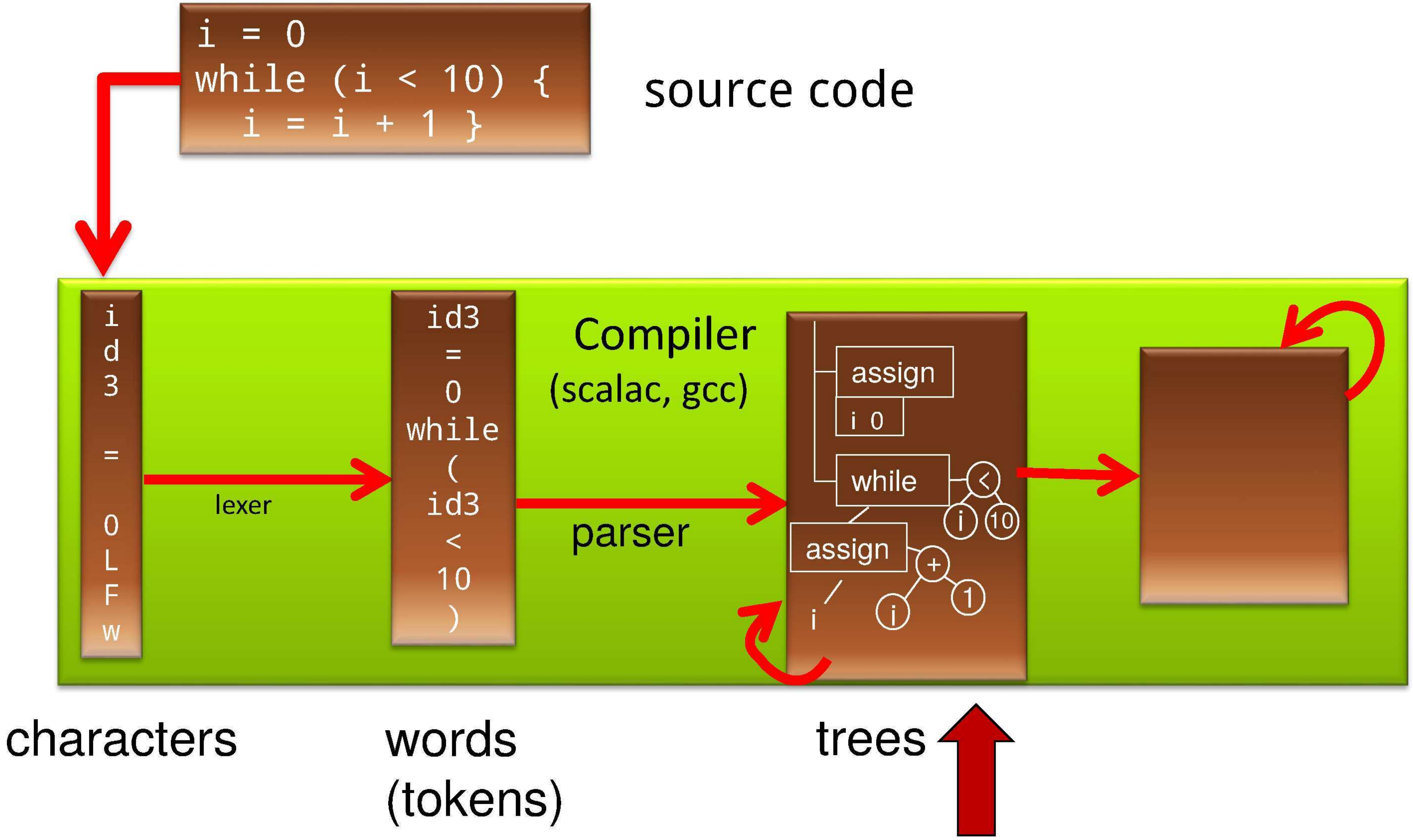


Type Checking

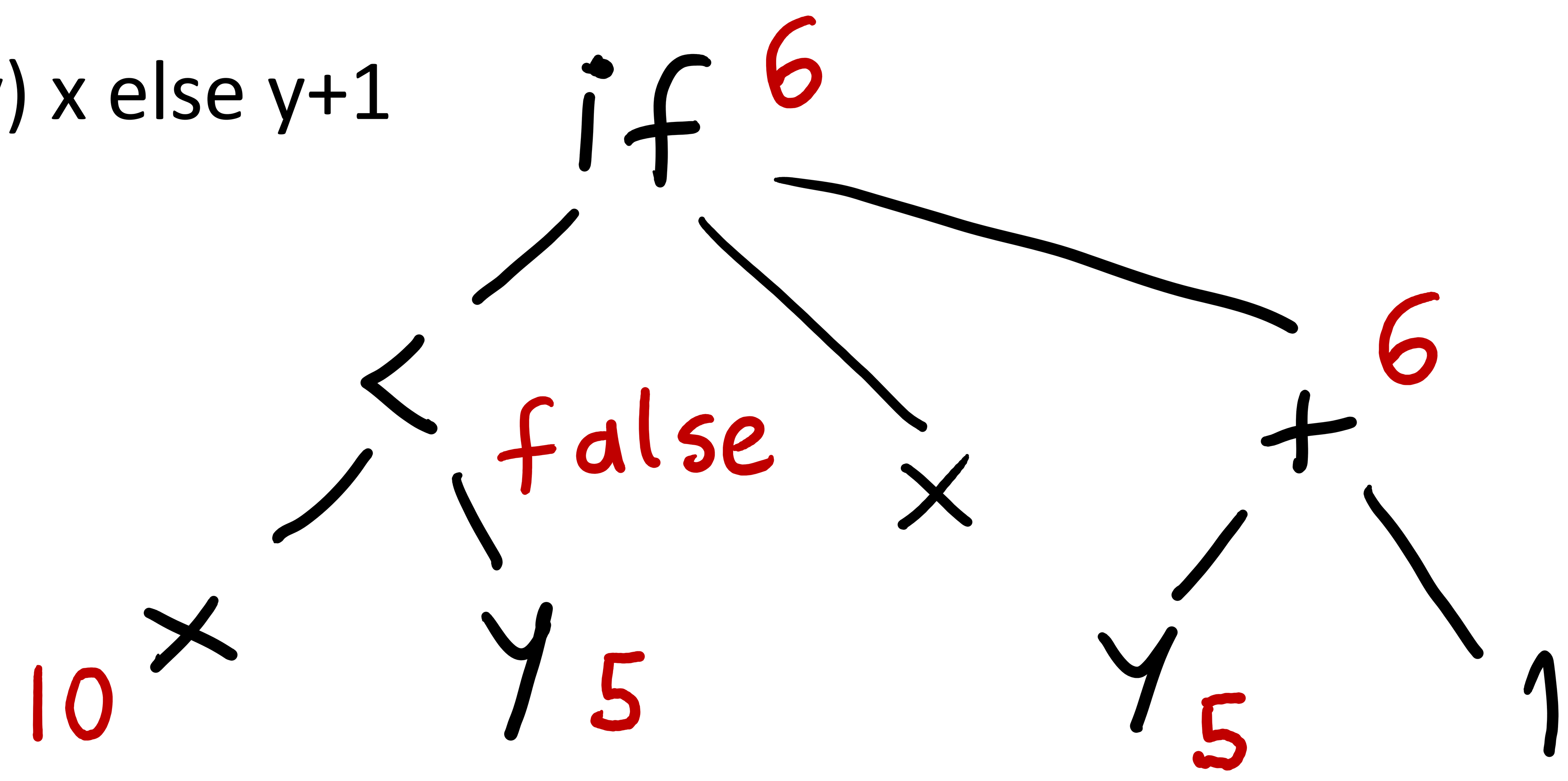
Evaluating an Expression

scala prompt:

>def min1(x : Int, y : Int) : Int = { if (x < y) x else y+1 }
min1: (x: Int,y: Int)Int
>min1(10,5)
res1: Int = 6

How can we think about this evaluation?

