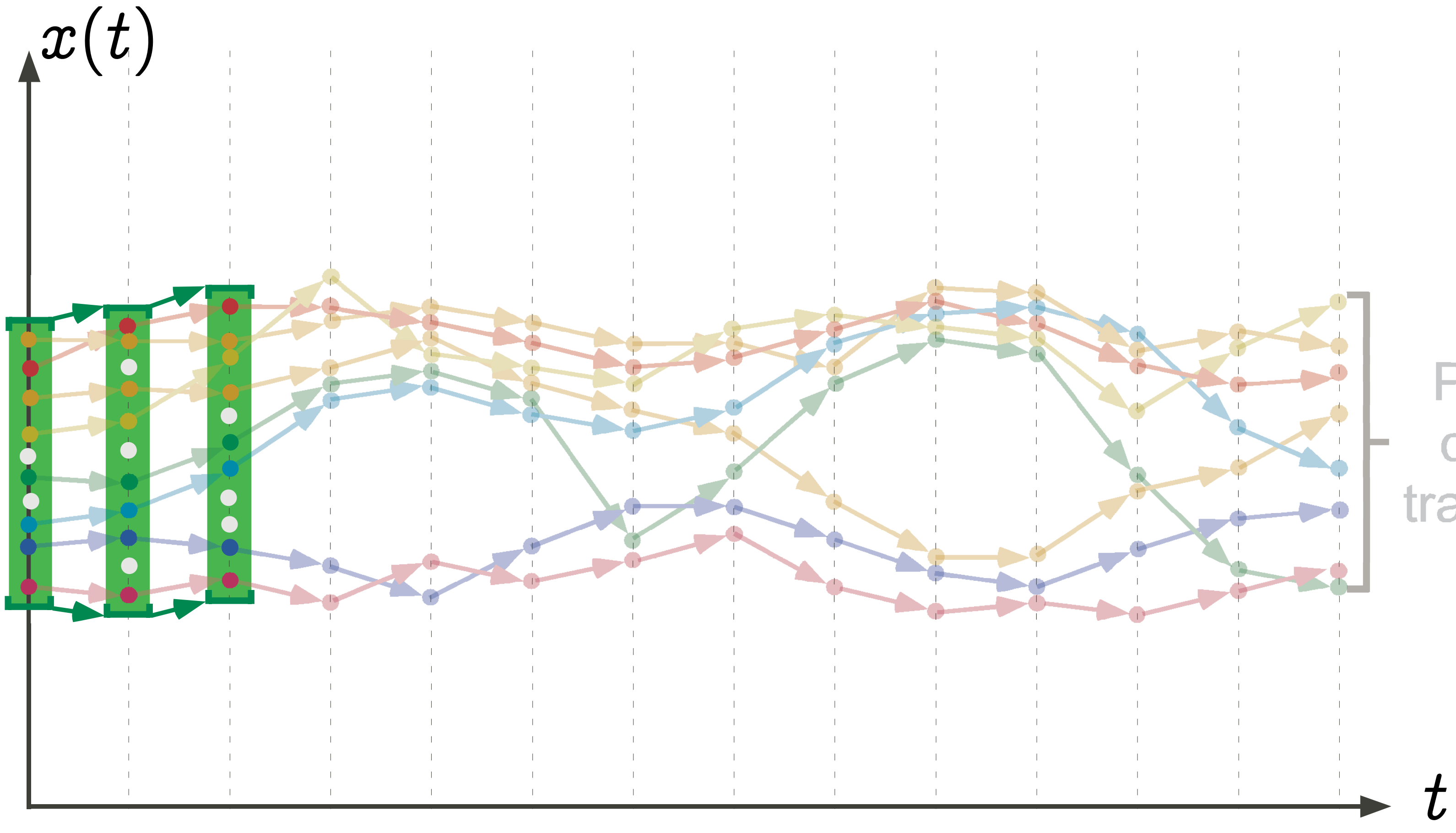
Graphic example: traces of intervals in fixpoint form



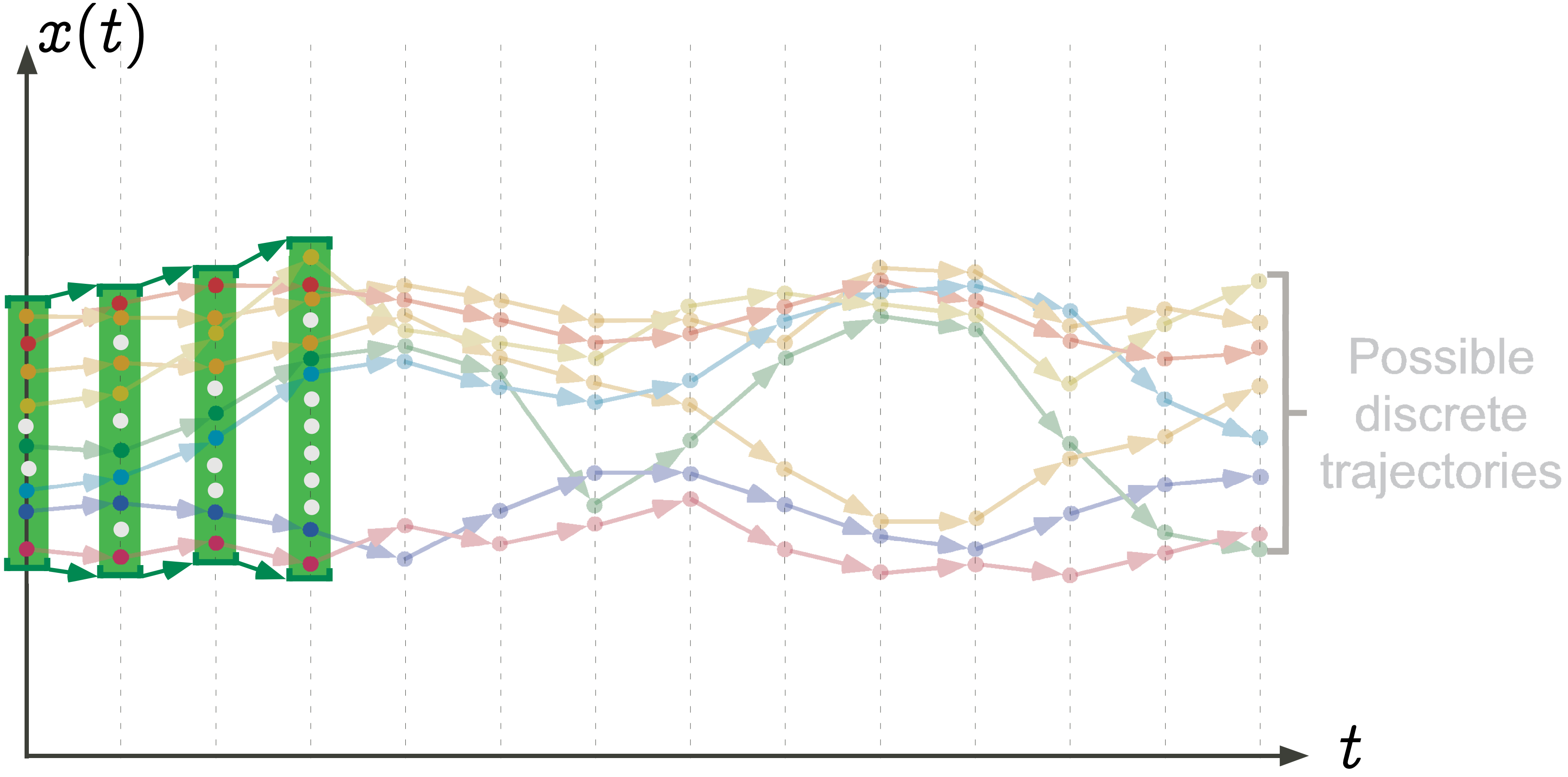
Graphic example: traces of intervals in fixpoint form



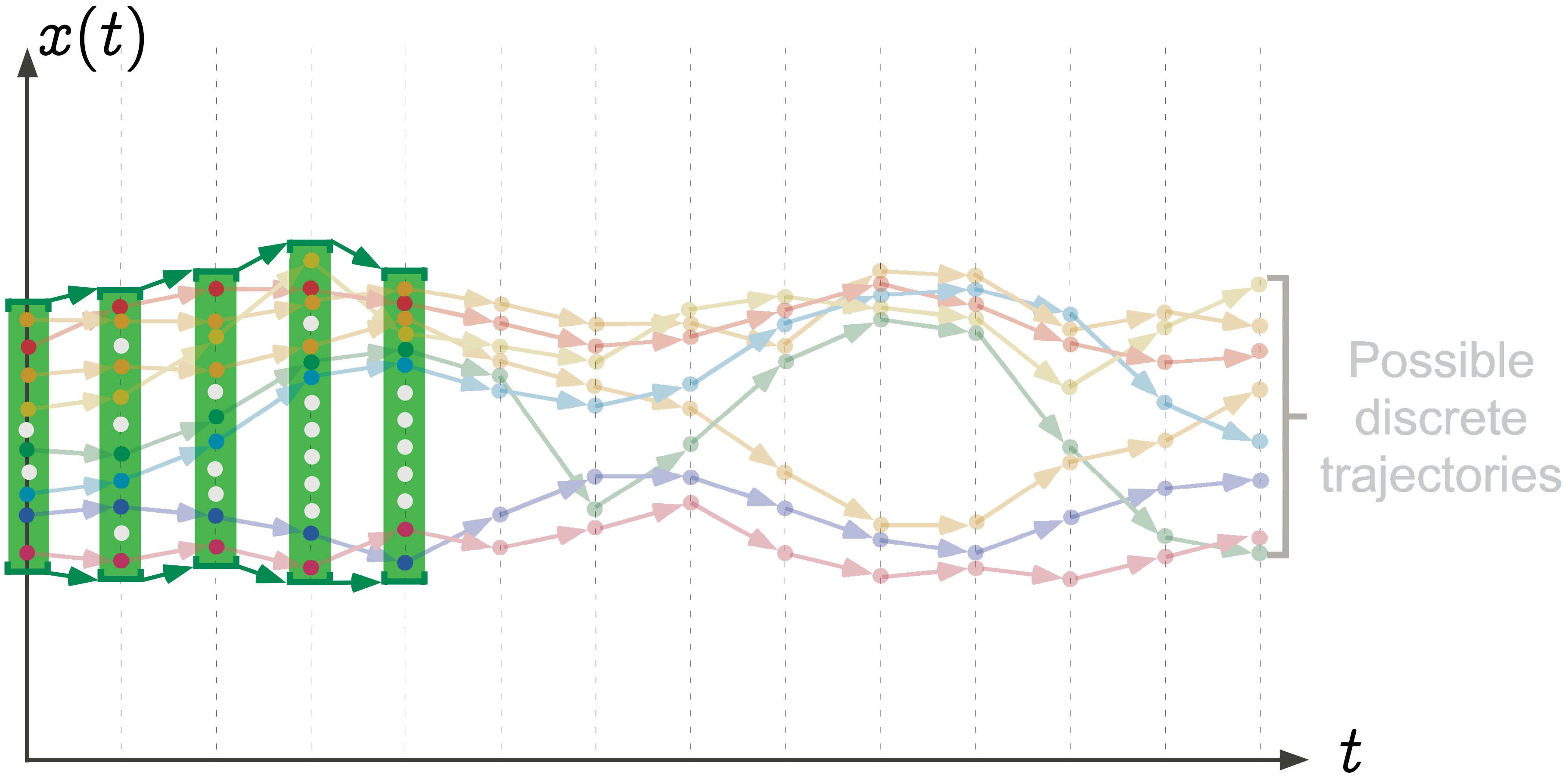
Graphic example: traces of intervals in fixpoint form



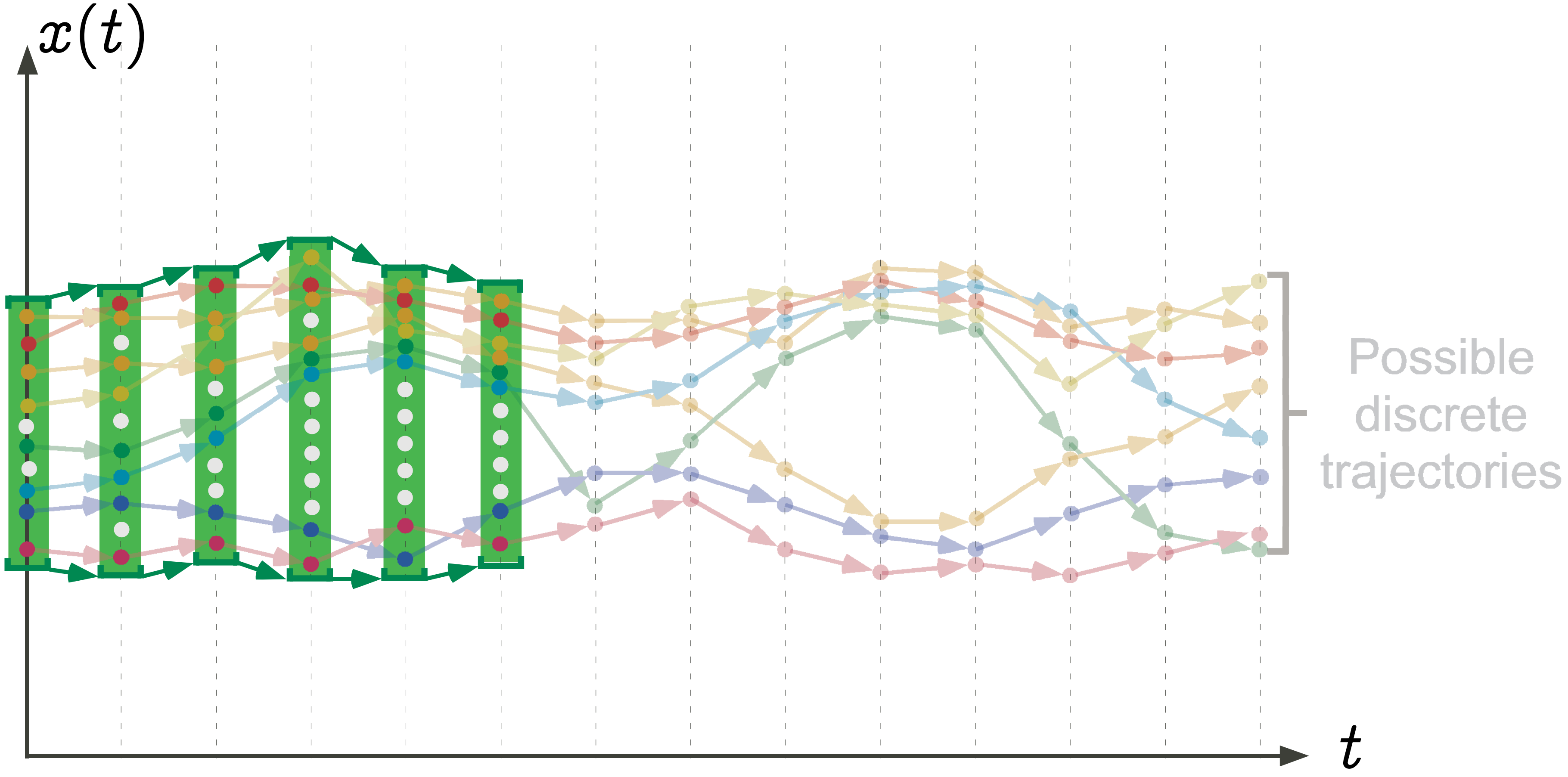
Graphic example: traces of intervals in fixpoint form



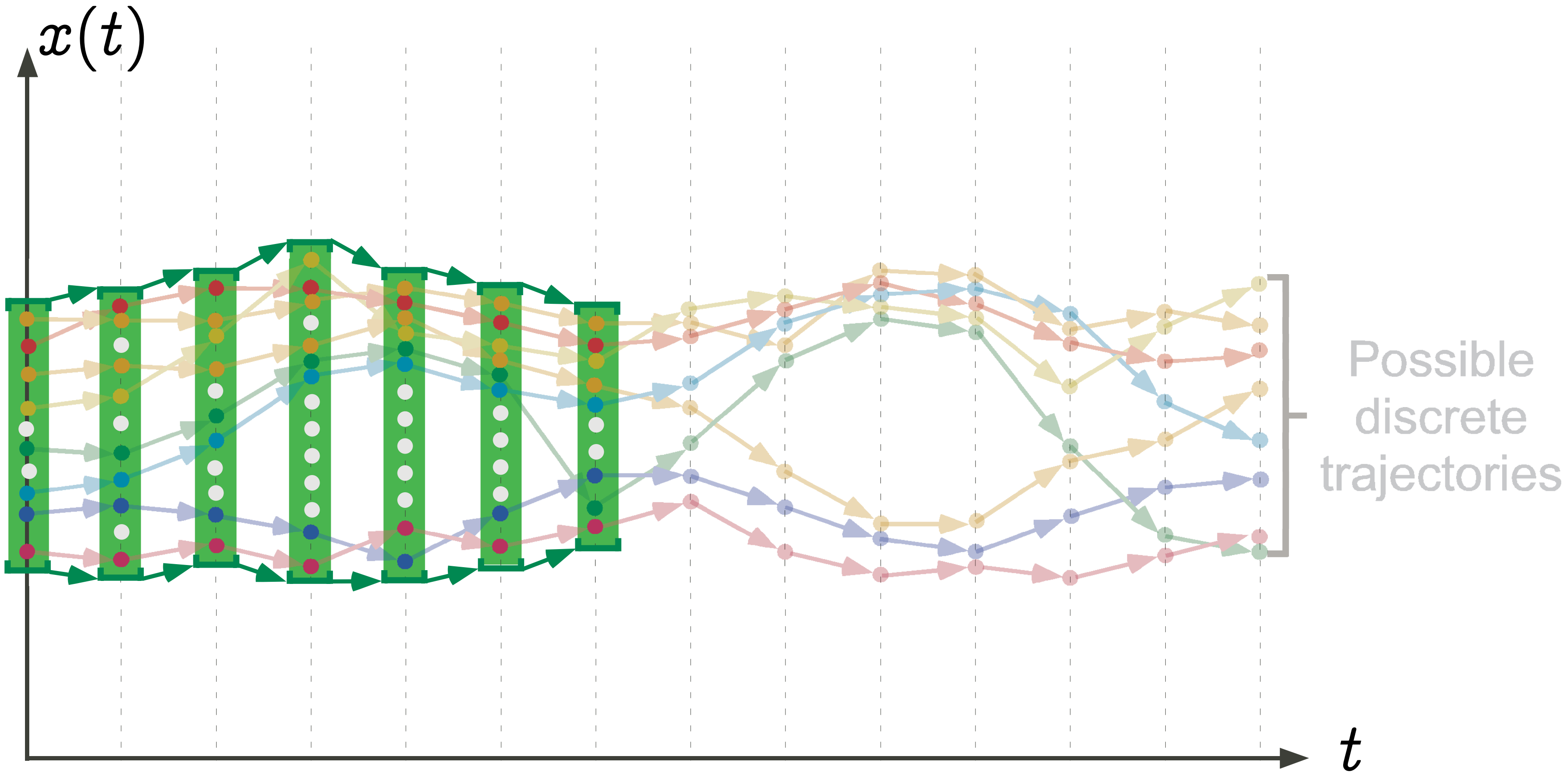
Graphic example: traces of intervals in fixpoint form



