# Using First-order Theorem Provers in Verification

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**Our result**

We formally verified a checker for priority queues in the first-order theorem prover Saturate.

# Priority Queues

## Definition

`p_queue<E,I>` is a standard data structure, where $E$ is a linearly ordered set (priorities), and $I$ is the type of data (items) stored in the queue. It supports the following operations:

- create
- insert$(e, i)$
- delete$(e, i)$
- find_min
- del_min

## Note

The results returned by find_min and del_min have to be checked

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## Problem

This is too costly. PQ finds minimal element in time $O(\log(n))$. Going through all elements takes time $O(n)$.

## Solution

Postpone detection of errors. An error is detected, not when a non-minimal element is returned, but at the moment when a smaller element is returned.

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- during the execution of a correct priority queue, one collects knowledge about elements in it.
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- the lower bound of an element is the maximal priority reported by all `del_min` operations after the insertion of the element.

➜ priority queue checker maintains system of lower bounds

# Priority Queue Checker

## Example

PRIORITY QUEUE OPERATIONS:

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Q.insert(3);
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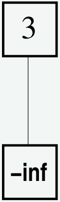

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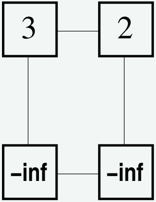

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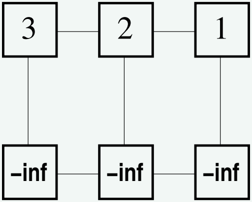

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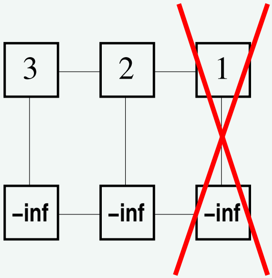

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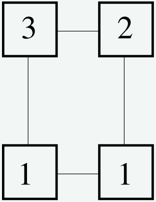

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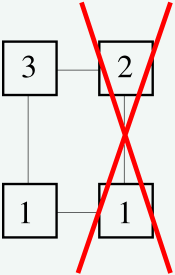

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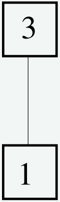

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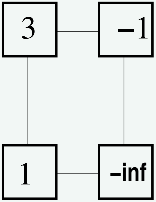

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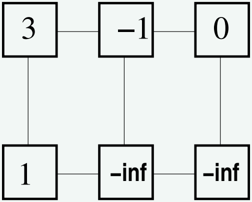

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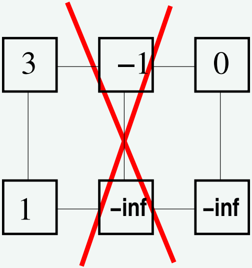

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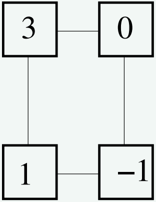

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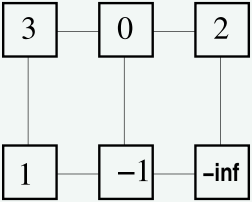

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**Priority queue operations:**

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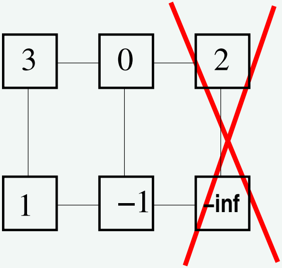
**Lower bounds system:**



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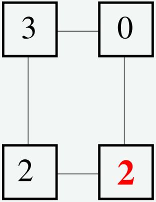

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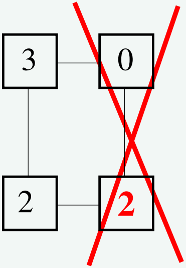

# Verification Process

## Ideal workflow

1. express correctness theorem as first-order formula $F$
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## Problems in real life

- expressing correctness is non-trivial
  - what is a priority queue?
  - what is the checker doing?
- theorem prover cannot handle induction
  ➜ generate induction hypotheses by hand
- theorem prover runs out of memory
  ➜ guide theorem prover by manually inserting lemmas

# Priority Queues in FOL

## Inductive definition of priority queues

Let $E$ be a linearly ordered set. The set of priority queues $PQ$ is the smallest set satisfying:

- $\texttt{empty} \in PQ$
- if $q \in PQ$ and $e \in E$ then $\texttt{insert}(q, e) \in PQ$

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## Specification of delete

$$\texttt{delete}(\texttt{empty}, e) = \texttt{empty}$$
$$\texttt{delete}(\texttt{insert}(q, e), e) = q$$
$$e_1 \neq e_2 \rightarrow$$
$$\quad \texttt{delete}(\texttt{insert}(q, e_1), e_2) = \texttt{insert}(\texttt{delete}(q, e_2), e_1)$$

# Lower Bounds System in FOL

## Definition of lower bound systems

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LBS $= (E \times E)^*$

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## Specification of `adjust`

$\texttt{adjust}(\epsilon, e) = \epsilon$
$e_3 < e_1 \rightarrow \texttt{adjust}(l.(e_2, e_3), e_1) = \texttt{adjust}(l, e_1).(e_2, e_3)$
$e_1 \leq e_3 \rightarrow \texttt{adjust}(l.(e_2, e_3), e_1) = \texttt{adjust}(l, e_1).(e_2, e_3)$

# Defining the Goal

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- a sequence is alert free if every time the element is accessed, it is greater or equal to its lower bound.
- a sequence of commands is complete if after its execution the queue is empty.
- a sequence of commands is correct if every `del_min` operation returns indeed the minimal element.

# Alert free Sequences

## Sequence of Operations

Sequence of operations = word over the alphabet

$$\Sigma = \{\text{ins}(e), \text{del}(e), \text{dmin} \mid e \in E\}$$

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## Alert free sequence

$\texttt{alert\_free}(\epsilon)$
$\texttt{alert\_free}(s.\text{ins}(e)) \Leftrightarrow \texttt{alert\_free}(s)$
$\texttt{alert\_free}(s.\text{del}(e)) \Leftrightarrow$
$\quad \texttt{alert\_free}(s) \wedge \texttt{lower\_bound}(e, \texttt{ex1}(s, \epsilon)) \leq e$
$\texttt{alert\_free}(s.\text{dmin}) \Leftrightarrow$
$\quad \texttt{alert\_free}(s) \wedge \texttt{lower\_bound}(e^*, \texttt{ex1}(s, \epsilon)) \leq e^*$
where $e^* = \texttt{elem}(\texttt{ex}(s.\text{dmin}, \mathrm{empty}))$

# Correctness Theorem

**Theorem**

*Let* $s \in \Sigma^*$ *then*
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**Proof**

The theorem is not inductive, so we cannot prove it. We need to formulate a more general (but inductive) theorem.

# Correctness Theorem

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*Let* $s \in \Sigma^*$ *then*

$$\texttt{alert\_free}(s) \wedge$$
$$\forall e \Big( \texttt{contains}(\texttt{queue}(\texttt{ex}(s, \texttt{empty})), e) \Rightarrow$$
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**Proof**

By induction on the length of word $s$.

# Further Contributions

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- We developed a specification that closely follows the concrete implementation and data structure in LEDA.
- We developed a framework that allows partial functions and non-deterministic behaviour
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## TPTP 2006

- The problem is being added to TPTP library
- Currently it is the only benchmark for lemma handling