x → 10
y → 5
if (x < y) x else y+1



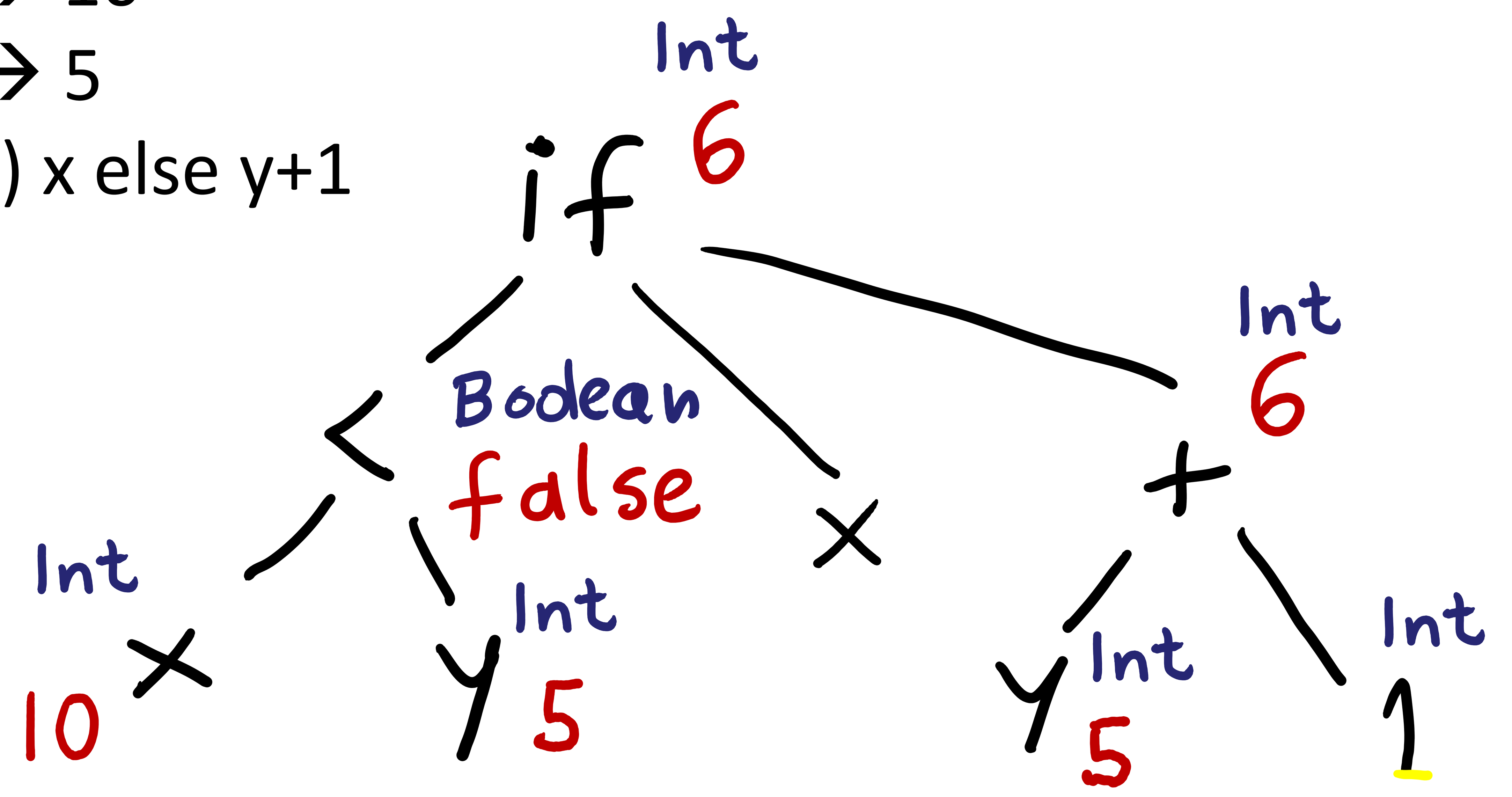
Computing types using the evaluation tree

scala prompt:

>def min1(x : Int, y : Int) : Int = { if (x < y) x else y+1 }
min1: (x: Int,y: Int)Int
>min1(10,5)
res1: Int = 6

How can we think about this evaluation?