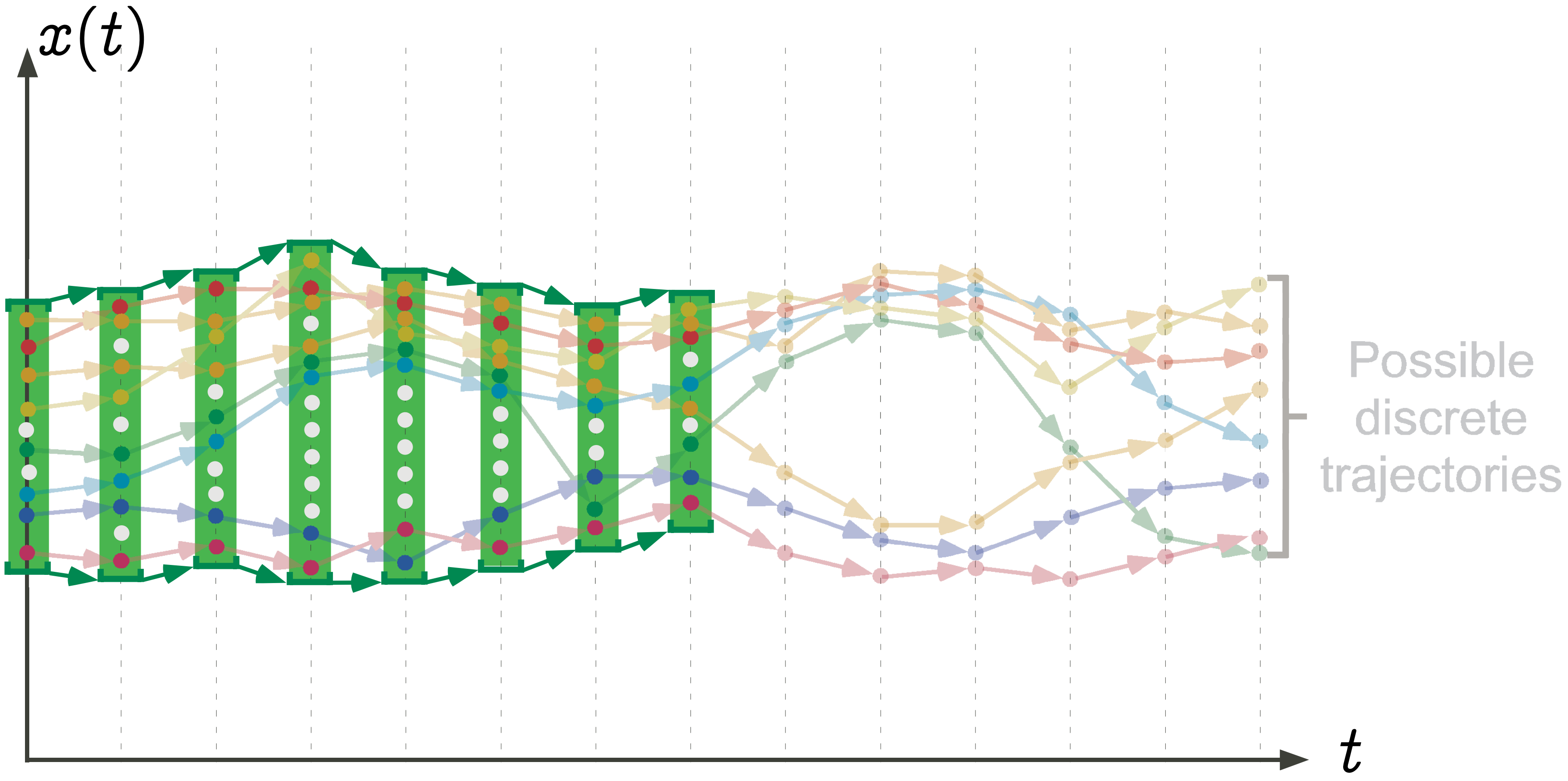
Graphic example: traces of intervals in fixpoint form



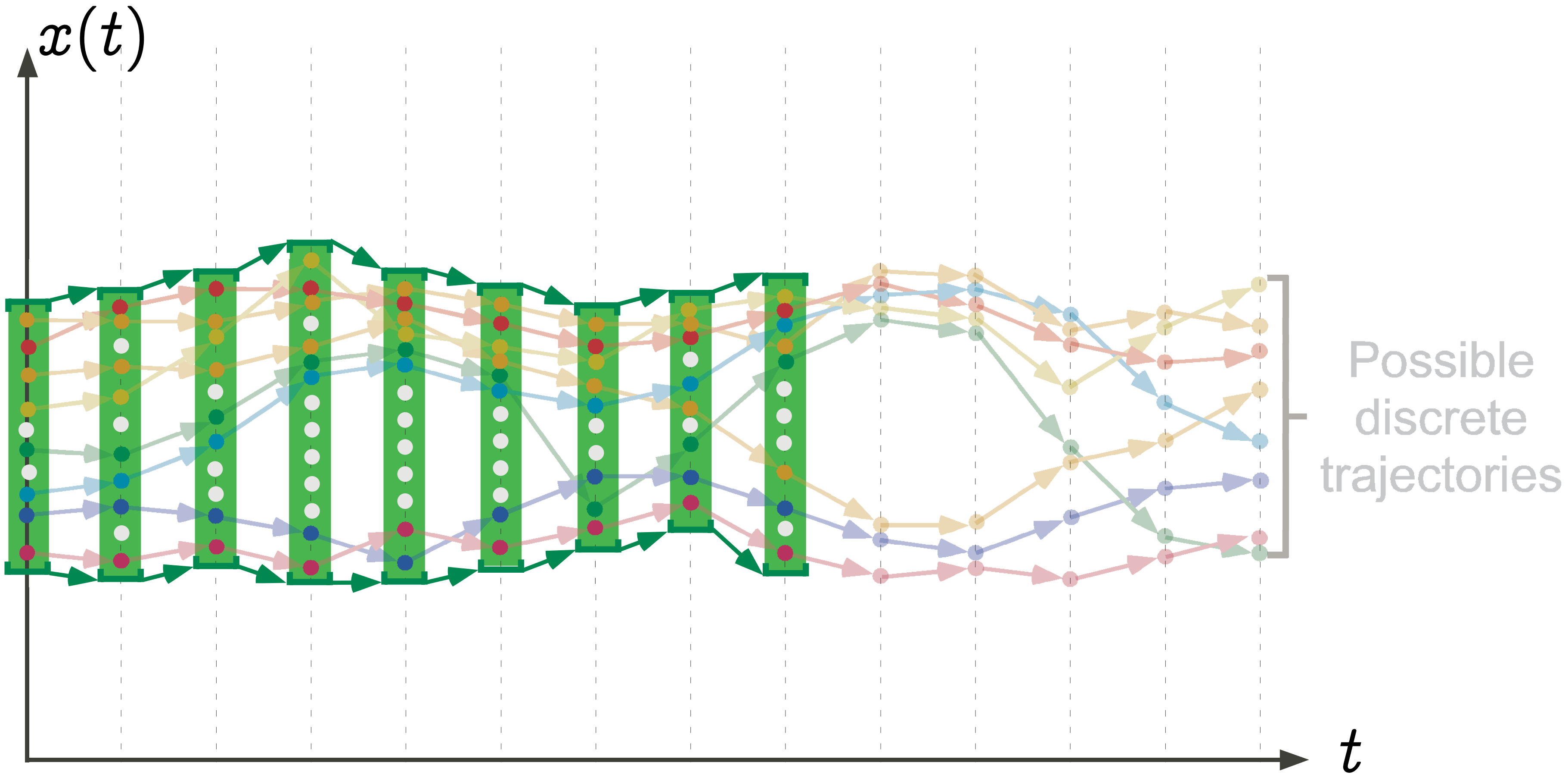
Graphic example: traces of intervals in fixpoint form



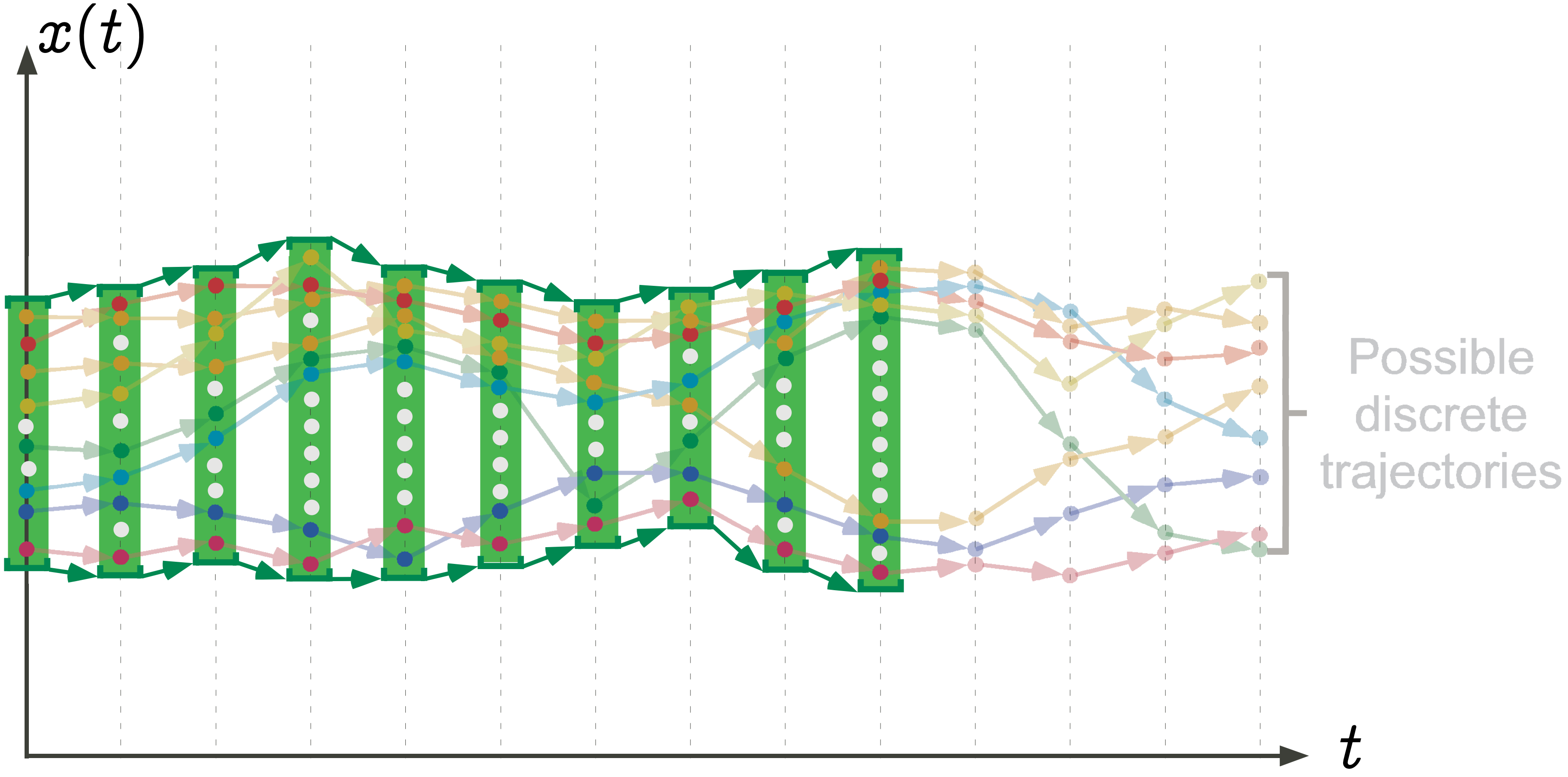
Graphic example: traces of intervals in fixpoint form



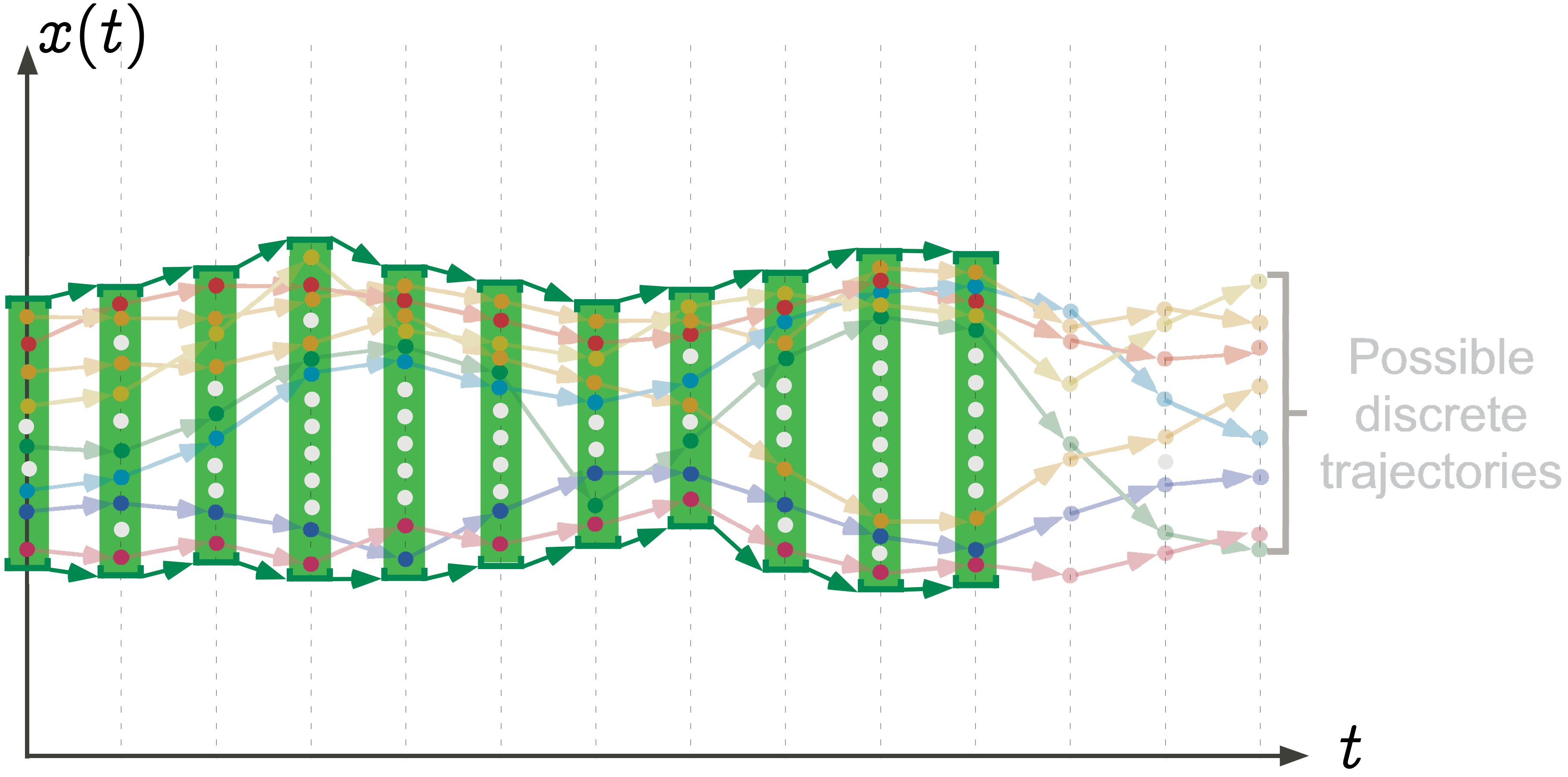
Graphic example: traces of intervals in fixpoint form



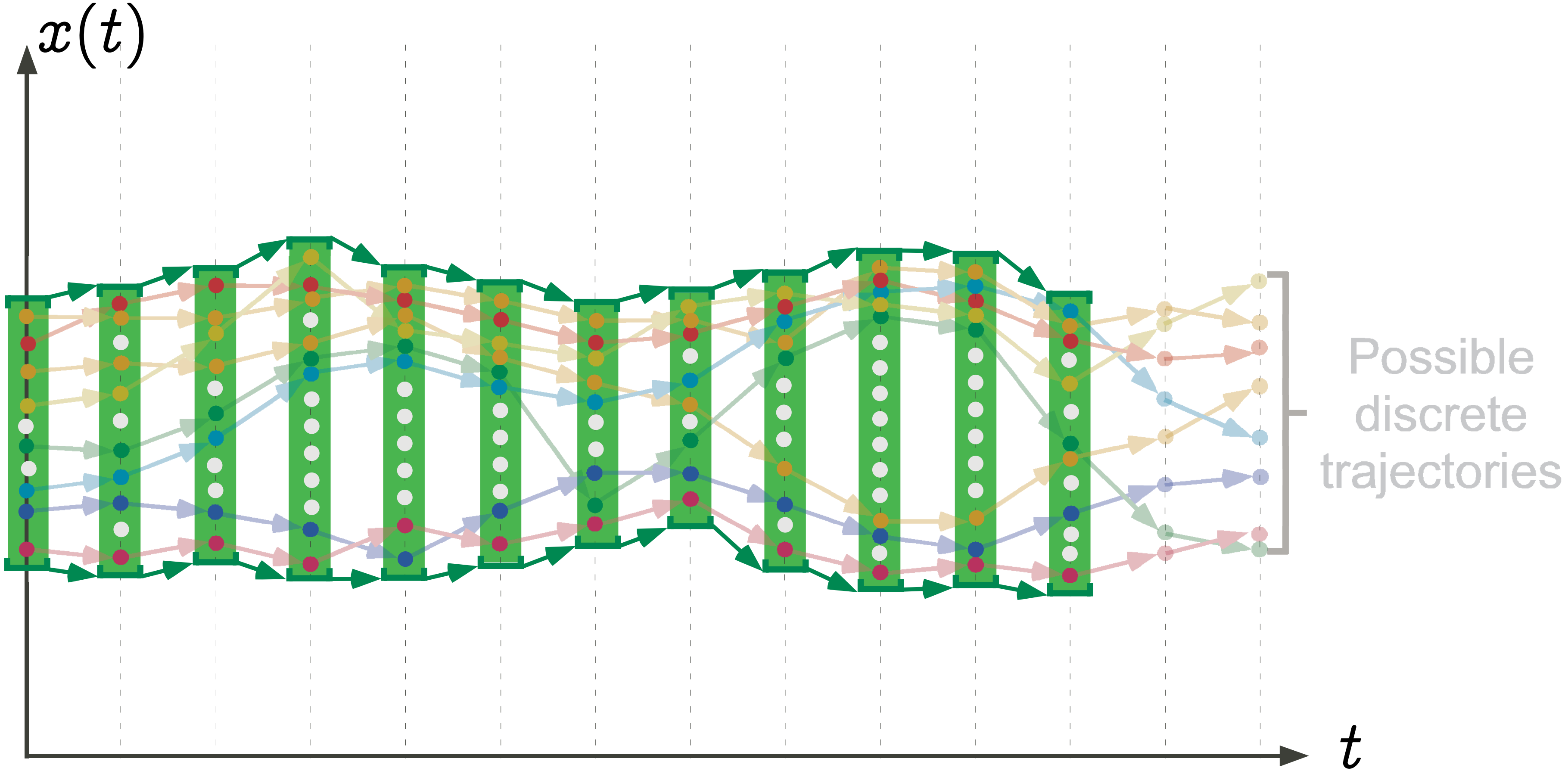
Graphic example: traces of intervals in fixpoint form



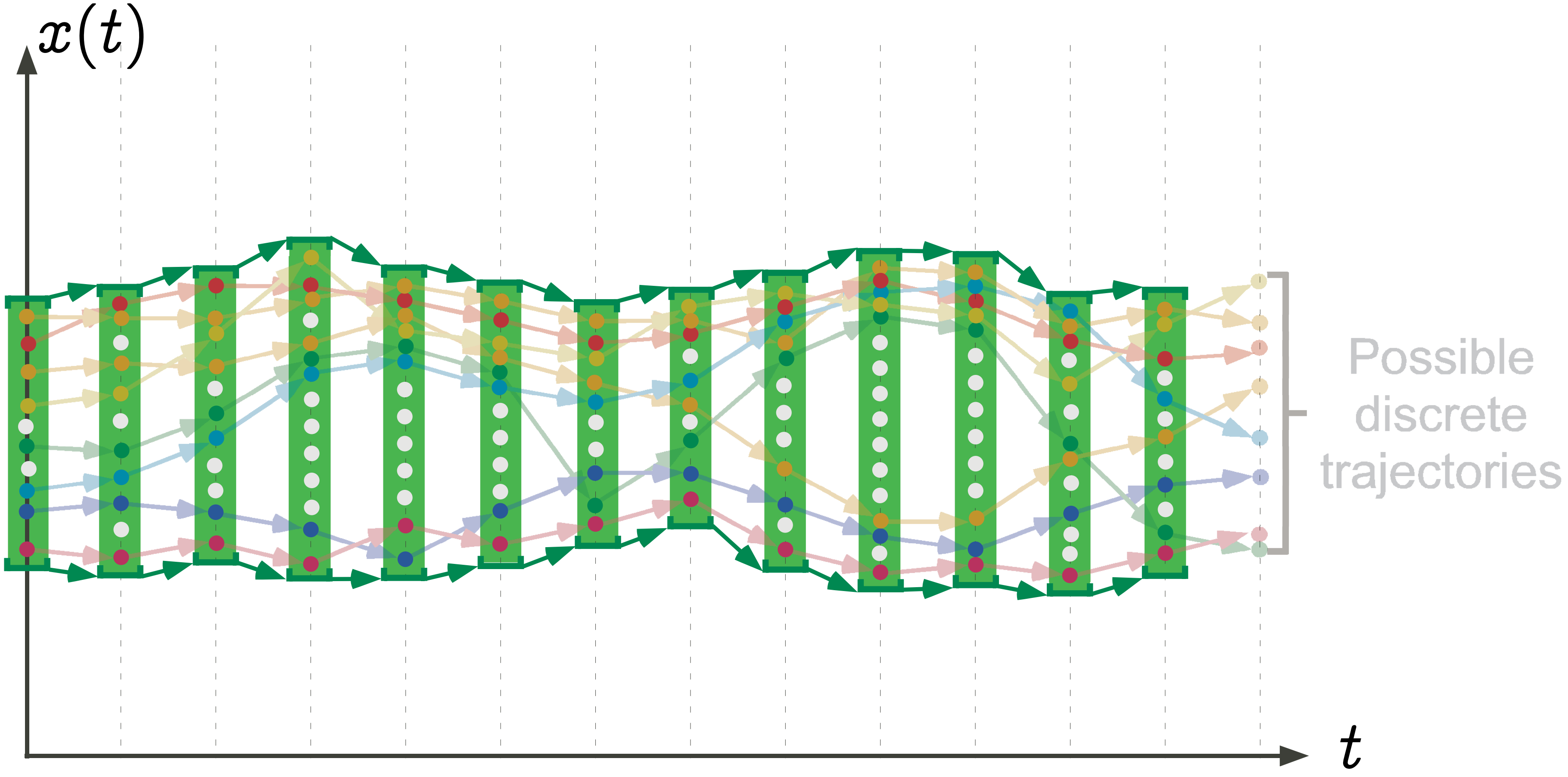
Graphic example: traces of intervals in fixpoint form



