for some matrix A and vector b, i.e. if P is the intersection of finitely many affine half-spaces. Here an *affine half-space* is a set of the form $\{x \mid wx \leqslant \delta\}$ for some nonzero row vector w and some number δ. If (9) holds, we say that $Ax \leqslant b$ *defines* or *determines* P. Trivially, each polyhedral cone is a polyhedron.

A set of vectors is a (*convex*) *polytope* if it is the convex hull of finitely many vectors.

It is intuitively obvious that the concepts of polyhedron and of polytope are related. This is made more precise in the Decomposition theorem for polyhedra (Motzkin [1936]) and in its direct corollary, the Finite basis theorem for polytopes (Minkowski [1896], Steinitz [1916], Weyl [1935]).

Corollary 7.1b (Decomposition theorem for polyhedra). *A set P of vectors in Euclidean space is a polyhedron, if and only if $P = Q + C$ for some polytope Q and some polyhedral cone C.*

Proof. I. First let $P = \{x \mid Ax \leqslant b\}$ be a polyhedron in $\mathbb{R}^n$. By Corollary 7.1a the polyhedral cone

$$(10) \qquad \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \middle| x \in \mathbb{R}^n; \lambda \in \mathbb{R}; \lambda \geqslant 0; Ax - \lambda b \leqslant 0 \right\}$$

is generated by finitely many vectors, say by $\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix}$. We may assume that each λ_i is 0 or 1. Let Q be the convex hull of the x_i with $\lambda_i = 1$, and let C be the cone generated by the x_i with $\lambda_i = 0$. Now $x \in P$, if and only if $\begin{pmatrix} x \\ 1 \end{pmatrix}$ belongs to (10), and hence, if and only if $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \text{cone} \left\{ \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix} \right\}$. It follows directly that $P = Q + C$.

II. Let $P = Q + C$ for some polytope Q and some polyhedral cone C. Say $Q = \text{conv.hull}\{x_1, \ldots, x_m\}$ and $C = \text{cone}\{y_1, \ldots, y_t\}$. Then a vector x_0 belongs to P, if and only if

$$(11) \qquad \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \in \text{cone} \left\{ \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ 1 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} y_t \\ 0 \end{pmatrix} \right\}.$$

By Corollary 7.1a, the cone in (11) is equal to $\left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \middle| Ax + \lambda b \leqslant 0 \right\}$ for some matrix A and vector b. Hence $x_0 \in P$, if and only if $Ax_0 \leqslant -b$, and therefore P is a polyhedron. □

We shall say that P is *generated by the points* $x_1, \ldots, x_m$ and *by the directions* $y_1, \ldots, y_t$ if.

$$(12) \qquad P = \text{conv.hull}\{x_1, \ldots, x_m\} + \text{cone}\{y_1, \ldots, y_t\}.$$

This gives a 'parametric' description of the solution set of a system of linear inequalities. For more about decomposition of polyhedra, see Section 8.9 below.