x : Int → 10
y : Int → 5
if (x < y) x else y+1



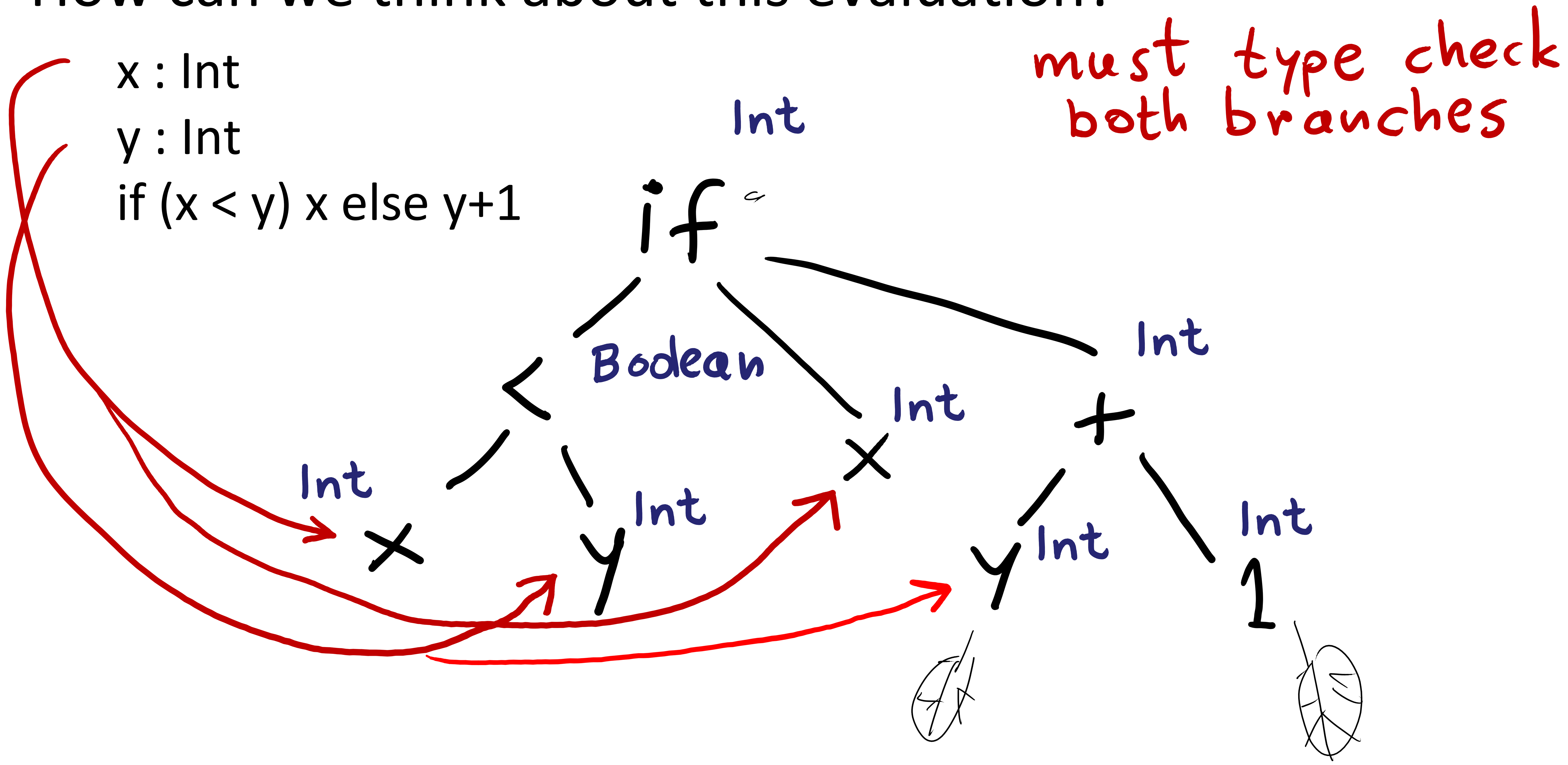
We can compute types without values

scala prompt:

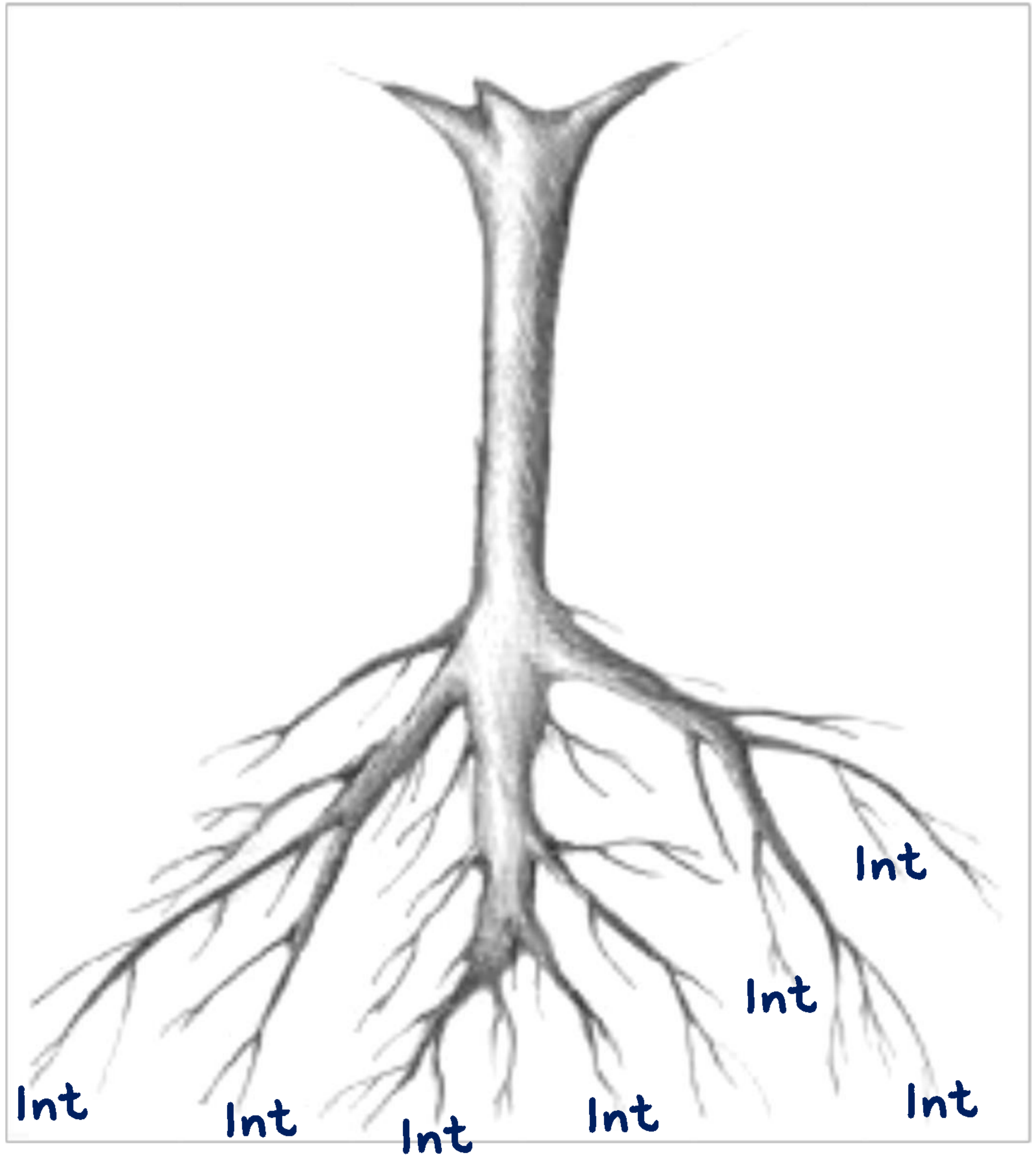
```
>def min1(x : Int, y : Int) : Int = { if (x < y) x else y+1 }
min1: (x: Int,y: Int)Int
>min1(10,5)
res1: Int = 6
```