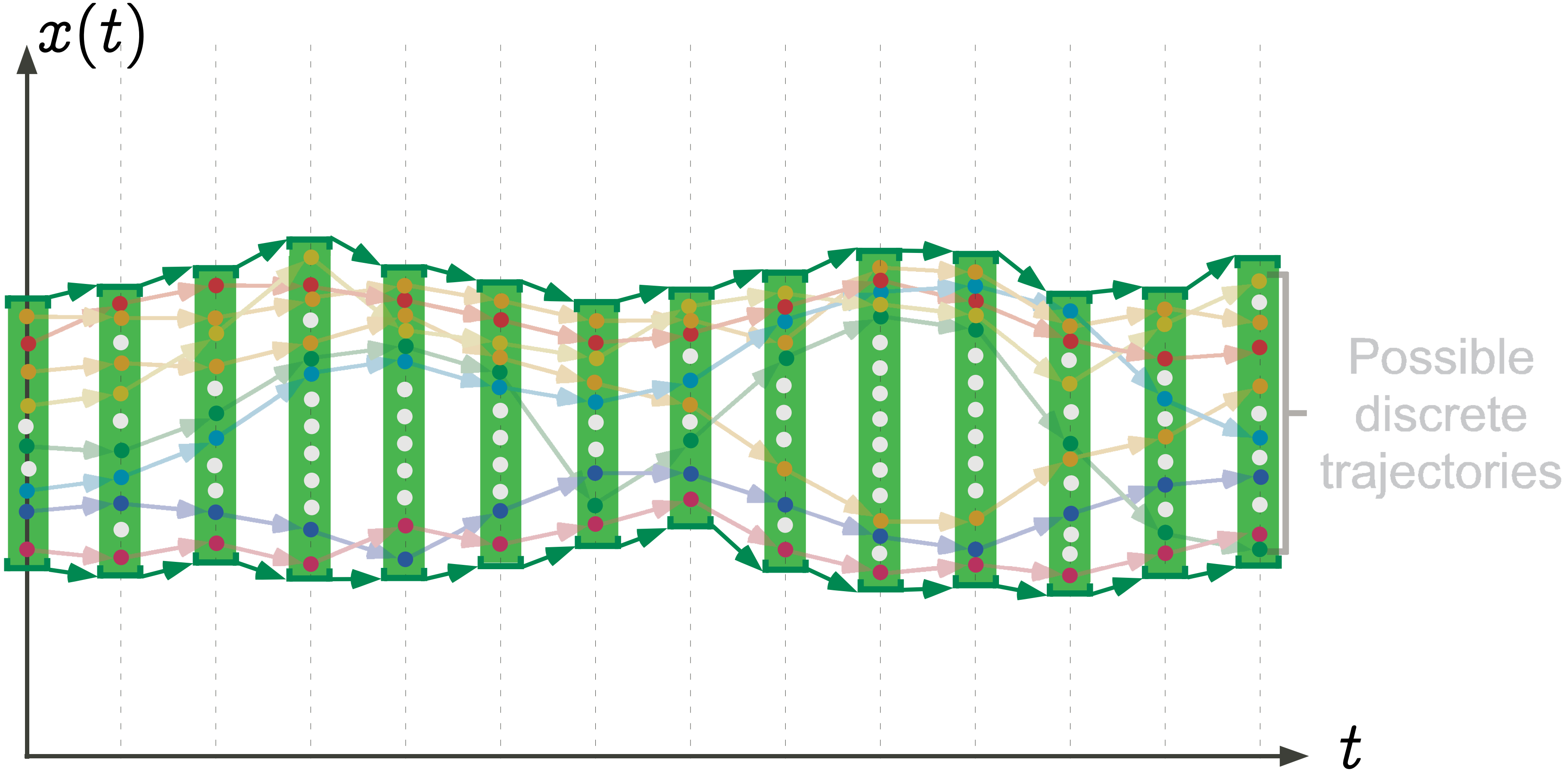
Graphic example: traces of intervals in fixpoint form



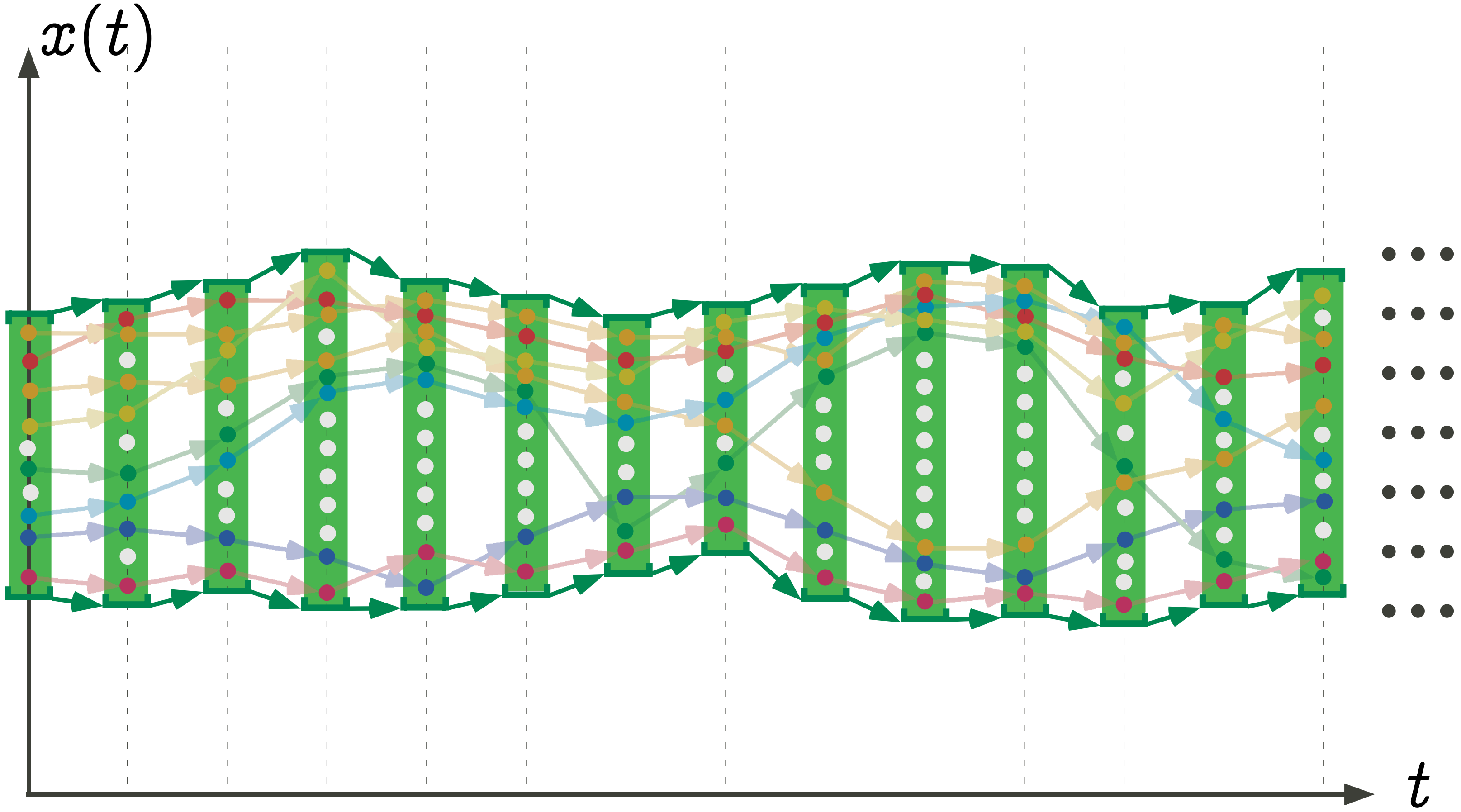
Graphic example: traces of intervals in fixpoint form



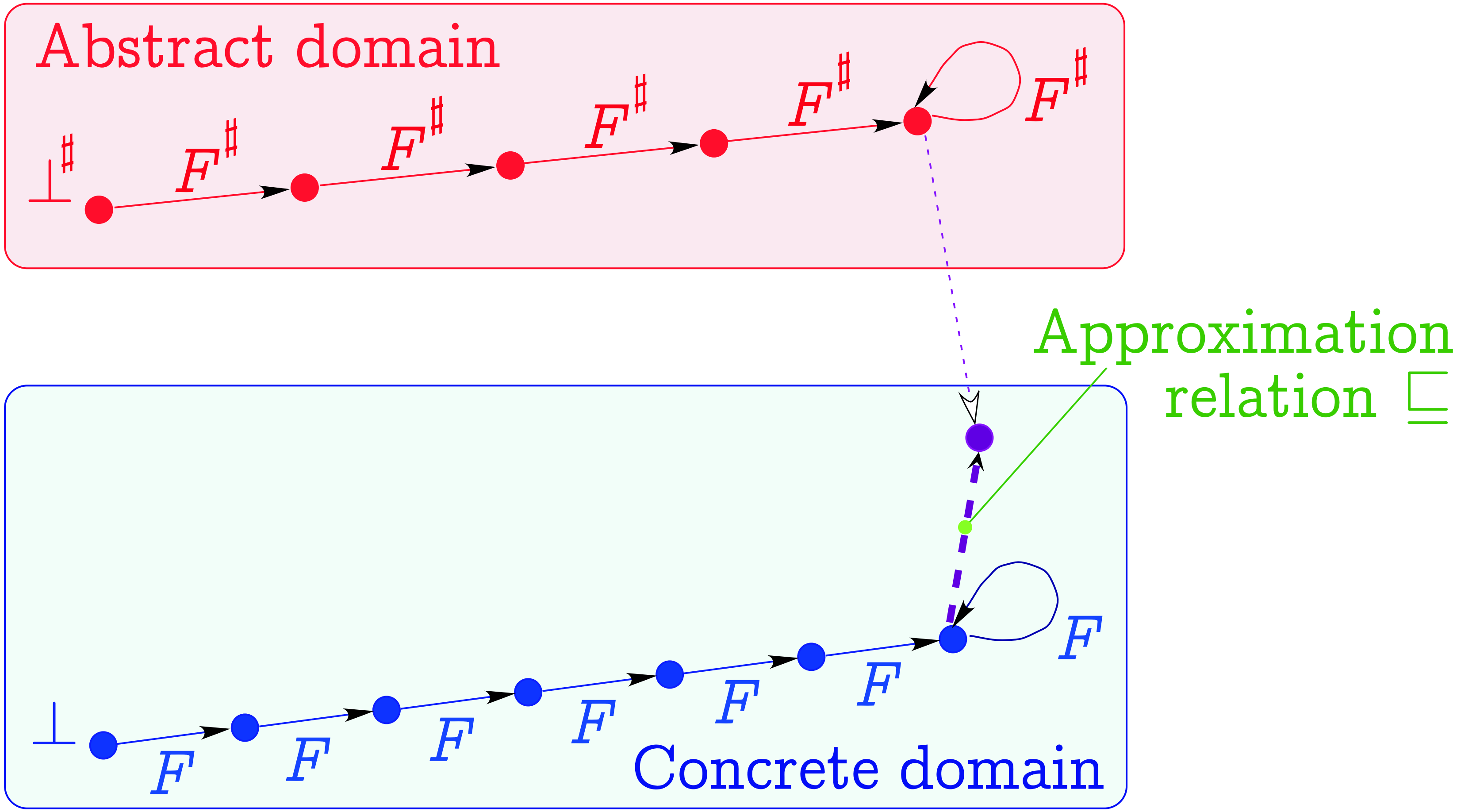
Graphic example: traces of intervals in fixpoint form



Graphic example: traces of intervals in fixpoint form



Approximate fixpoint abstraction



$$\alpha(\mathsf{lfp}\ F) \sqsubseteq \mathsf{lfp}\ F^\sharp$$

approximate/exact fixpoint abstraction

Exact Abstraction:

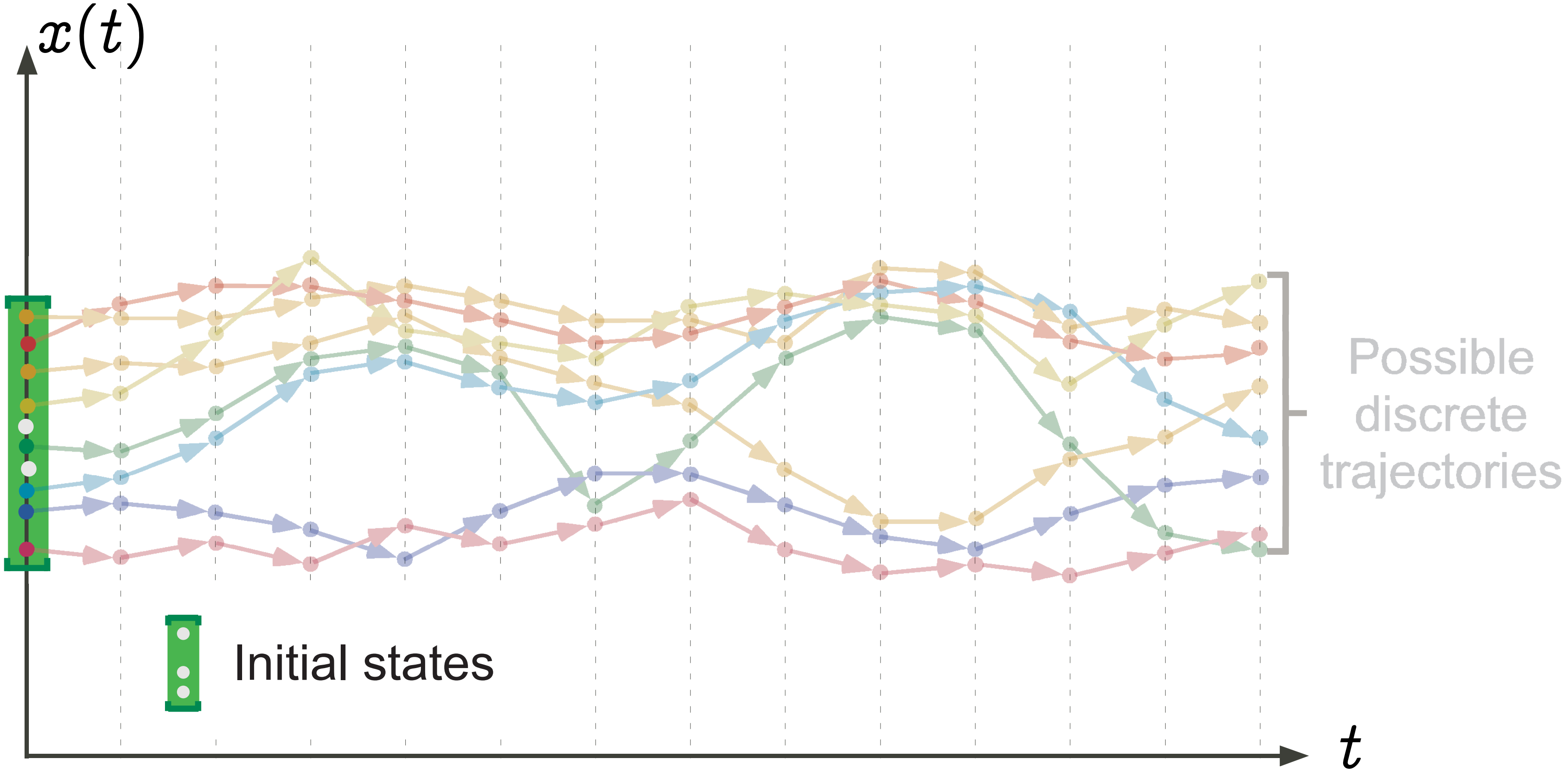
$$\alpha(\mathsf{lfp}\ F) = \mathsf{lfp}\ F^\sharp$$

Approximate Abstraction:

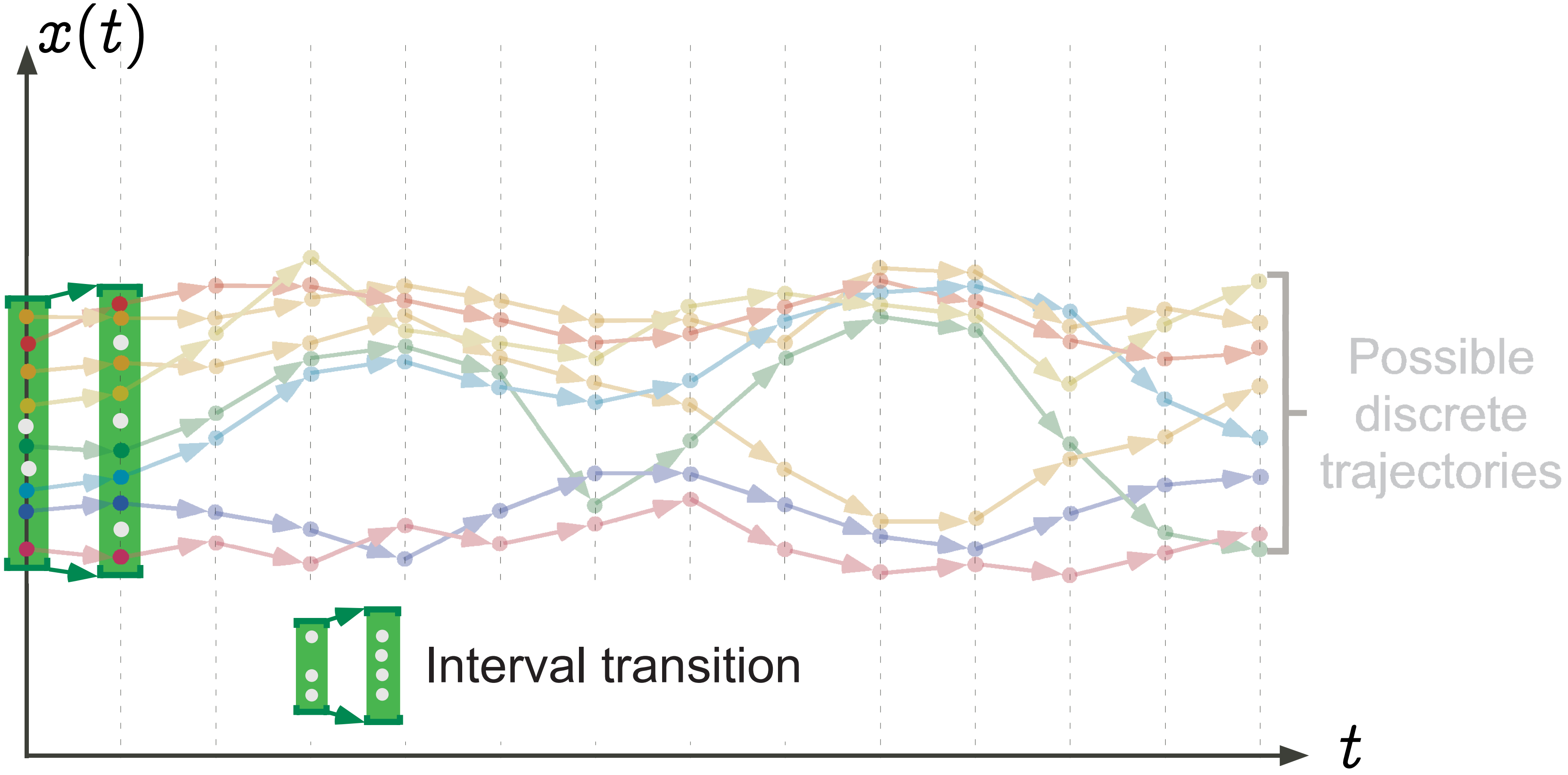
$$\alpha(\mathsf{lfp}\ F) \sqsubseteq^\sharp \mathsf{lfp}\ F^\sharp$$

Convergence acceleration by widening/narrowing

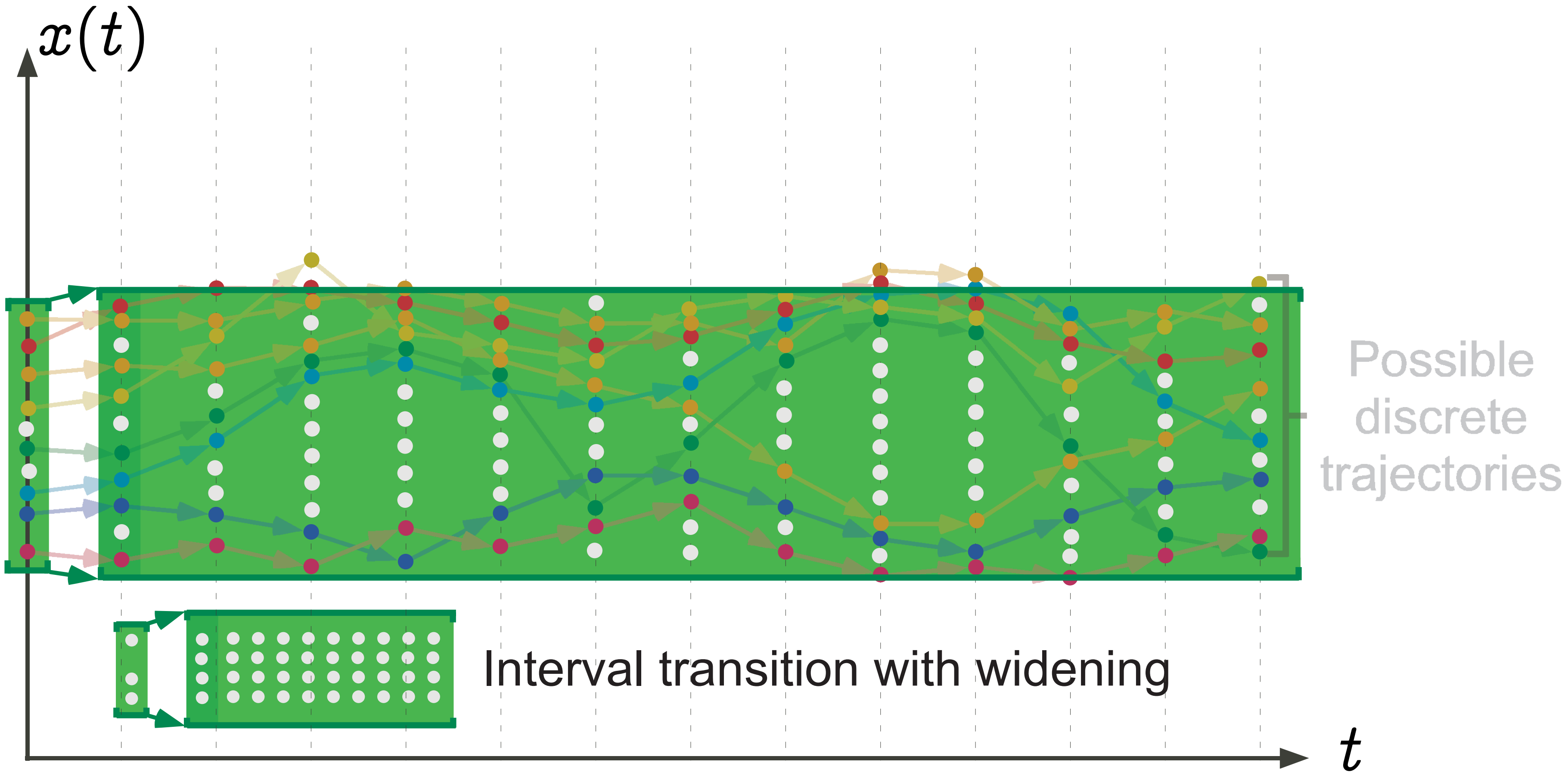
Graphic example: upward iteration with widening



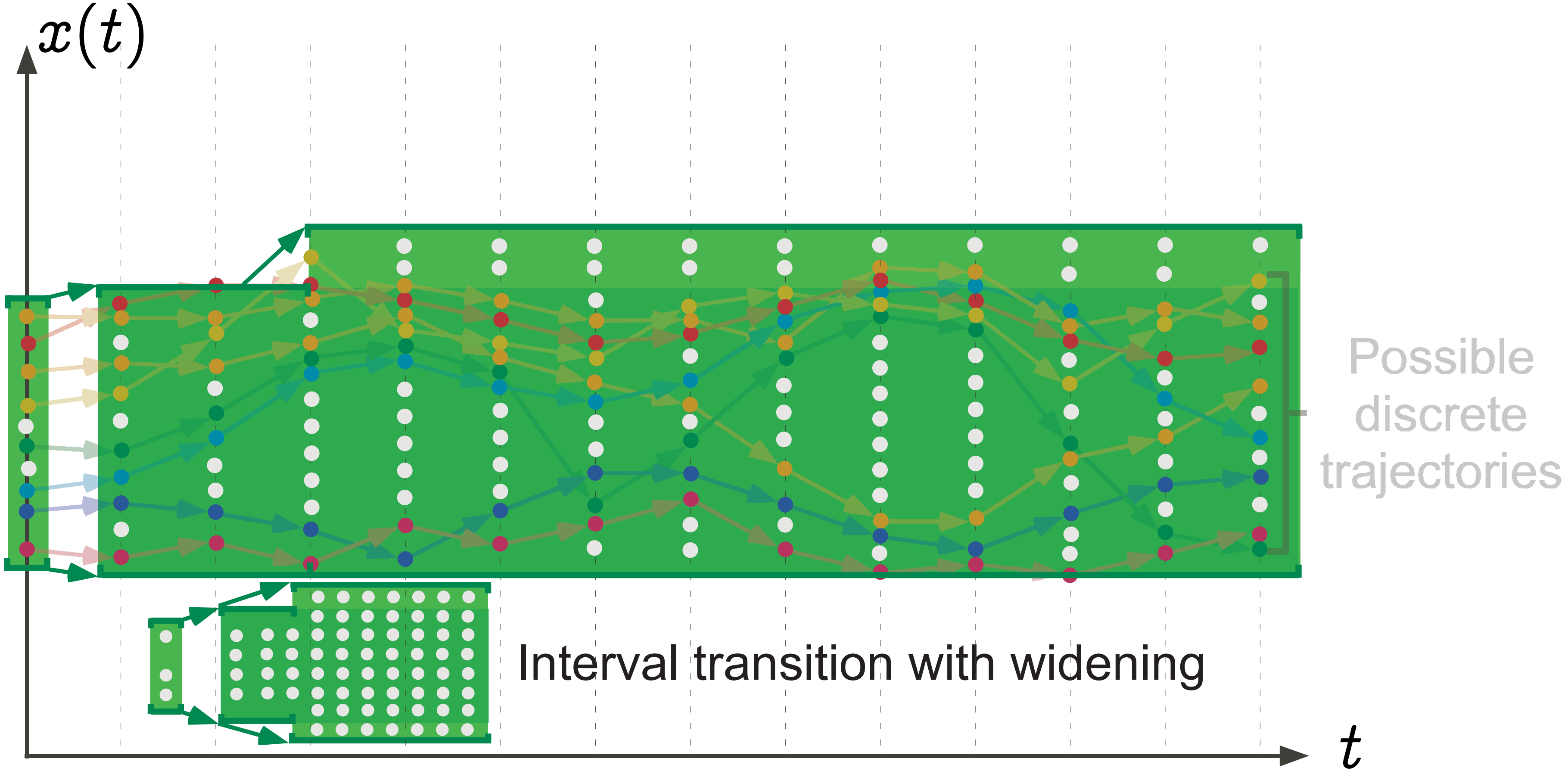
Graphic example: upward iteration with widening



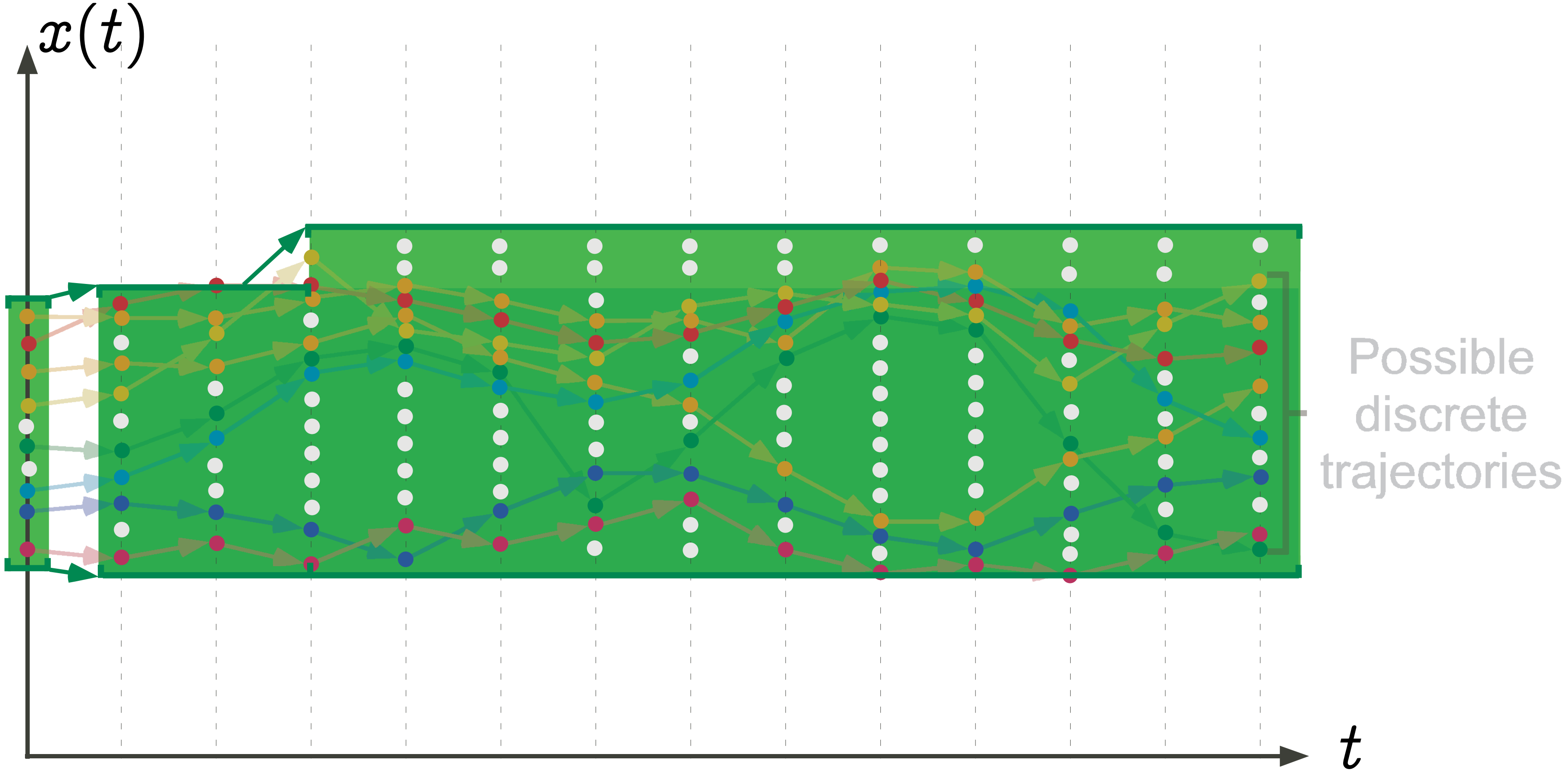
Graphic example: upward iteration with widening



Graphic example: upward iteration with widening



Graphic example: stability of the upward iteration



Convergence acceleration with widening

Widening operator

A widening operator $\nabla \in \overline{L} \times \overline{L} \mapsto \overline{L}$ is such that:

– Correctness:

  - $\forall x, y \in \overline{L} : \gamma(x) \sqsubseteq \gamma(x \mathbin{\nabla} y)$
  - $\forall x, y \in \overline{L} : \gamma(y) \sqsubseteq \gamma(x \mathbin{\nabla} y)$

– Convergence:

  - for all increasing chains $x^0 \sqsubseteq x^1 \sqsubseteq \ldots$, the increasing chain defined by $y^0 = x^0, \ldots, y^{i+1} = y^i \mathbin{\nabla} x^{i+1}, \ldots$ is not strictly increasing.

Fixpoint approximation with widening

The upward iteration sequence with widening:

- $\hat{X}^0 = \overline{\bot}$ (infimum)
- $\hat{X}^{i+1} = \hat{X}^i$ if $\overline{F}(\hat{X}^i) \mathrel{\underline{\sqsubseteq}} \hat{X}^i$

  $= \hat{X}^i \mathbin{\nabla} F(\hat{X}^i)$ otherwise

is ultimately stationary and its limit $\hat{A}$ is a sound upper approximation of $\mathsf{lfp}^{\overline{\bot}}\ \overline{F}$:

$$\mathsf{lfp}^{\overline{\bot}}\ \overline{F} \mathrel{\underline{\sqsubseteq}} \hat{A}$$

Interval widening

- $\overline{L} = \{\bot\} \cup \{[\ell, u] \mid \ell, u \in \mathbb{Z} \cup \{-\infty\} \wedge u \in \mathbb{Z} \cup \{\} \wedge \ell \leq u\}$
- The widening extrapolates unstable bounds to infinity:

$$\begin{aligned}
\bot \nabla X &= X \\
X \nabla \bot &= X \\
[\ell_0,\ u_0] \nabla [\ell_1,\ u_1] &= [\text{if } \ell_1 < \ell_0 \text{ then } -\infty \text{ else } \ell_0, \\
&\qquad \text{if } \mathrm{u}_1 > \mathrm{u}_0 \text{ then } +\infty \text{ else } \mathrm{u}_0]
\end{aligned}$$

Not monotone. For example $[0,\ 1] \sqsubseteq [0,\ 2]$ but $[0,\ 1] \nabla [0,\ 2] = [0,\ +\infty] \not\sqsubseteq [0,\ 2] = [0,\ 2] \nabla [0,\ 2]$

Example: Interval analysis (1975)

Program to be analyzed:

```
    x := 1;
1:
    while x < 10000 do
2:
        x := x + 1
3:
    od;
4:
```

Example: Interval analysis (1975)

Equations (abstract interpretation of the semantics):

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

Example: Interval analysis (1975)

Resolution by chaotic increasing iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = \emptyset \\ X_2 = \emptyset \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
    x := 1;
1:
    while x < 10000 do
2:
        x := x + 1
3:
    od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = \emptyset \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,1] \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, 1] \\ X_3 = [2, 2] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
    x := 1;
1:
    while x < 10000 do
2:
        x := x + 1
3:
    od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, 2] \\ X_3 = [2, 2] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !**

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, 2] \\ X_3 = [2, 3] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!**

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,3] \\ X_3 = [2,3] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!**

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,3] \\ X_3 = [2,4] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, 4] \\ X_3 = [2, 4] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,4] \\ X_3 = [2,5] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,5] \\ X_3 = [2,5] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!!!!**

```
    x := 1;
1:
    while x < 10000 do
2:
        x := x + 1
3:
    od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,5] \\ X_3 = [2,6] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Convergence speed-up by widening:

```
    x := 1;
1:
    while x < 10000 do
2:
        x := x + 1
3:
    od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, +\infty] \quad \Leftarrow \text{widening} \\ X_3 = [2, 6] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, +\infty] \\ X_3 = [2, +\infty] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
         x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1, 1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1, 1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1, 1] \\ X_2 = [1, 9999] \\ X_3 = [2, +\infty] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Final solution:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = [+10000, +10000] \end{cases}$$

Example: Interval analysis (1975)

Result of the interval analysis:

```
   x := 1;
1: $\{x = 1\}$
   while x < 10000 do
2: $\{x \in [1, 9999]\}$
      x := x + 1
3: $\{x \in [2, +10000]\}$
   od;
4: $\{x = 10000\}$
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = [+10000, +10000] \end{cases}$$

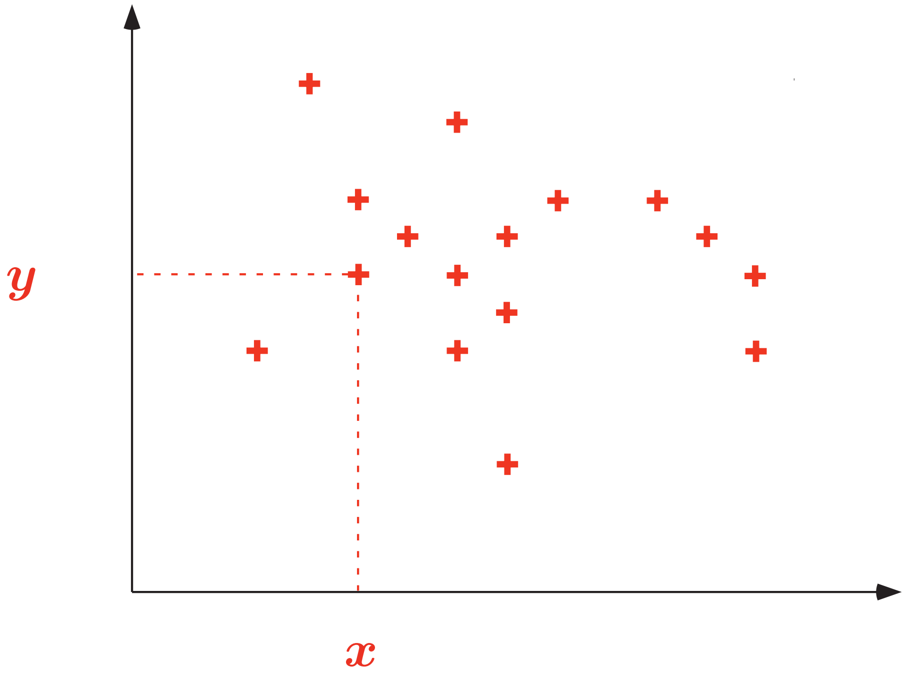
Example: Interval analysis (1975)

Checking absence of runtime errors with interval analysis:

```
   x := 1;
1: {x = 1}
   while x < 10000 do
2: {x ∈ [1, 9999]}
        x := x + 1          ⟵ no overflow
3: {x ∈ [2, +10000]}
   od;
4: {x = 10000}
```

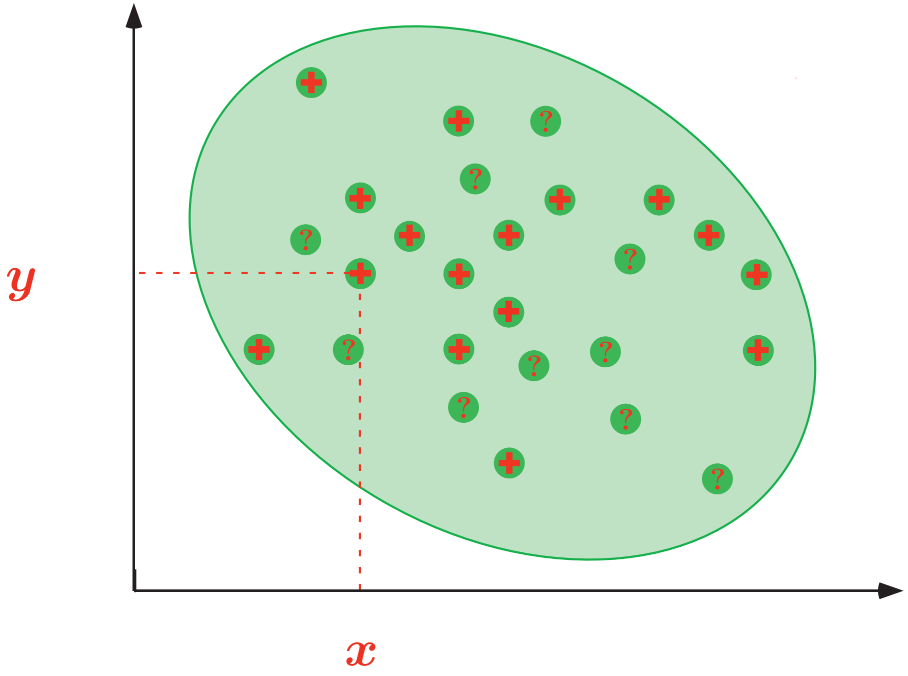
Refinement of abstractions

Approximations of an [in]finite set of points:



$$\{\ldots, \langle 19,\ 77\rangle, \ldots,\\ \langle 20,\ 03\rangle, \ldots\}$$

Approximations of an [in]finite set of points: from above



$\{\ldots, \langle 19,\ 77\rangle, \ldots,$

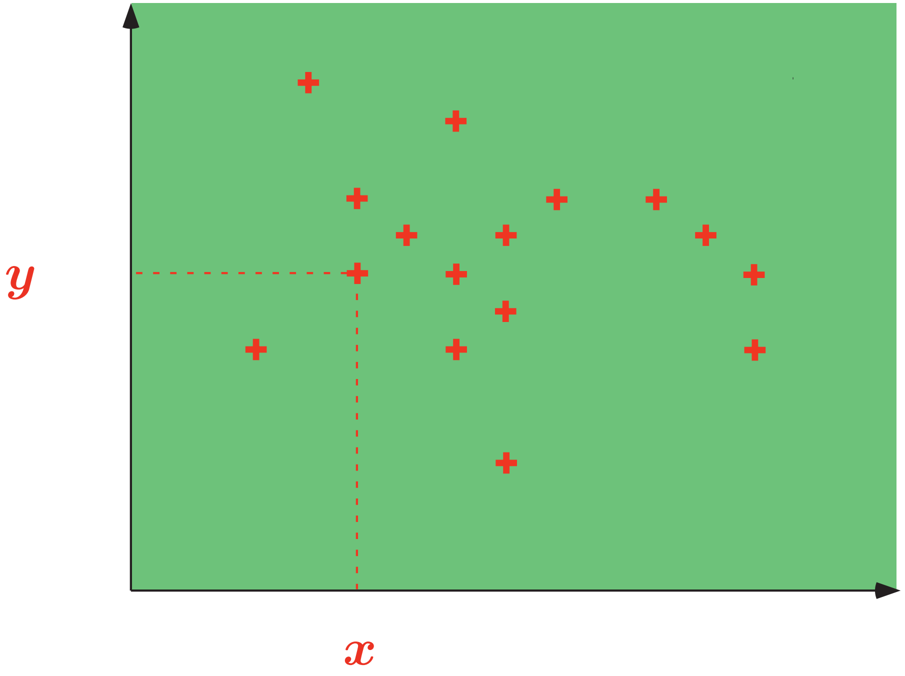
$\langle 20,\ 03\rangle, \langle ?,\ ?\rangle, \ldots\}$

From Below: dual[2] + combinations.

---

[2] Trivial for finite states (liveness model-checking), more difficult for infinite states (variant functions).

Effective computable approximations of an [in]finite set of points; Signs[3]

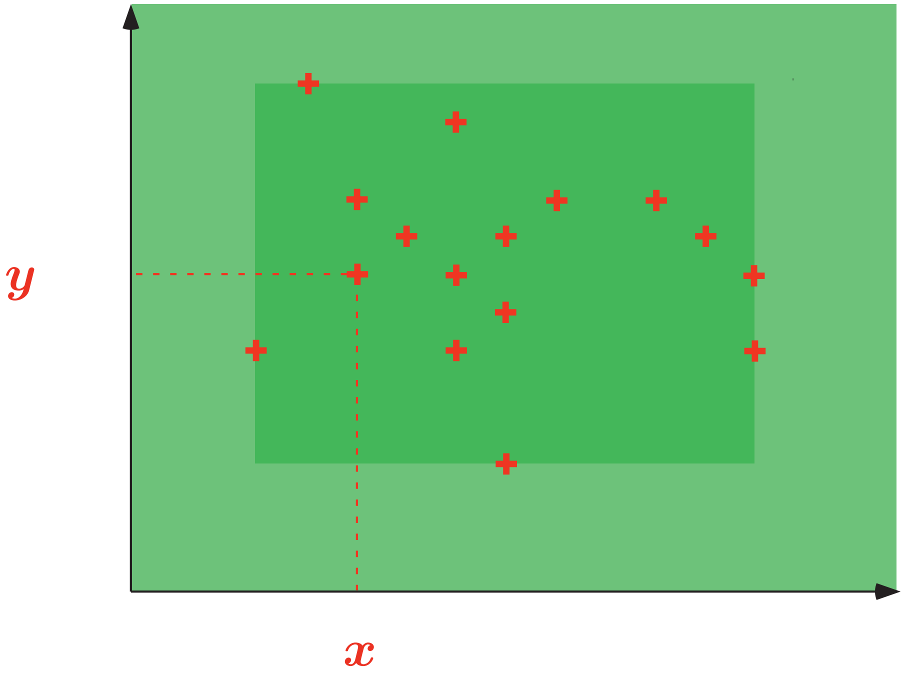


$$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$$

---

[3] P. Cousot & R. Cousot. *Systematic design of program analysis frameworks.* ACM POPL'79, pp. 269–282, 1979.

Effective computable approximations of an [in]finite set of points; Intervals [4]

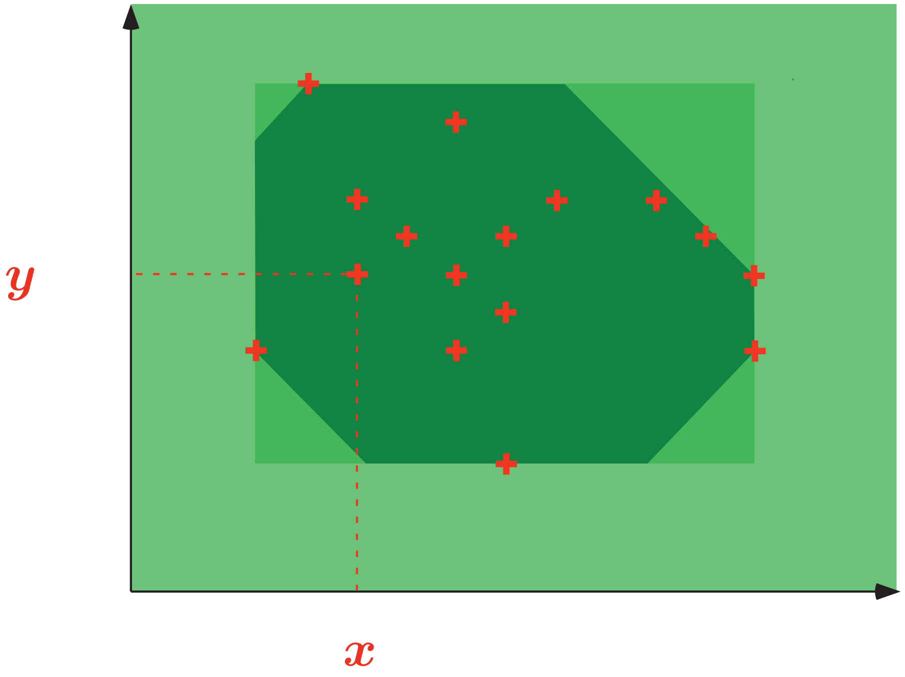


$$\begin{cases} x \in [19,\ 77] \\ y \in [20,\ 03] \end{cases}$$

---

[4] P. Cousot & R. Cousot. *Static determination of dynamic properties of programs*. Proc. 2^{nd} Int. Symp. on Programming, Dunod, 1976.

Effective computable approximations of an [in]finite set of points; Octagons [5]

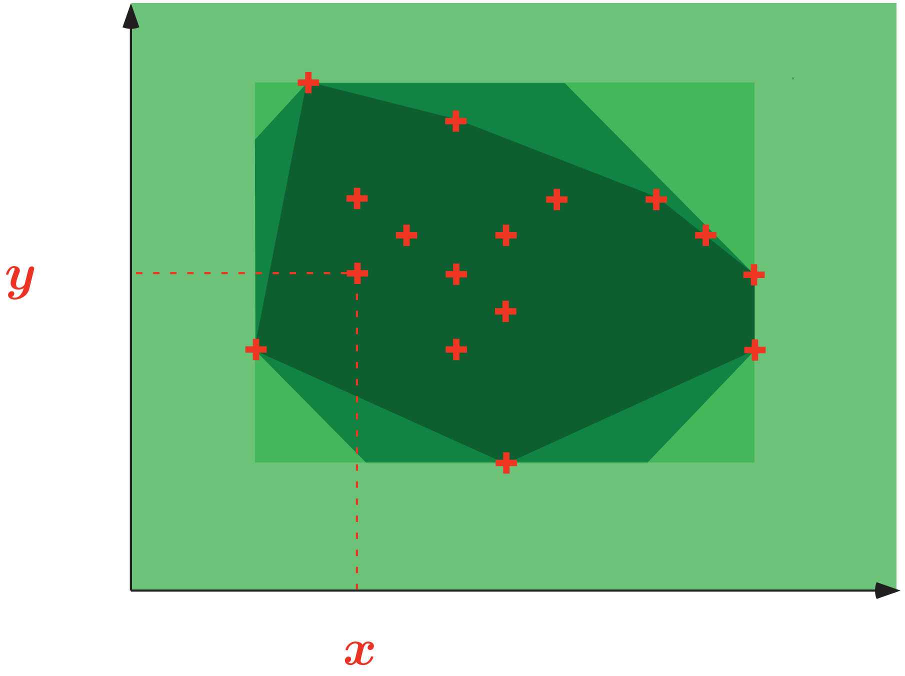


$$\begin{cases} 1 \leq x \leq 9 \\ x + y \leq 77 \\ 1 \leq y \leq 9 \\ x - y \leq 99 \end{cases}$$

---

[5] A. Miné. *A New Numerical Abstract Domain Based on Difference-Bound Matrices.* PADO '2001. LNCS 2053, pp. 155–172. Springer 2001. See the *The Octagon Abstract Domain Library* on `http://www.di.ens.fr/~mine/oct/`

Effective computable approximations of an [in]finite set of points; Polyhedra[6]

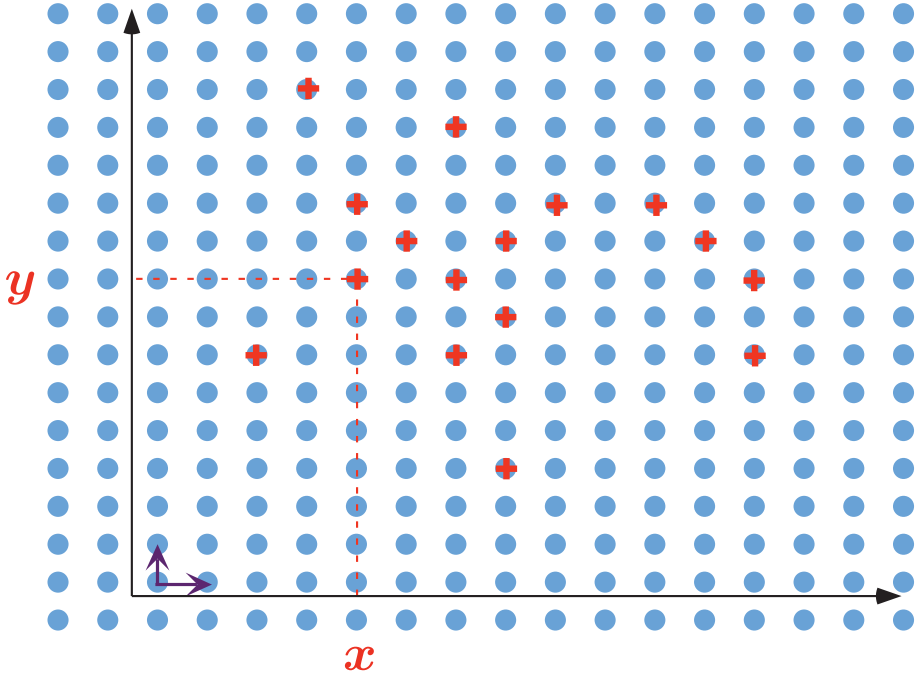