How can we think about this evaluation?

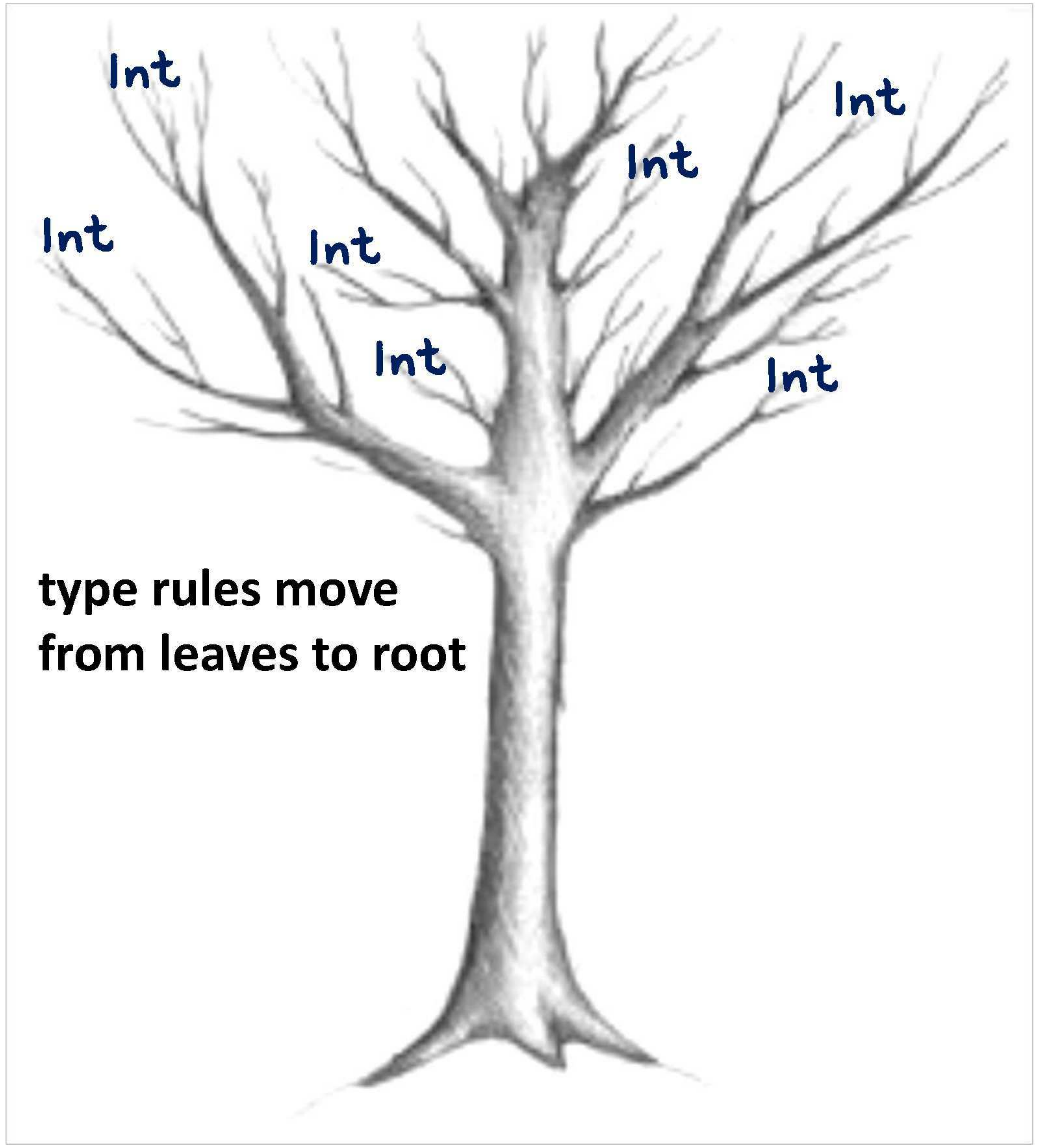
x : Int
y : Int
if (x < y) x else y+1

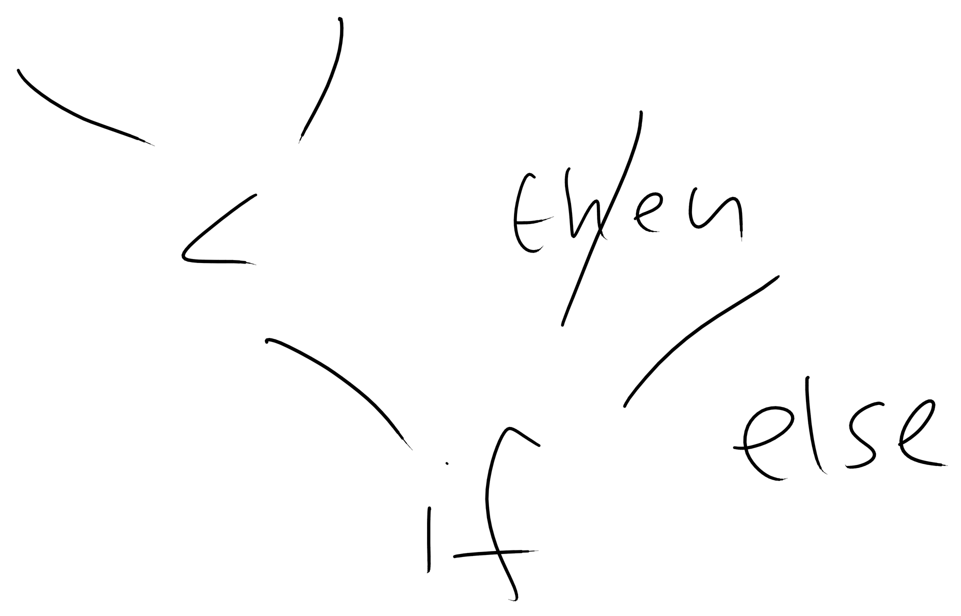


We do not like trees upside-down



Leaves are Up





Type Judgements and Type Rules

Int   Int

<: boolean

- e type checks to T under Γ (type environment)

$$\Gamma \vdash e : T$$

  - Types of constants are predefined
  - Types of variables are specified in the source code
- If e is composed of sub-expressions

$$\frac{\Gamma \vdash e_1 : T_1 \cdots \Gamma \vdash e_n : T_n}{\Gamma \vdash e : T}$$

type check from leaves

Type Judgements and Type Rules

$$\Gamma \vdash e : T$$

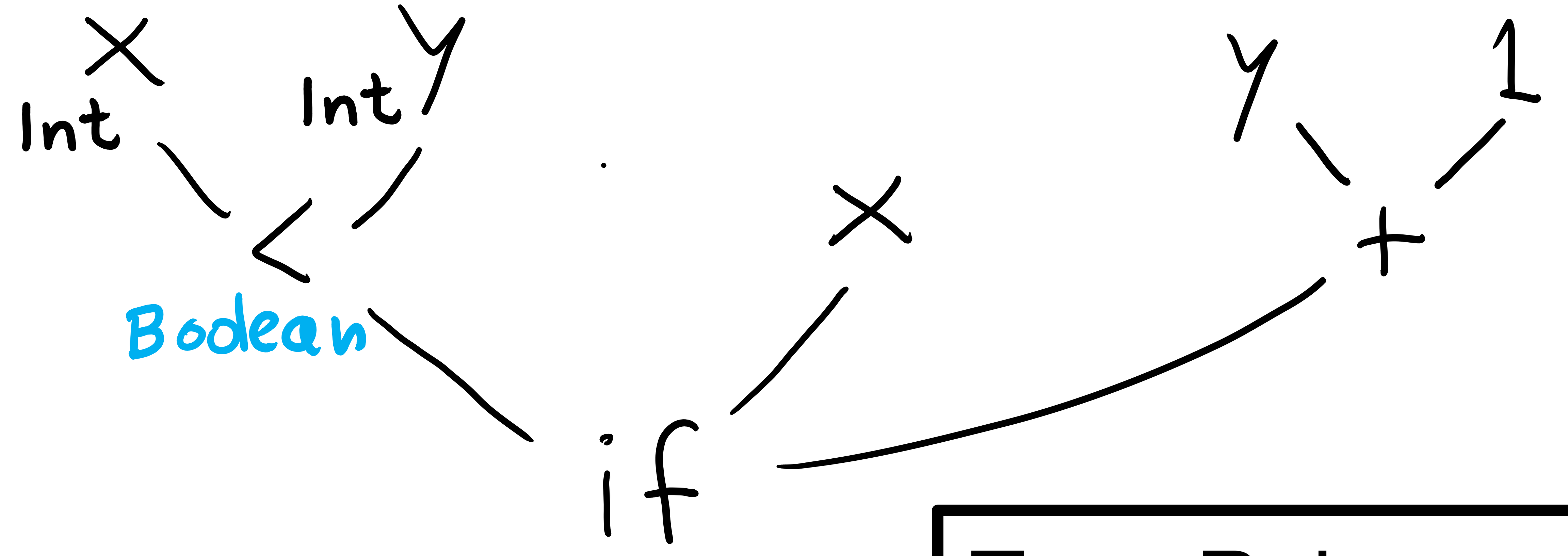
if the (free) variables of e have types given by the type environment gamma, then e (correctly) type checks and has type T

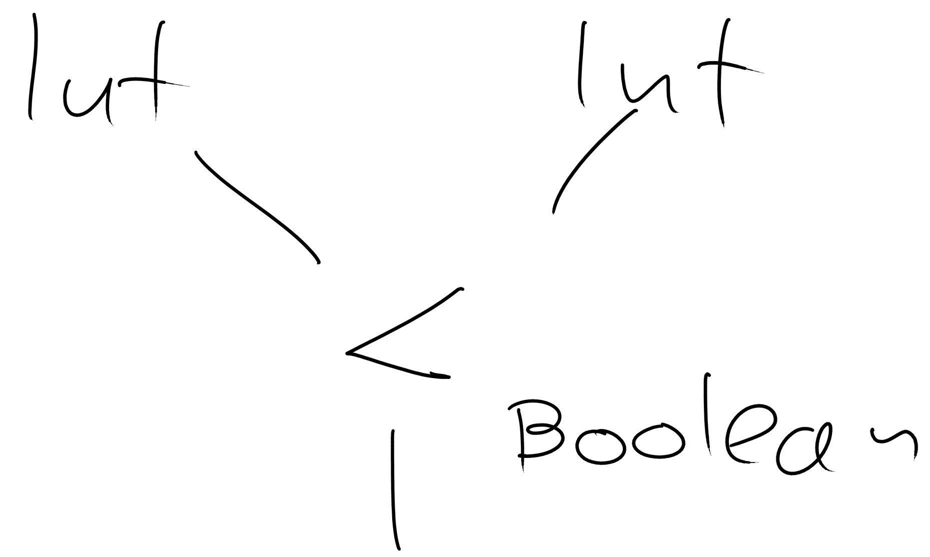
type rule
$$\frac{\Gamma \vdash e_1 : T_1 \quad \cdots \quad \Gamma \vdash e_n : T_n}{\Gamma \vdash e : T}$$