$$\begin{cases} 19x + 77y \leq 2004 \\ 20x + 03y \geq 0 \end{cases}$$

---

[6] P. Cousot & N. Halbwachs. *Automatic discovery of linear restraints among variables of a program*. ACM POPL, 1978, pp. 84–97.

Effective computable approximations of an [in]finite set of points; Simple congruences [7]

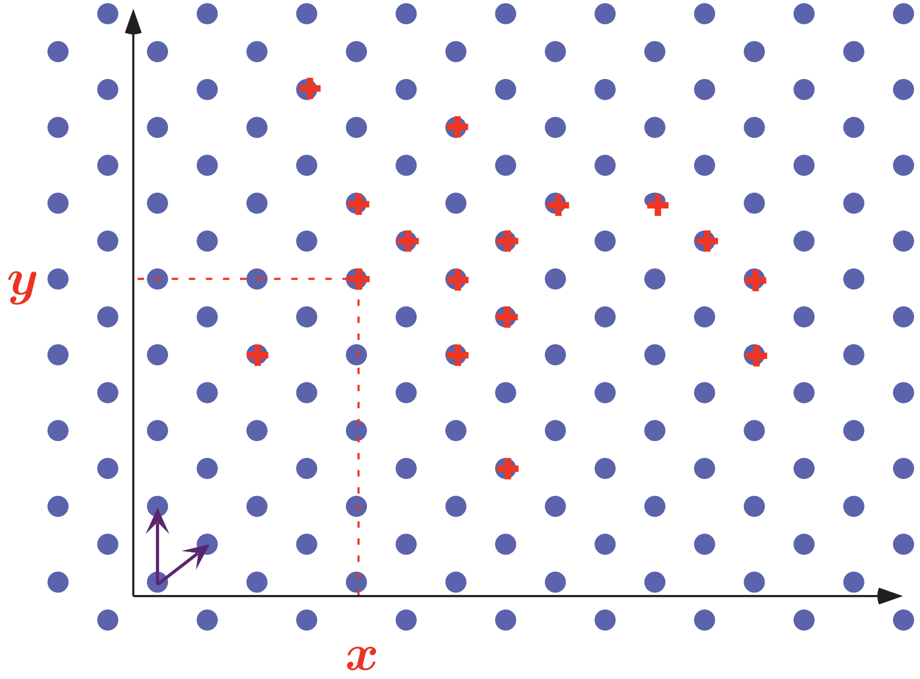


$$\begin{cases} x = 19 \bmod 77 \\ y = 20 \bmod 99 \end{cases}$$

---

[7] Ph. Granger. *Static Analysis of Arithmetical Congruences*. Int. J. Comput. Math. 30, 1989, pp. 165–190.

Effective computable approximations of an [in]finite set of points; Linear congruences[8]

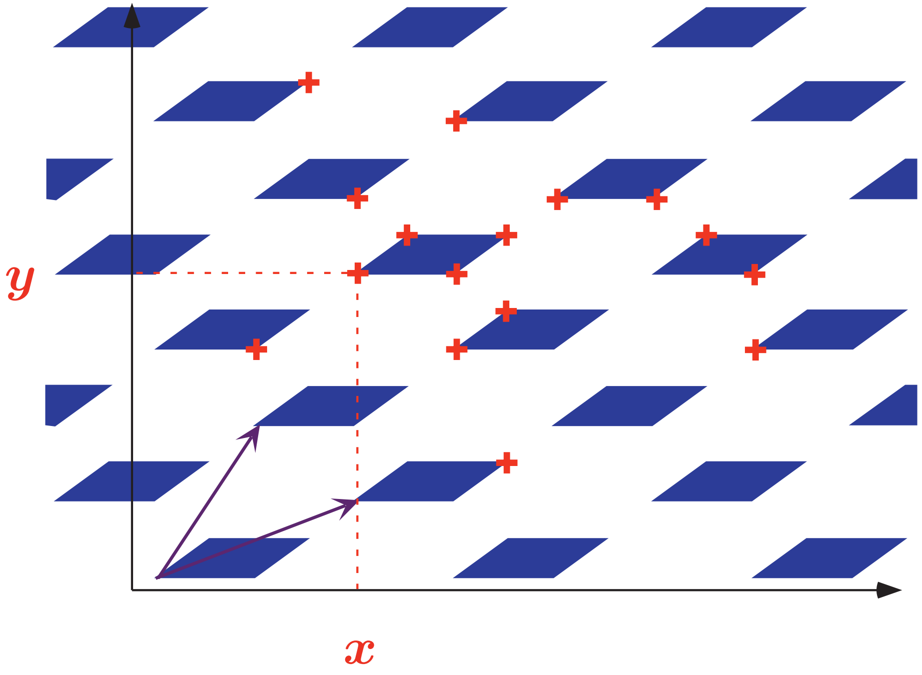


$$\begin{cases} 1x + 9y = 7 \bmod 8 \\ 2x - 1y = 9 \bmod 9 \end{cases}$$

---

[8] Ph. Granger. *Static Analysis of Linear Congruence Equalities among Variables of a Program.* TAPSOFT '91, pp. 169–192. LNCS 493, Springer, 1991.

Effective computable approximations of an [in]finite set of points; Trapezoidal linear congruences[9]



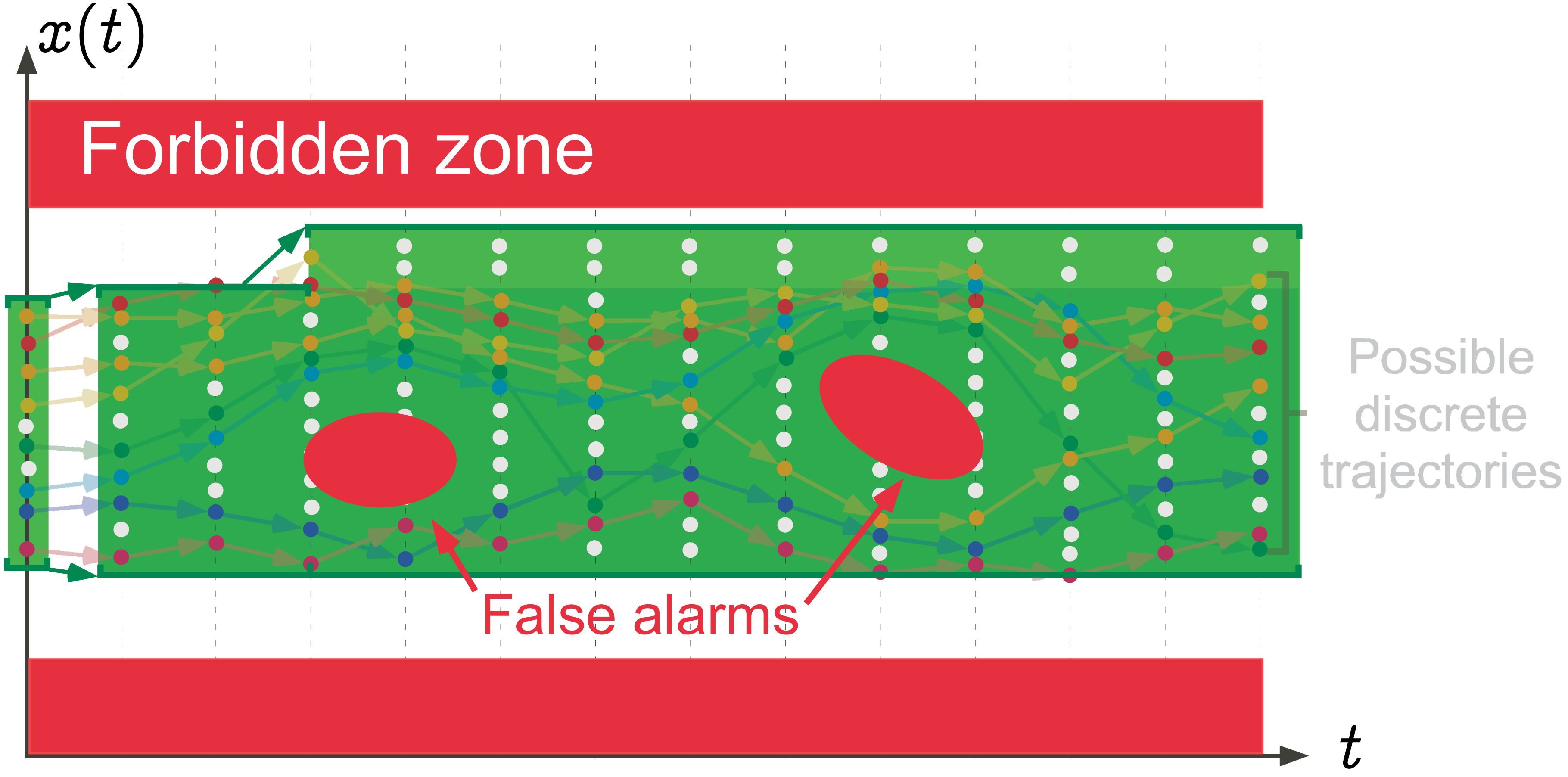
$$\begin{cases} 1x + 9y \in [0, 77] \bmod 10 \\ 2x - 1y \in [0, 99] \bmod 11 \end{cases}$$

---

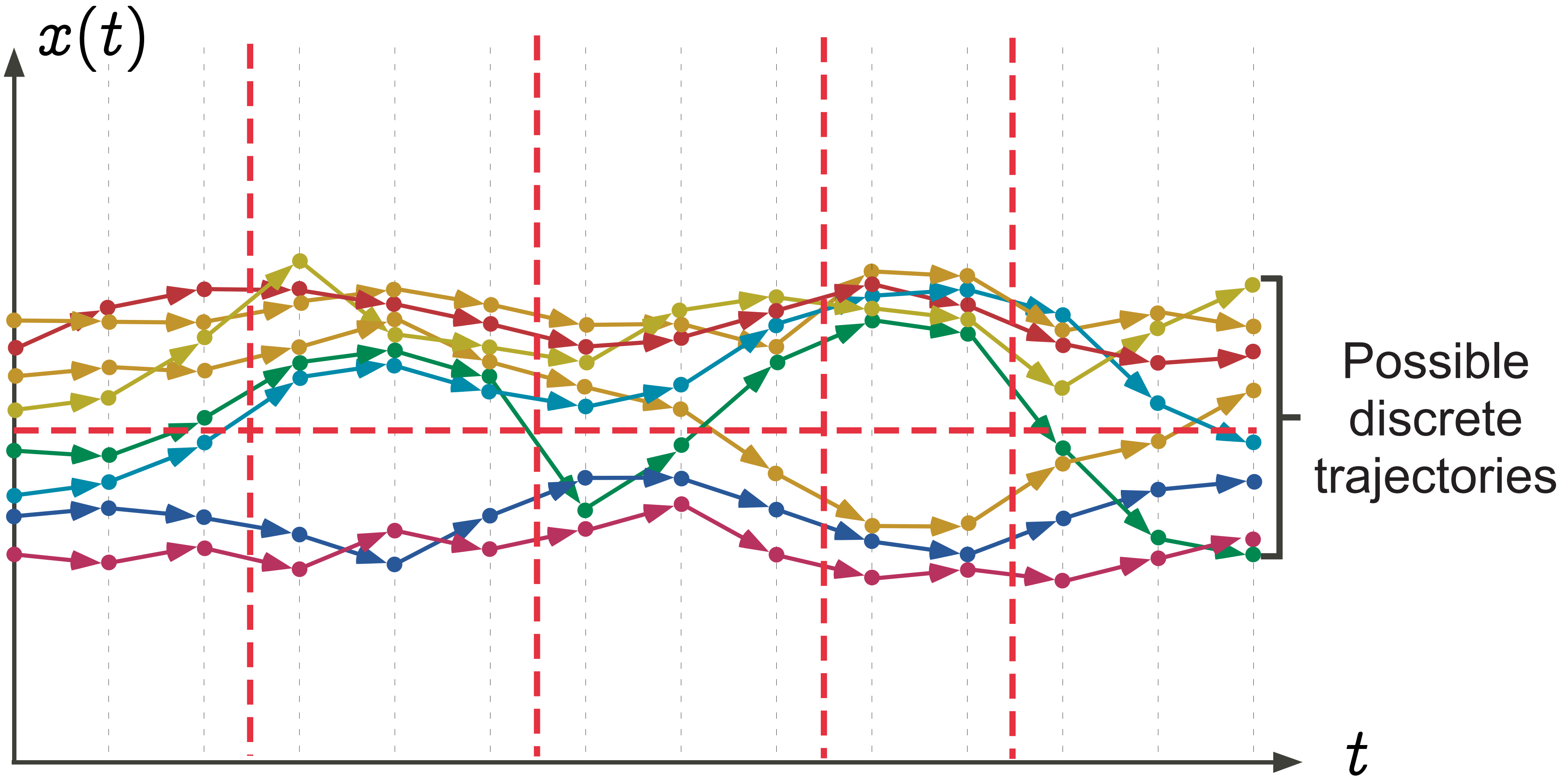
[9] F. Masdupuy. *Array Operations Abstraction Using Semantic Analysis of Trapezoid Congruences*. ACM ICS '92.

Refinement of iterates

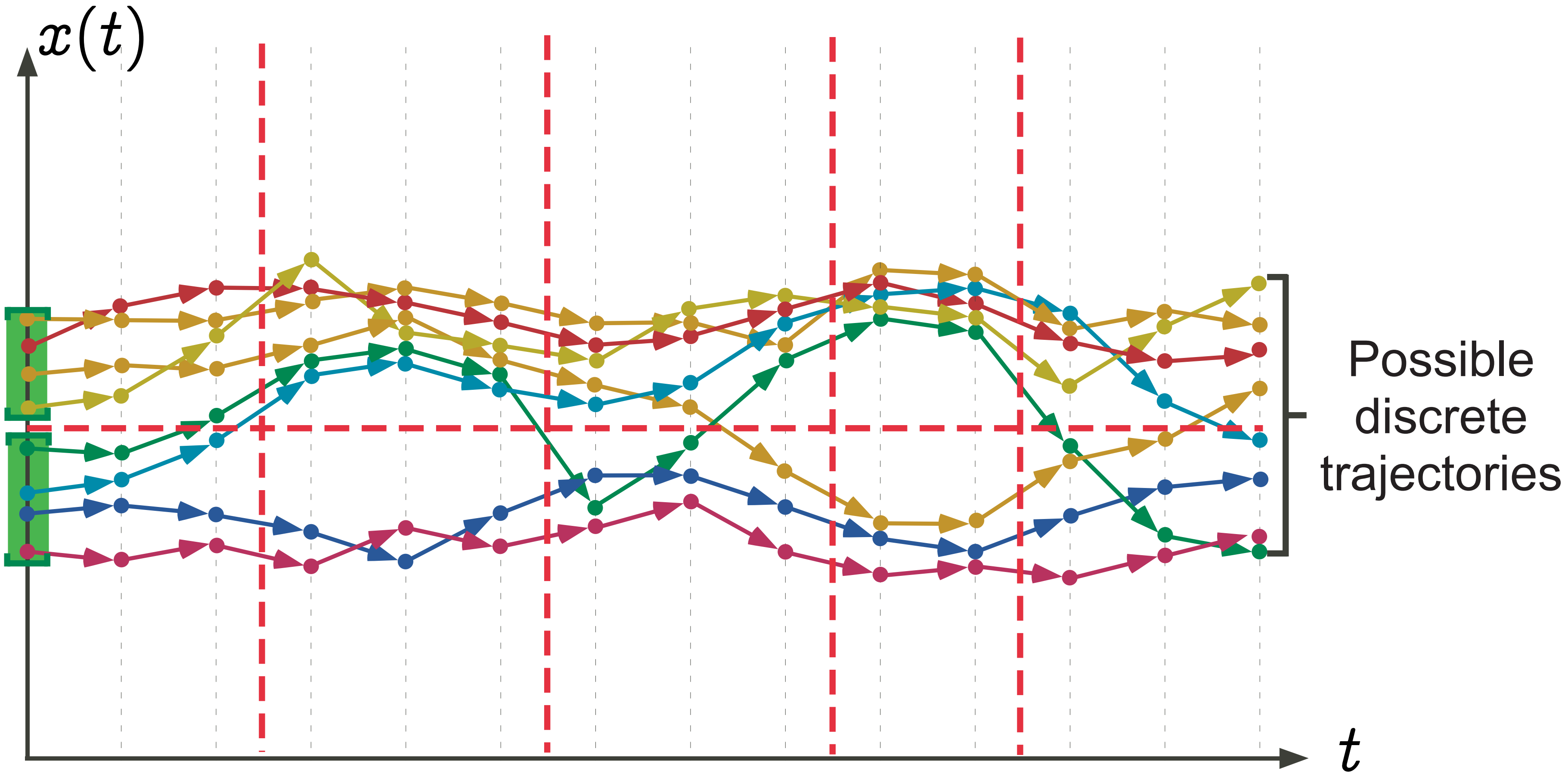
Graphic example: Refinement required by false alarms



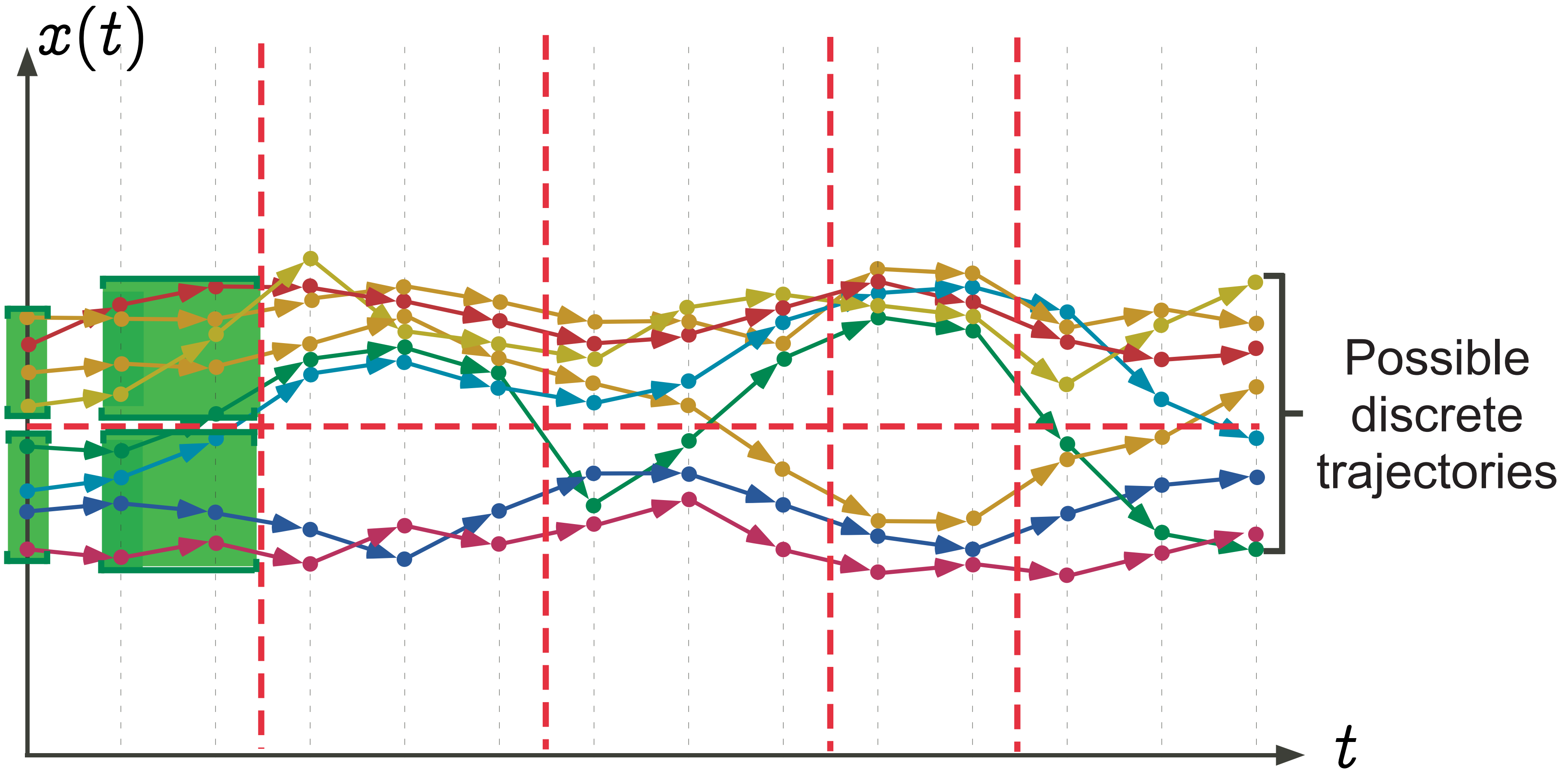
Graphic example: Partitionning



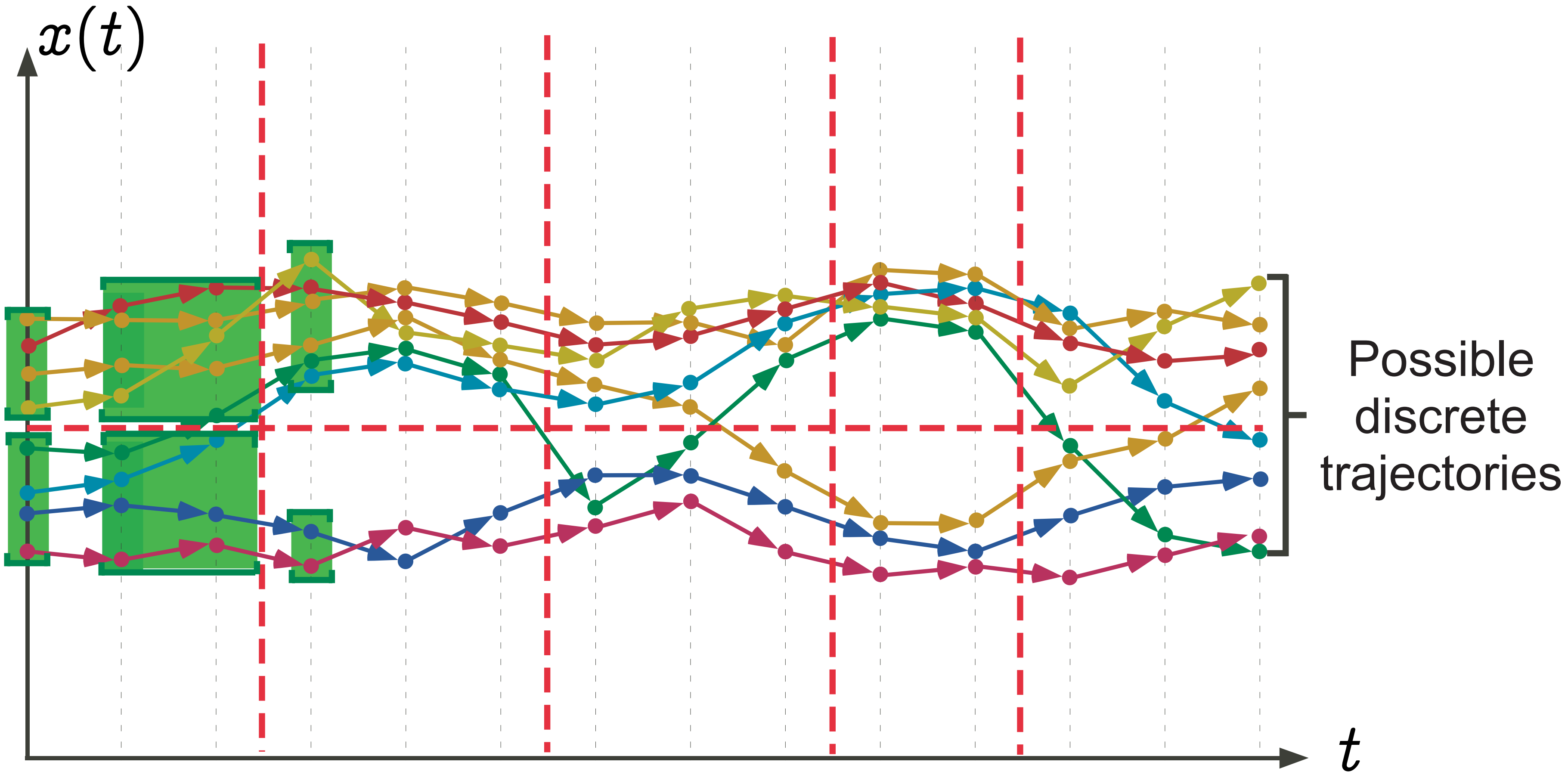
Graphic example: partitionned upward iteration with widening



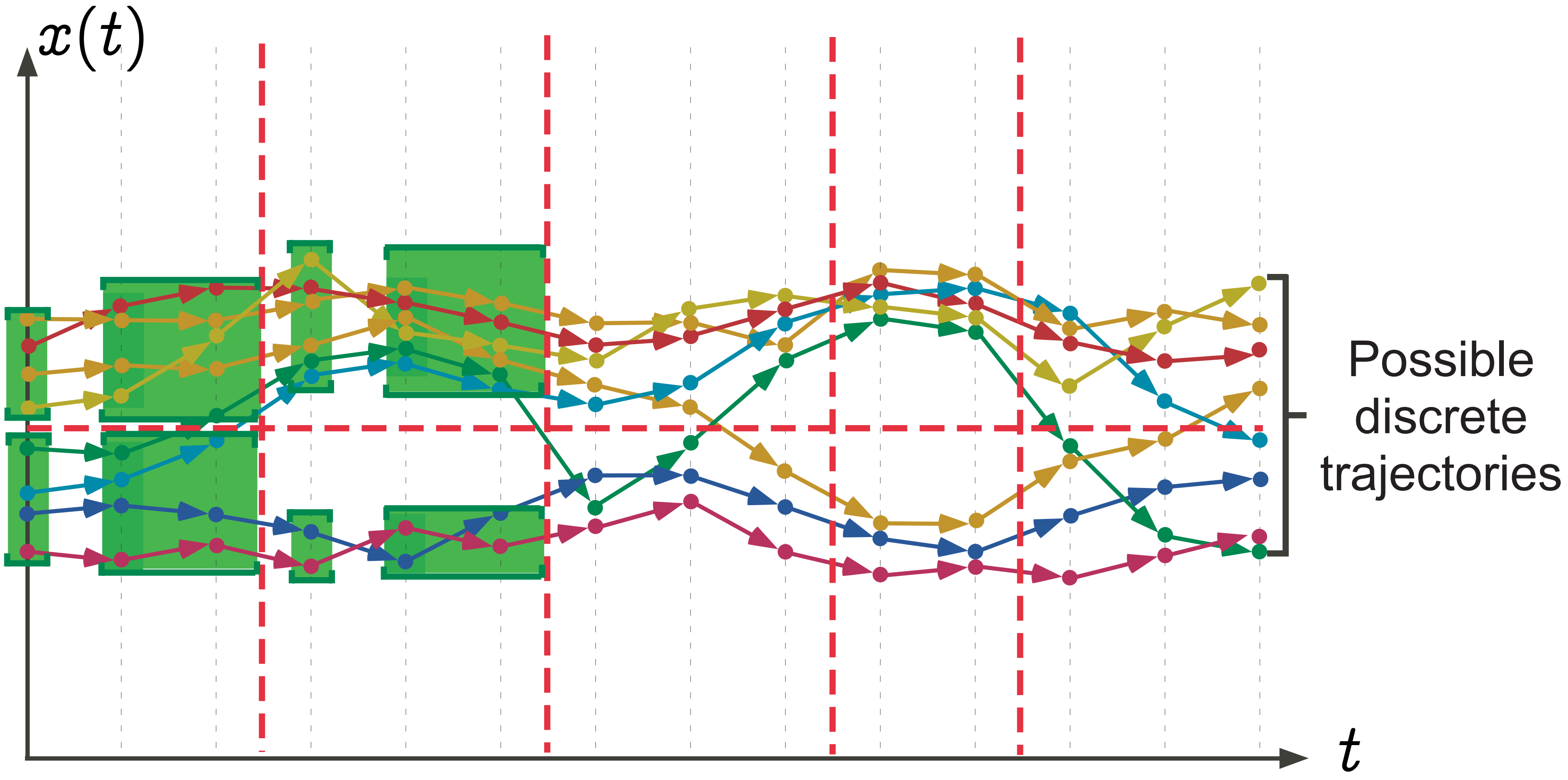
Graphic example: partitionned upward iteration with widening



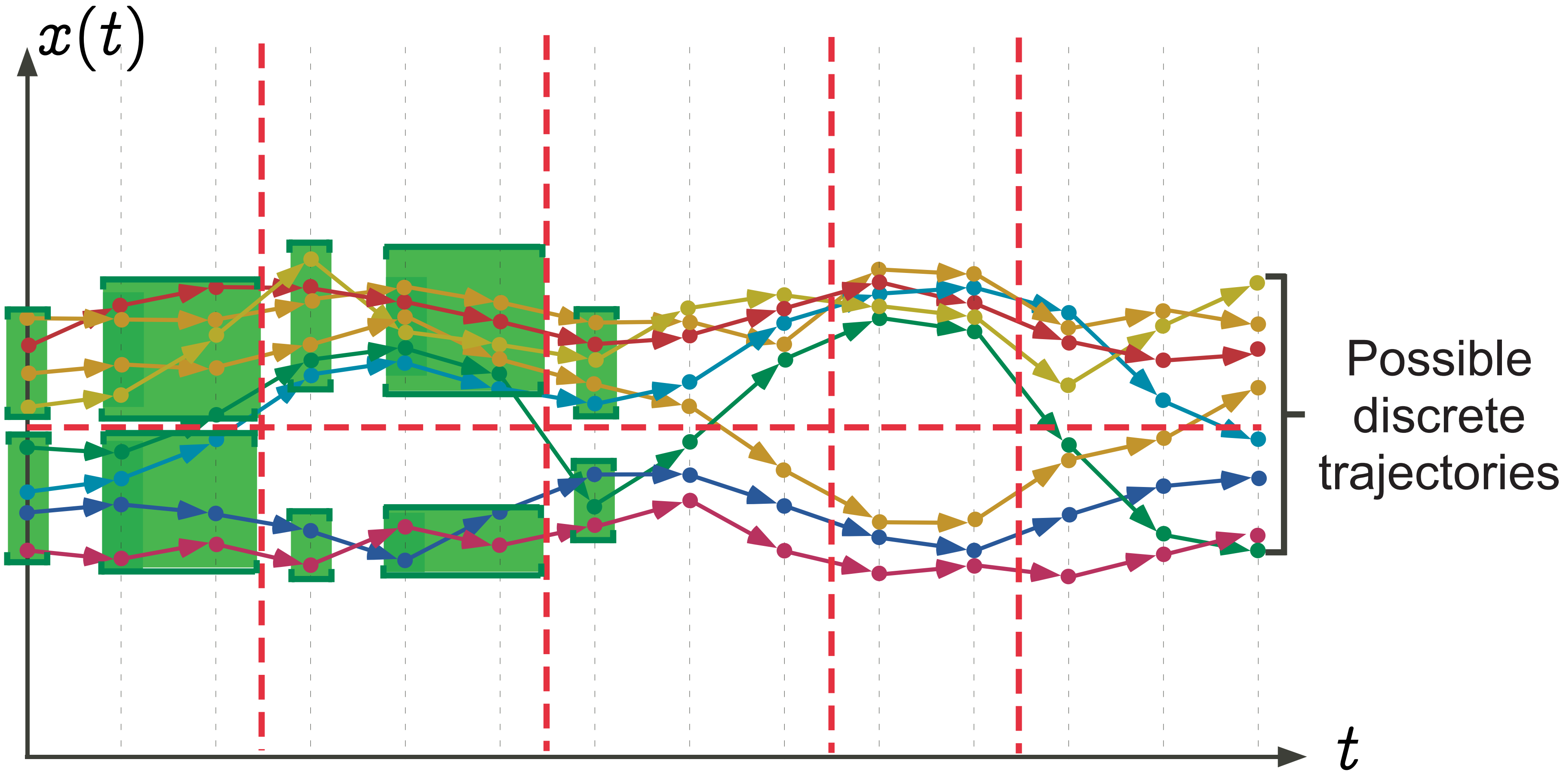
Graphic example: partitionned upward iteration with widening



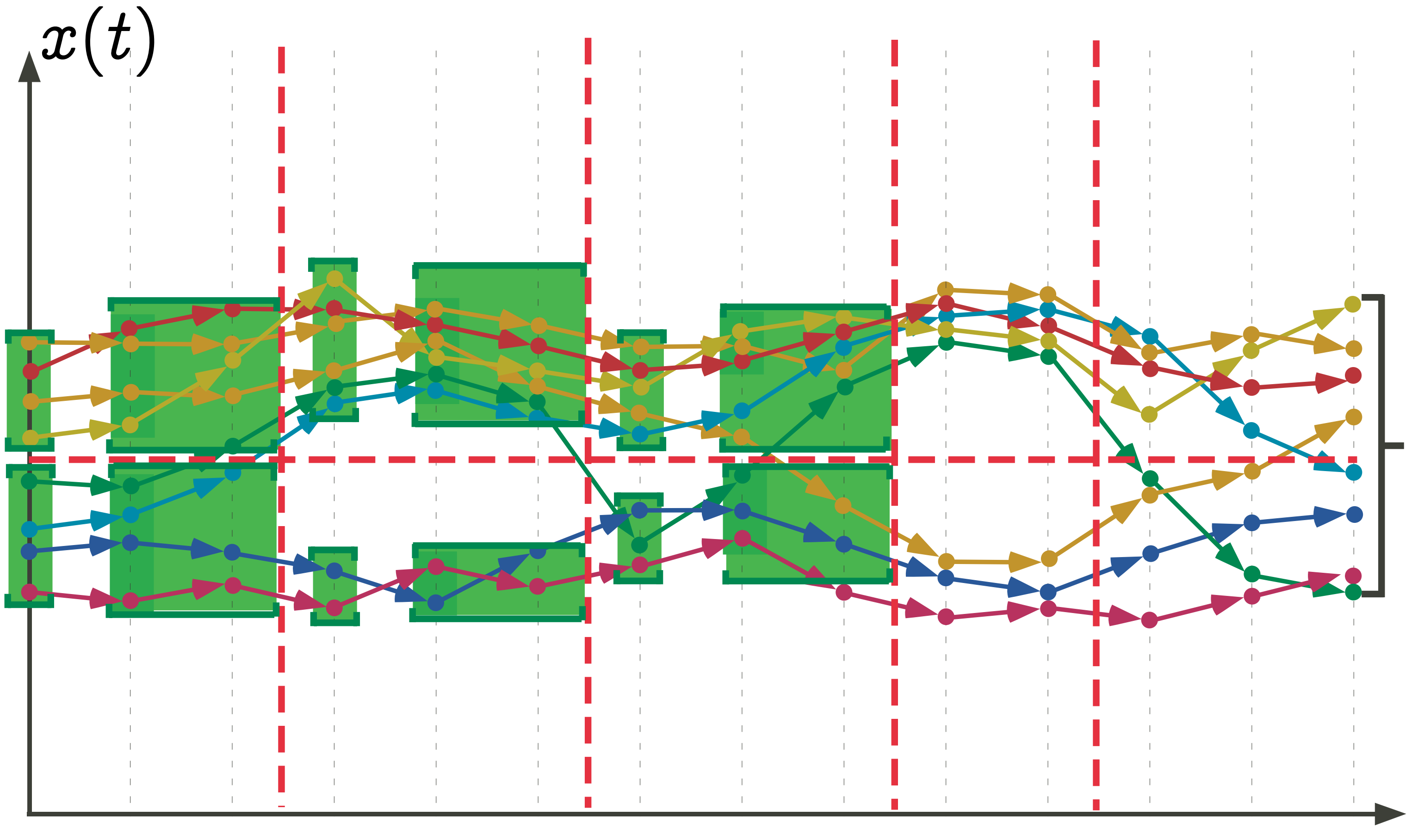
Graphic example: partitionned upward iteration with widening



Graphic example: partitionned upward iteration with widening