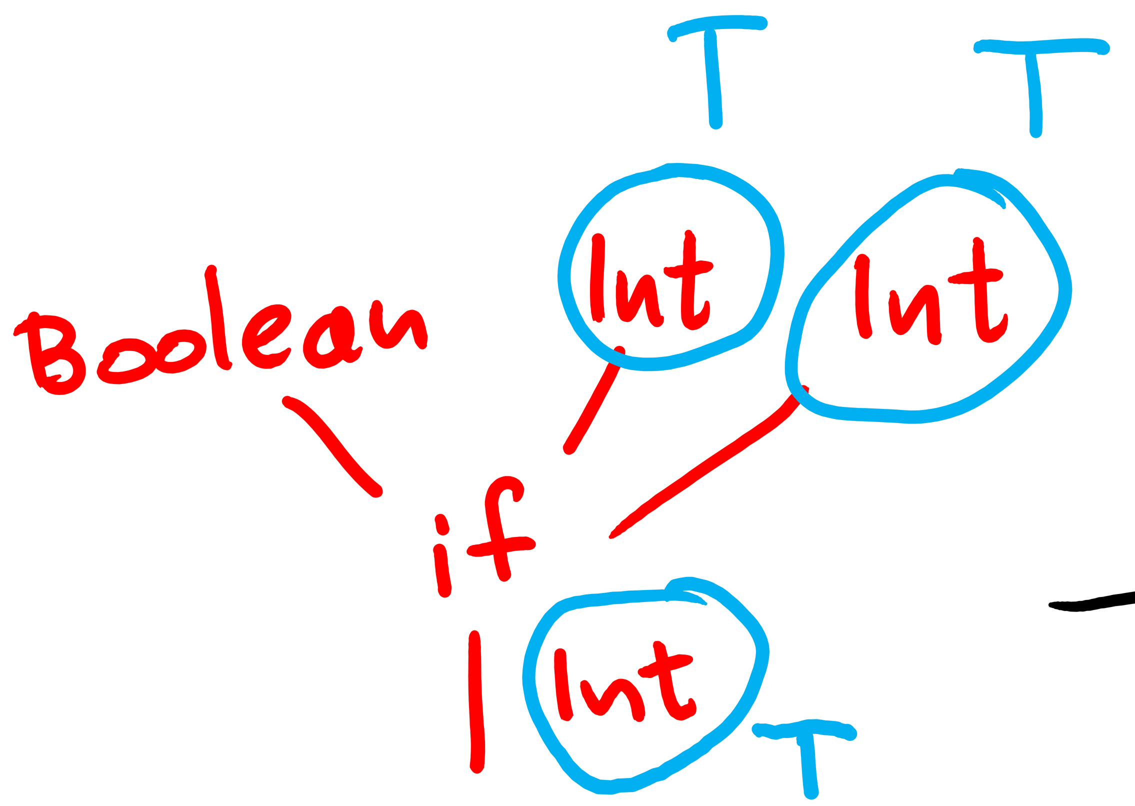
If e_1 type checks in gamma and has type T_1 and ...
and e_n type checks in gamma and has type T_n
then e type checks in gamma and has type T

Type Rules as Local Tree Constraints

x : Int
y : Int







Type Rules

$$\frac{e_1 : \mathrm{Int} \qquad e_2 : \mathrm{Int}}{e_1 < e_2 : \mathrm{Boolean}}$$

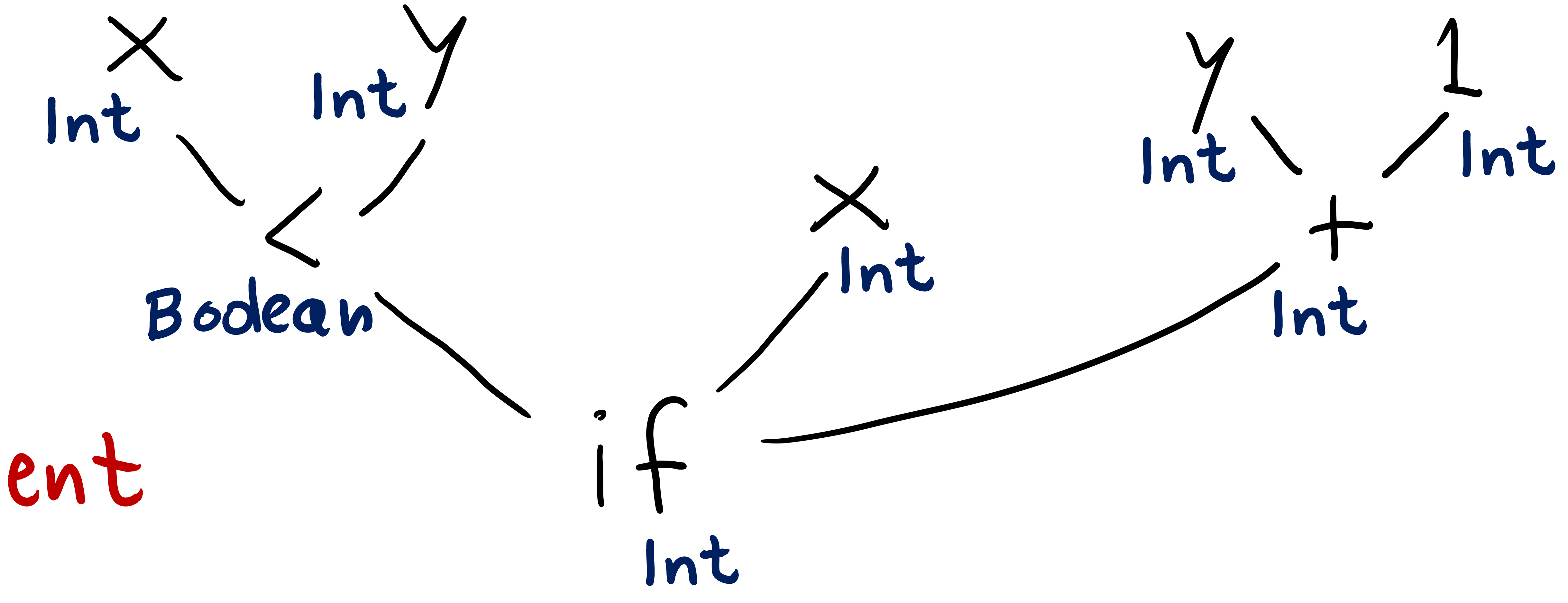
for every type T, if
b has type Boolean, and ...
then

$$\frac{b : \mathrm{Boolean} \qquad e_1 : T \qquad e_2 : T}{\mathrm{if}(b)\ e_1\ \mathrm{else}\ e_2 : T}$$

Type Rules with Environment

x : Int
y : Int

type environment
Γ



Type Rules

$$\frac{(x:T)\in\Gamma}{\Gamma\vdash x:T}$$

$$\frac{}{\mathrm{IntConst}(k):\mathrm{Int}}$$

$$\frac{\Gamma\vdash e_1:\mathrm{Int} \qquad \Gamma\vdash e_2:\mathrm{Int}}{\Gamma\vdash (e_1<e_2):\mathrm{Boolean}}$$

...(then) in the (same) environment Γ the expression $e_1<e_2$ has type Bool.

$$\frac{\Gamma\vdash e_1:\mathrm{Int} \qquad \Gamma\vdash e_2:\mathrm{Int}}{\Gamma\vdash (e_1+e_2):\mathrm{Int}}$$

$$\frac{\Gamma\vdash b:\mathrm{Boolean} \qquad \Gamma\vdash e_1:T \qquad \Gamma\vdash e_2:T}{\Gamma\vdash (\mathrm{if}\ (b)\ e_1\ \mathrm{else}\ e_2):T}$$

Type Checker Implementation Sketch

def typeCheck(Γ : Map[ID, ~~Type~~ TypeTree], e : ExprTree) : TypeTree = {

e **match** {

    case Var(id) => { ?? }

    case If(c,e1,e2) => { ?? }

    ...

}}

    case Var(id) => { Γ(id) **match**

      case Some(t) => t

      case None => error(UnknownIdentifier(id,id.pos))

    }

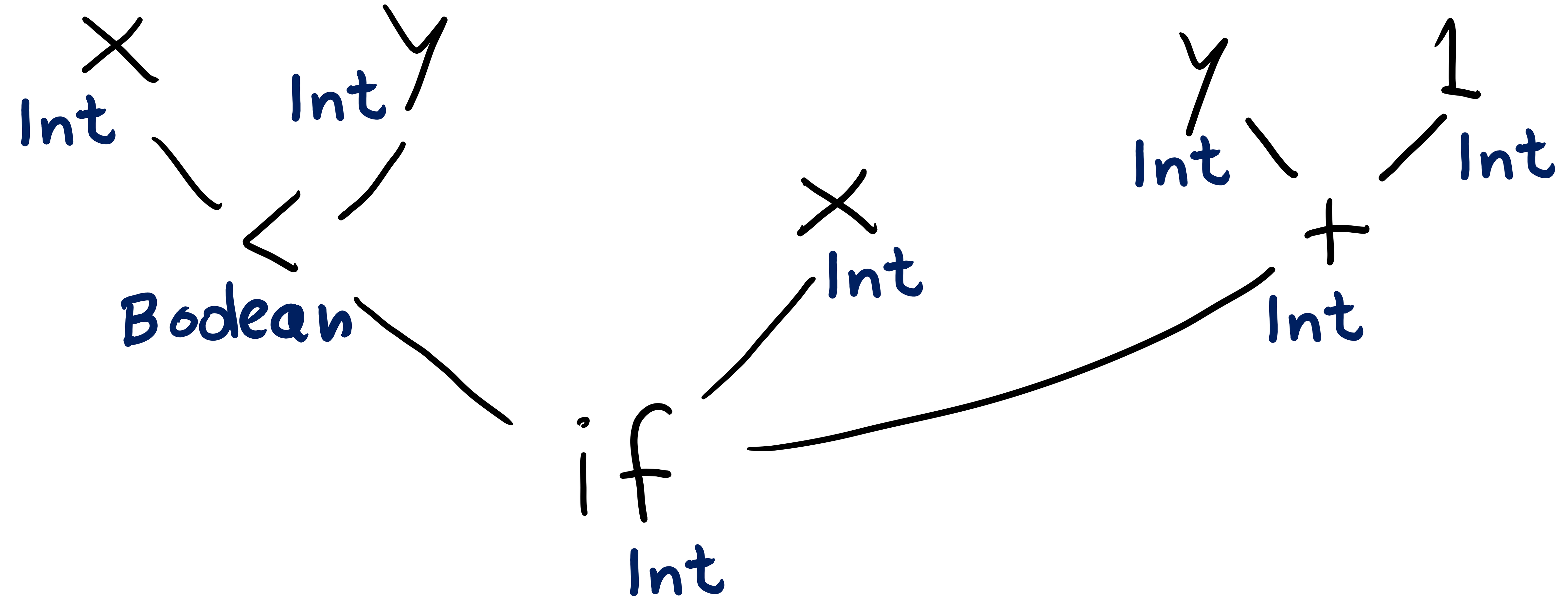
Type Checker Implementation Sketch

- **case** If(c,e1,e2) => {

```
val tc = typeCheck(Γ,c)
if (tc != BooleanType) error(IfExpectsBooleanCondition(e.pos))
val t1 = typeCheck(Γ, e1); val t2 = typeCheck(Γ, e2)
if (t1 != t2) error(IfBranchesShouldHaveSameType(e.pos))
t1
}
```

Derivation Using Type Rules

x : Int
y : Int



Let $\Gamma = \{(x,Int),(y,Int)\}$

$$\dfrac{\dfrac{\dfrac{(x,Int)\in\Gamma}{\Gamma\vdash x:Int}\qquad\dfrac{(y,Int)\in\Gamma}{\Gamma\vdash y:Int}}{\Gamma\vdash (x<y):Boolean}\qquad \boxed{\dfrac{(x,Int)\in\Gamma}{\Gamma\vdash x:Int}}\qquad \dfrac{\dfrac{(y,Int)\in\Gamma}{\Gamma\vdash y:Int}\qquad \dfrac{}{\Gamma\vdash 1:Int}}{\Gamma\vdash (y+1):Int}}{\Gamma\vdash \underbrace{(\text{if}(x<y)\ x\ \text{else}\ y+1)}:Int}$$

Type Rule for Function Application

$$\frac{\Gamma \vdash e_1 : T_1 \quad \cdots \quad \Gamma \vdash e_n : T_n \quad \Gamma \vdash f : (T_1 \times \cdots \times T_n) \to T}{\Gamma \vdash f(e_1, \cdots, e_n) : T}$$

Type Rule for Function Application [Cont.]