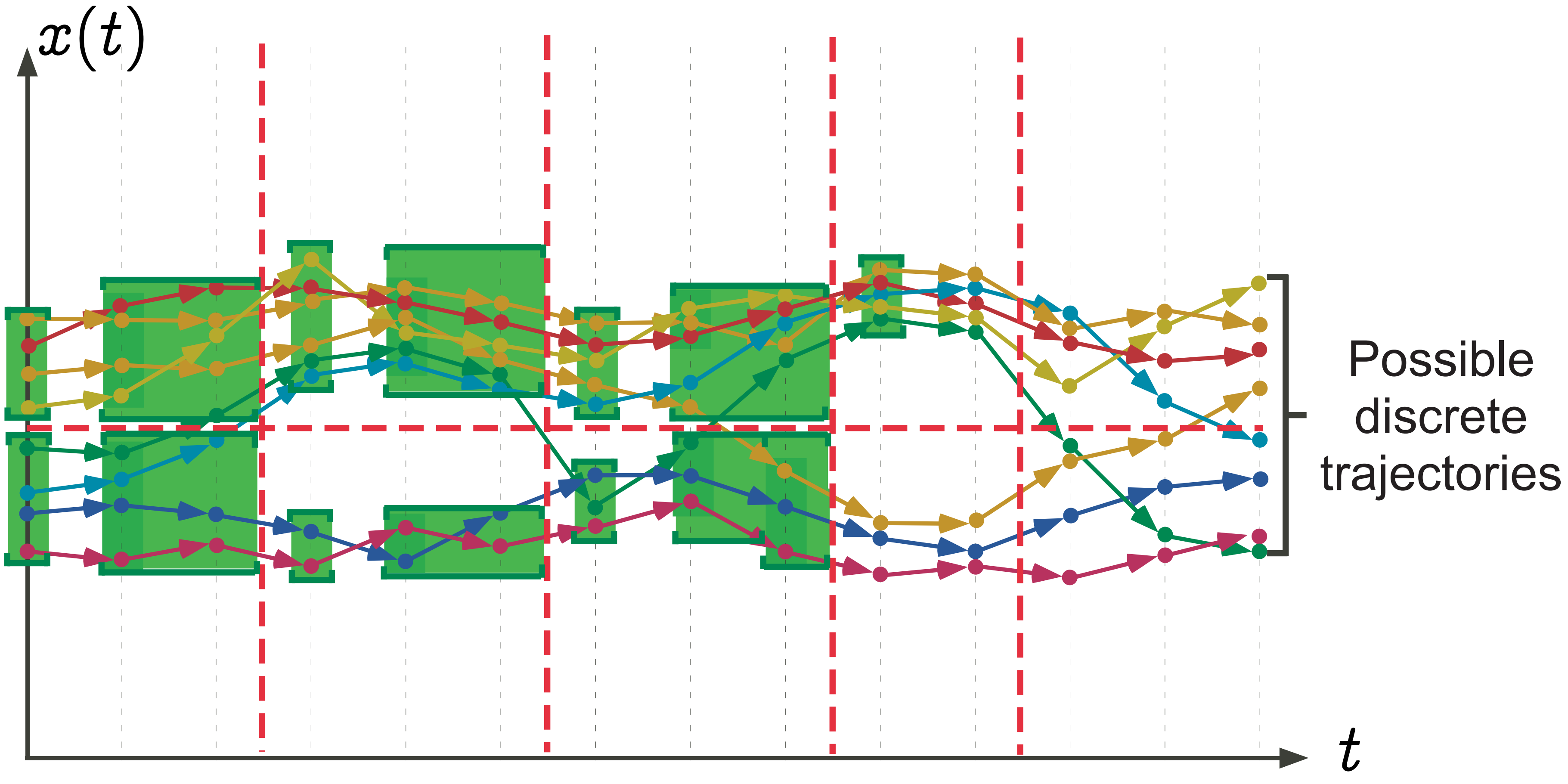


Graphic example: partitionned upward iteration with widening

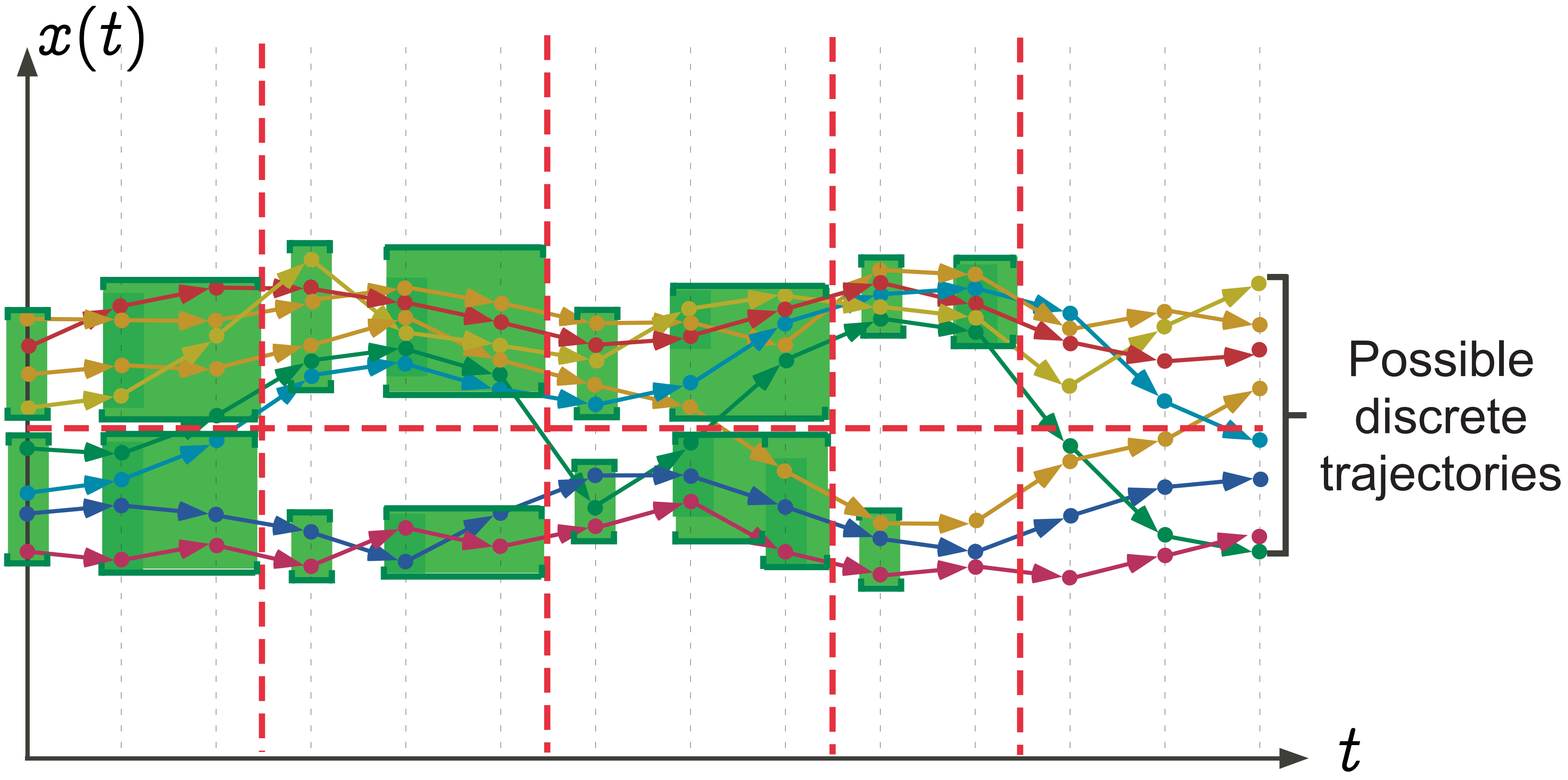


Possible discrete trajectories

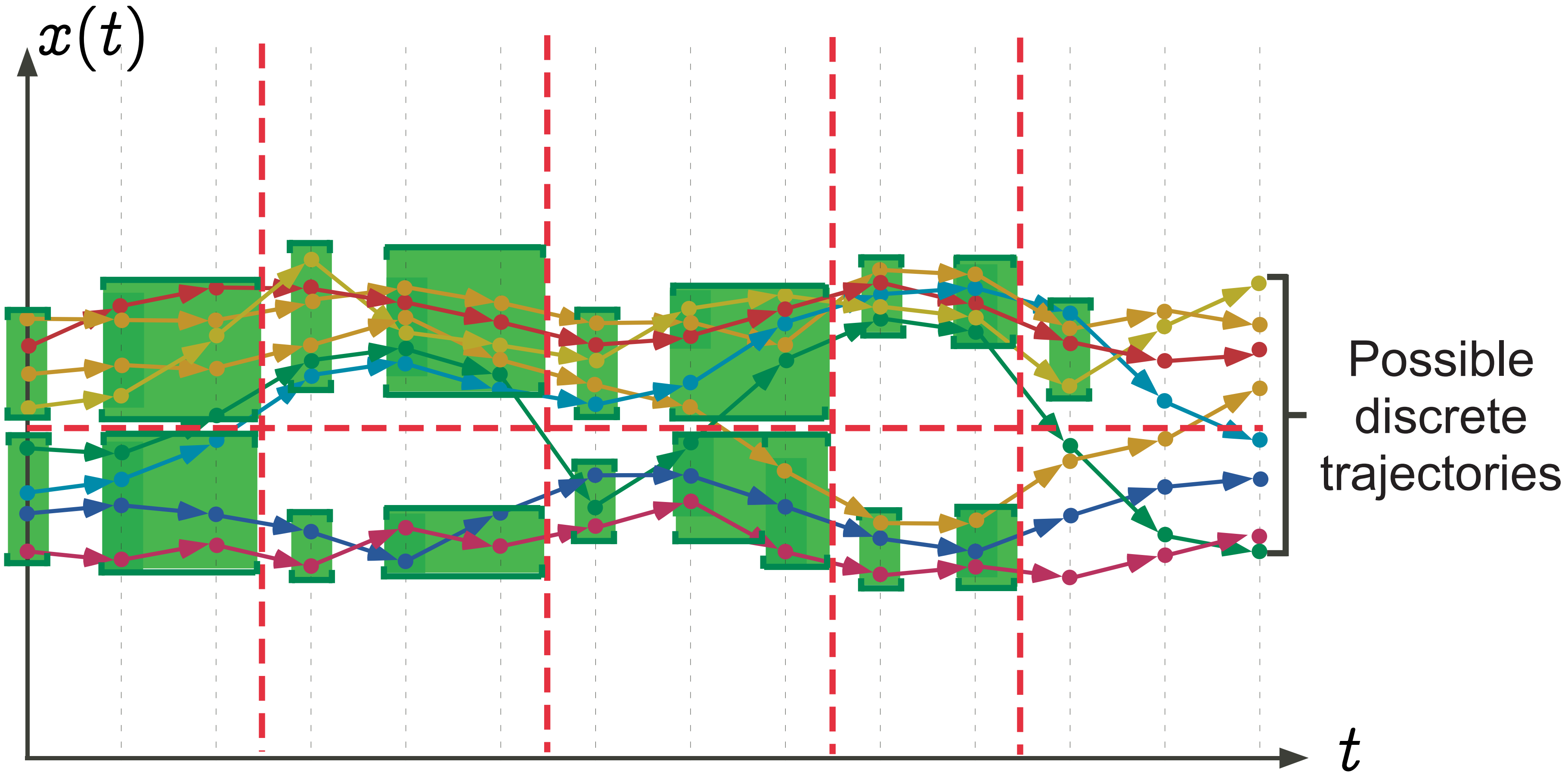
Graphic example: partitionned upward iteration with widening



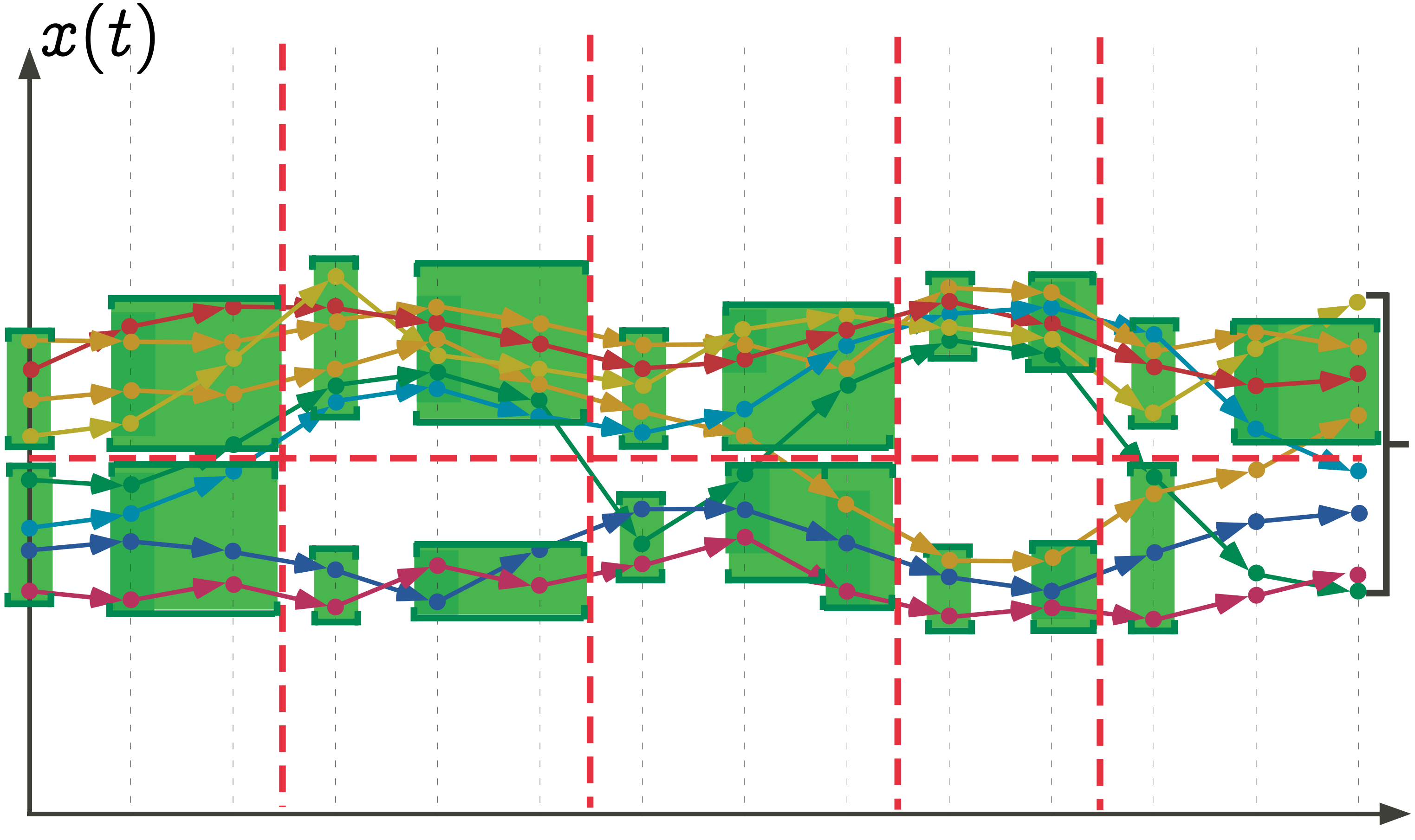
Graphic example: partitionned upward iteration with widening



Graphic example: partitionned upward iteration with widening

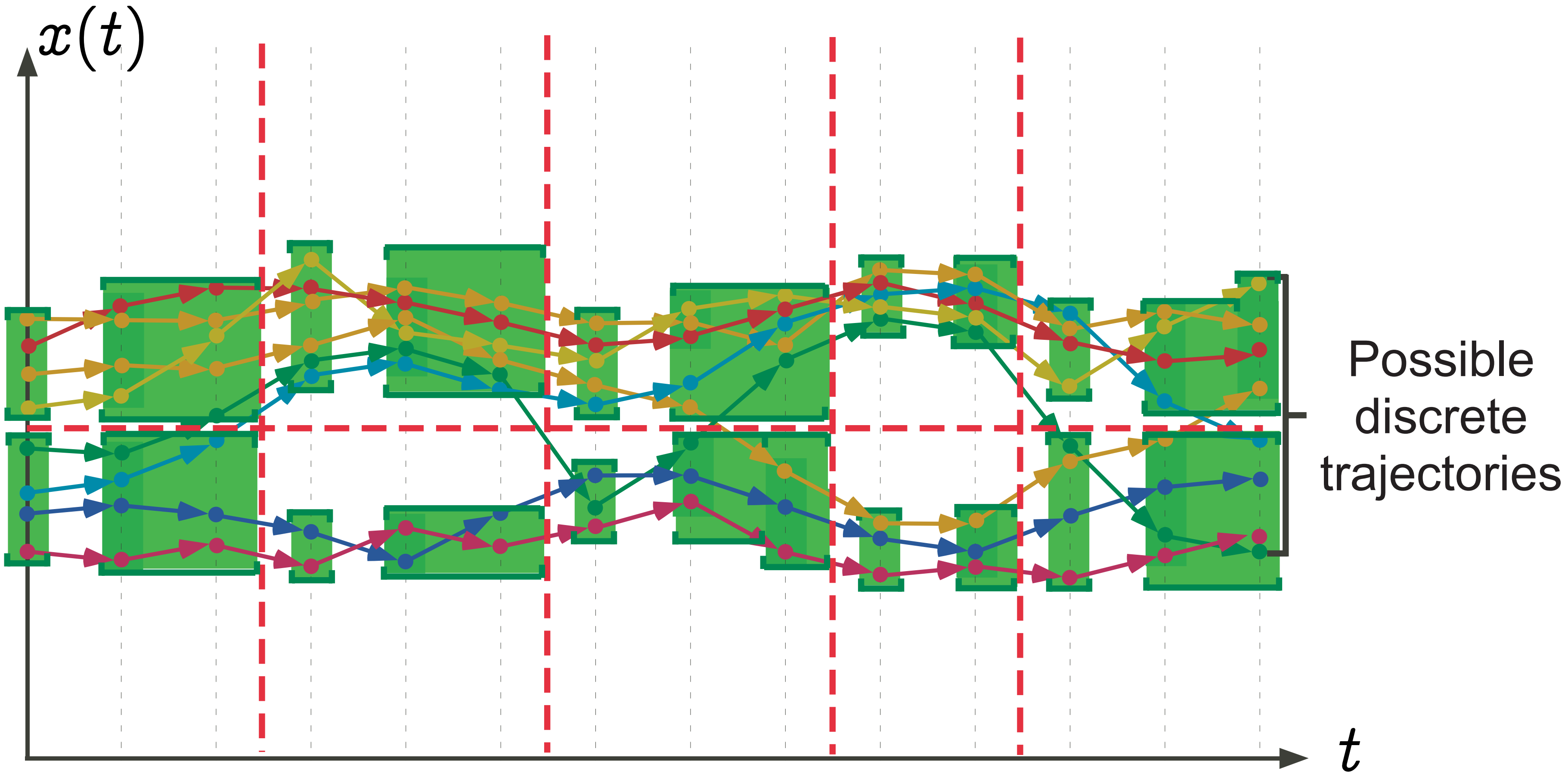


Graphic example: partitionned upward iteration with widening

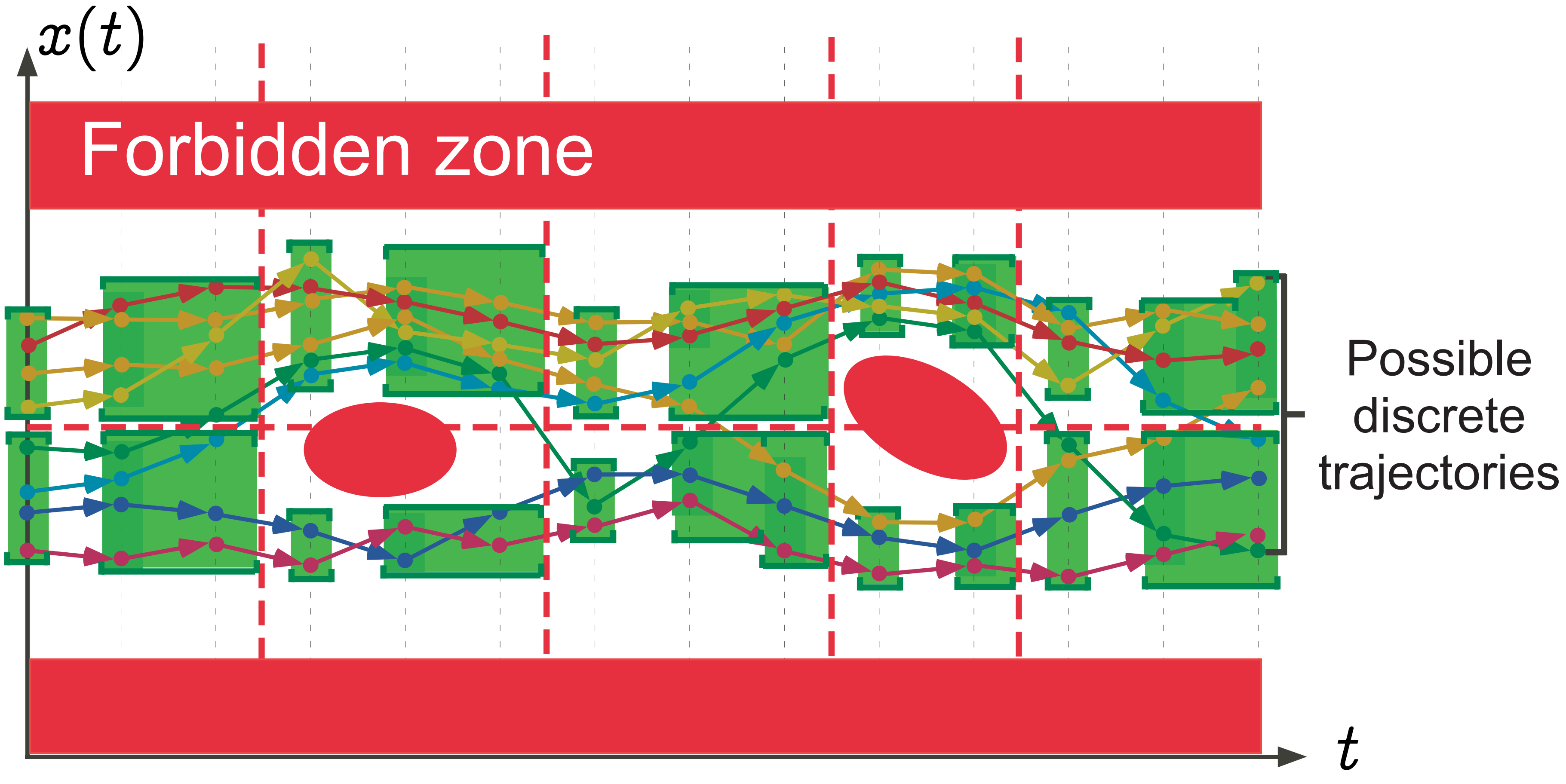


Possible
discrete
trajectories

Graphic example: partitionned upward iteration with widening



Graphic example: safety verification



Examples of partitionnings

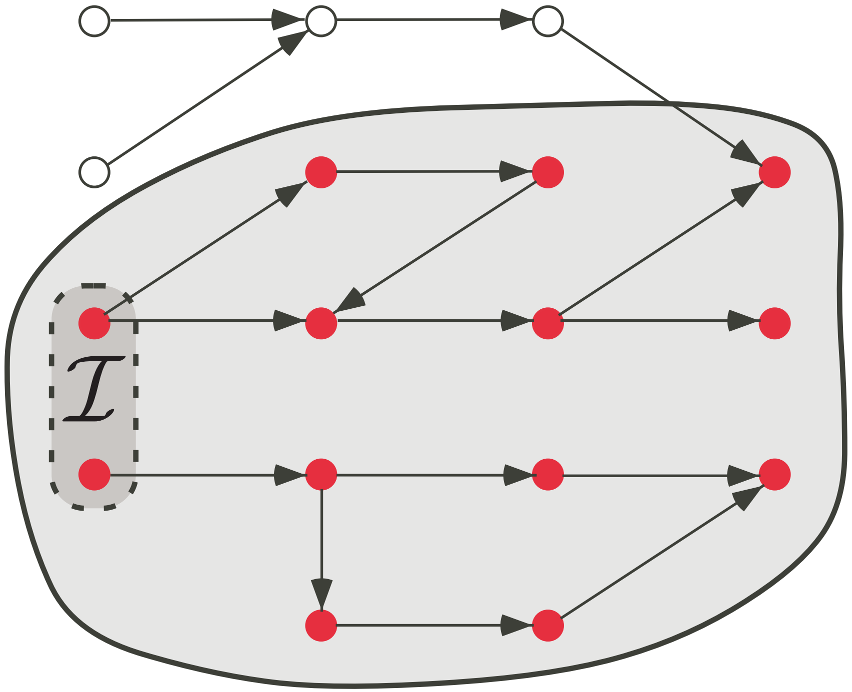
- sets of control states: attach local information to program points instead of global information for the whole program/procedure/loop
- sets of data states:
  - case analysis (test, switches)
- fixpoint iterates:
  - widening with threshold set

Interval widening with threshold set

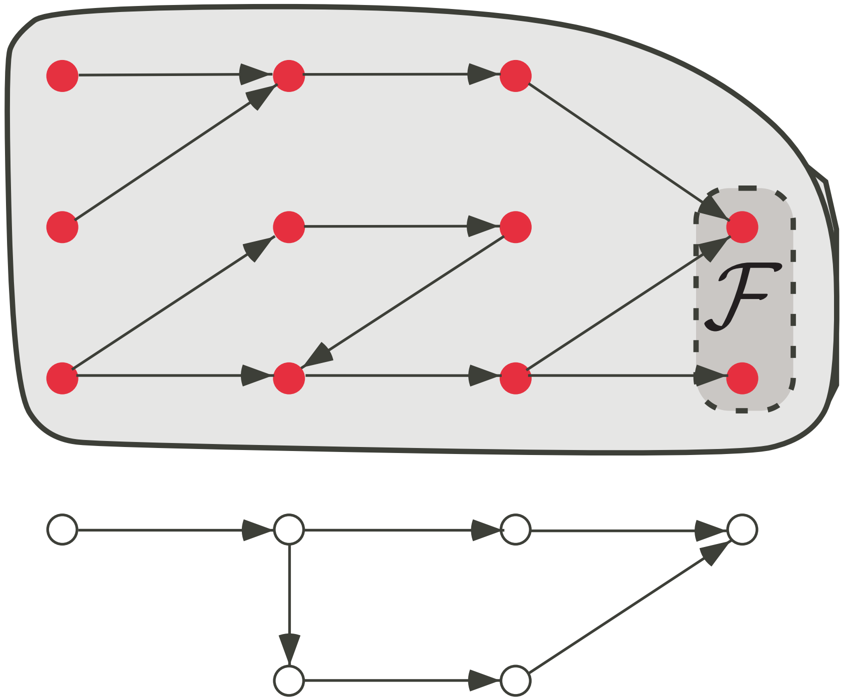
- The threshold set T is a finite set of numbers (plus $+\infty$ and $-\infty$),
- $$[a, b] \mathbin{\nabla_T} [a', b'] = [\begin{array}[t]{l} \mathit{if}\ a' < a\ \mathit{then}\ \max\{\ell \in T \mid \ell \leq a'\} \\ \quad \mathit{else}\ a, \\ \mathit{if}\ b' > b\ \mathit{then}\ \min\{h \in T \mid h \geq b'\} \\ \quad \mathit{else}\ b] \ . \end{array}$$
- Examples (intervals):
  - sign analysis: $T = \{-\infty, 0, +\infty\}$;
  - strict sign analysis: $T = \{-\infty, -1, 0, +1, +\infty\}$;
- T is a parameter of the analysis.

Combinations of abstractions

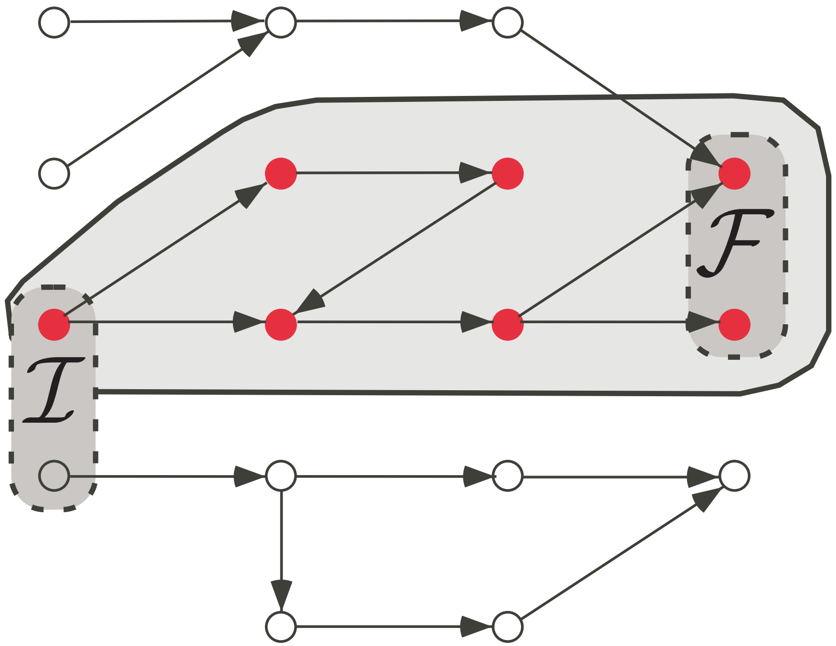
Forward/reachability analysis



Backward/ancestry analysis



Iterated forward/backward analysis



Example of iterated forward/backward analysis

Arithmetical mean of two integers x and y:

```
{x>=y}
  while (x <> y) do
    {x>=y+2}
       x := x - 1;
    {x>=y+1}
       y := y + 1
    {x>=y}
  od
{x=y}
```

Necessarily $x \geq y$ for proper termination

Example of iterated forward/backward analysis

Adding an auxiliary counter `k` decremented in the loop body and asserted to be null on loop exit:

```
{x=y+2k,x>=y}
  while (x <> y) do
    {x=y+2k,x>=y+2}
      k := k - 1;
    {x=y+2k+2,x>=y+2}
      x := x - 1;
    {x=y+2k+1,x>=y+1}
      y := y + 1
    {x=y+2k,x>=y}
  od
{x=y,k=0}
  assume (k = 0)
{x=y,k=0}
```

Moreover the difference of `x` and `y` must be even for proper termination

Bibliography

Seminal papers

- Patrick Cousot & Radhia Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In 4th Symp. on Principles of Programming Languages, pages 238—252. ACM Press, 1977.
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- Patrick Cousot & Radhia Cousot. Systematic design of program analysis frameworks. In 6th Symp. on Principles of Programming Languages pages 269—282. ACM Press, 1979.

Recent surveys

- Patrick Cousot. Interprétation abstraite. Technique et Science Informatique, Vol. 19, Nb 1-2-3. Janvier 2000, Hermès, Paris, France. pp. 155-164. 🇫🇷
- Patrick Cousot. Abstract Interpretation Based Formal Methods and Future Challenges. In Informatics, 10 Years Back — 10 Years Ahead, R. Wilhelm (Ed.), LNCS 2000, pp. 138-156, 2001.
- Patrick Cousot & Radhia Cousot. Abstract Interpretation Based Verification of Embedded Software: Problems and Perspectives. In Proc. 1st Int. Workshop on Embedded Software, EMSOFT 2001, T.A. Henzinger & C.M. Kirsch (Eds.), LNCS 2211, pp. 97–113. Springer, 2001.

Conclusion

Theoretical applications of abstract interpretation

- **Static Program Analysis** [POPL '77,78,79] inluding **Data-flow Analysis** [POPL '79,00], **Set-based Analysis** [FPCA '95], etc
- **Syntax Analysis** [TCS 290(1) 2002]
- **Hierarchies of Semantics (including Proofs)** [POPL '92, TCS 277(1–2) 2002]
- **Typing** [POPL '97]
- **Model Checking** [POPL '00]
- **Program Transformation** [POPL '02]
- **Software watermarking** [POPL '04]

Practical applications of abstract interpretation

- **Program analysis and manipulation**: a small rate of false alarms is acceptable
  - AiT: worst case execution time – Christian Ferdinand
- **Program verification**: no false alarms is acceptable
  - TVLA: A system for generating abstract interpreters – Mooly Sagiv
  - Astrée: verification of absence of run-time errors – Laurent Mauborgne

Industrial applications of abstract interpretation

- Both to Program analysis and verification
- Experience with the industrial use of abstract interpretation-based static analysis tools – Jean Souyris (Airbus France)

THE END

More references at URL www.di.ens.fr/~cousot.