$<$

f

We can treat operators as variables that have function type

$$+ : Int \times Int \rightarrow Int$$
$$< : Int \times Int \rightarrow Boolean$$
$$\&\& : Boolean \times Boolean \rightarrow Boolean$$

We can replace many previous rules with application rule:

$$\frac{\Gamma \vdash e_1 : T_1 \quad \ldots \quad \Gamma \vdash e_n : T_n \quad \Gamma \vdash f : ((T_1 \times \cdots \times T_n) \rightarrow T)}{\Gamma \vdash f(e_1, \ldots, e_n) : T}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \mathrm{Bool} \quad \Gamma \vdash \&\& : (\mathrm{Bool} \times \mathrm{Bool}) \rightarrow \mathrm{Bool}}{\Gamma \vdash e_1 \ \&\& \ e_2 : \mathrm{Bool}}$$

Computing the Environment of a Class

```
object World {
  var data : Int
  var name : String
  def m(x : Int, y : Int) : Boolean { ... }
  def n(x : Int) : Int {
    if (x > 0) p(x – 1) else 3
  }
  def p(r : Int) : Int = {
     var k = r + 2
     m(k, n(k))
  }
}
```

$\Gamma_0 = \{$

(data, Int),

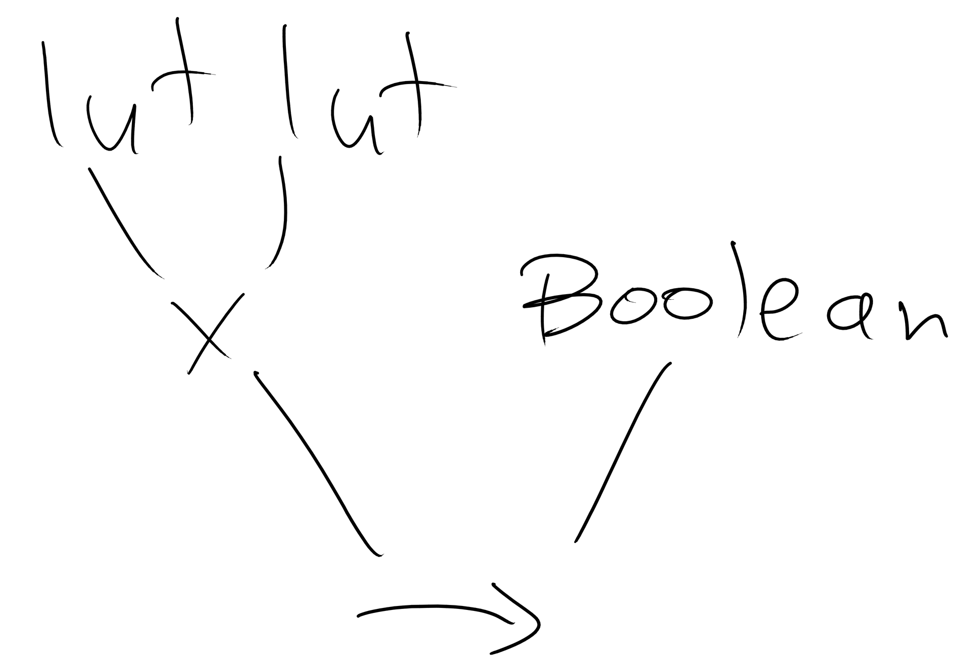
(name, String),

(m, Int × Int → Boolean),

(n, Int → Int),

(p, Int → Int)

$\}$



We can type check each function m,n,p in this global environment

Extending the Environment

$\Gamma_0 = \{$

$(data, Int),$

$(name, String),$

$(m, Int \times Int \to Boolean),$

$(n, Int \to Int),$

$(p, Int \to Int) \}$

```
class World {
  var data : Int
  var name : String
  def m(x : Int, y : Int) : Boolean { ... }
  def n(x : Int) : Int {
    if (x > 0) p(x – 1) else 3
  }
  def p(r : Int) : Int = {
    var k:Int
    k = r + 2
    m(k, n(k))
  }
}
```

$\leftarrow \Gamma_0$

$\leftarrow \Gamma_1 = \Gamma_0 \oplus \{(r, Int)\}$

$\leftarrow \Gamma_2 = \Gamma_1 \oplus \{(k, Int)\} = \Gamma_0 \cup \{(r, Int), (k, Int)\}$

Type Rule for Method Definitions $\text{def } m(x_1 : T_1, \cdots, x_n : T_n) : T = e$

$$\frac{\Gamma \oplus \{(x_1,T_1),\dots,(x_n,T_n)\} \vdash e : T}{\Gamma \vdash (\text{def } m(x_1:T_1,\dots,x_n:T_n) : T = e) : OK}$$

Type Rule for Assignments

$$\frac{(x,T) \in \Gamma \qquad \Gamma \vdash e : T}{\Gamma \vdash (x = e) : \text{void}}$$

Unit

Type Rules for Block: { var x_1:T_1 … var x_n:T_n; s_1; … s_m; e }

$$\frac{\underbrace{\Gamma \oplus \{(x_1,T_1),\dots,(x_n,T_n)\}}_{\Gamma_1} \qquad \begin{array}{l} \Gamma_1 \vdash s_1 : \text{void} \\ \dots \\ \Gamma_1 \vdash s_n : \text{void} \\ \Gamma_1 \vdash e : T \end{array}}{\Gamma \vdash \{\underbrace{\text{var } x_1 : T_1; \dots; \text{var } x_n : T_n}; \underbrace{s_1; \dots; s_n}; e\} : T}$$

Blocks with Declarations in the Middle

$$\frac{\Gamma \vdash e : T}{\Gamma \vdash \{e\} : T}$$

just expression

$$\frac{}{\Gamma \vdash \{\} : \mathrm{void}}$$

empty

$$\frac{\Gamma \oplus \{(x, T_1)\} \vdash \{t_2; \ldots; t_n\} : T}{\Gamma \vdash \{\mathrm{var}\ x : T_1; t_2; \ldots; t_n\} : T}$$

declaration is first

$$\frac{\Gamma \vdash s_1 : \mathrm{void} \qquad \Gamma \vdash \{t_2; \ldots; t_n\} : T}{\Gamma \vdash \{s_1; t_2; \ldots; t_n\} : T}$$

statement is first

Rule for While Statement

$$\frac{\Gamma \vdash b : \text{Boolean} \qquad \Gamma \vdash s : \text{void}}{\Gamma \vdash (\text{while}\ (b)\ s) : \text{void}}$$

Rule for a Method Call

```
class T0 {
  ...
  def m(x1:T1,...,xn:Tn):T = {
    ...
  }
  ...
}
```

$$\frac{\Gamma \vdash x : T_0 \qquad \Gamma_{T_0} \vdash m : T_0 \times T_1 \times \ldots \times T_n \to T \qquad \overset{\forall i \in \{1,2,\ldots,n\}}{\Gamma \vdash e_i : T_i}}{\Gamma \vdash x.m(e_1,\ldots,e_n) : T}$$

$m(x, e_1, \ldots, e_n)$