Ex 1

Part 1

$$\frac{}{\Gamma \vdash true : Boolean} \qquad \frac{}{\Gamma \vdash false : Boolean}$$

$$\frac{c_e \text{ is an integer literal}}{\Gamma \vdash c_e : Integer} \qquad \frac{\Gamma \vdash t_1 : Integer \quad \Gamma \vdash t_2 : Integer}{\Gamma \vdash t_1 == t_2 : Boolean}$$

$$\frac{\Gamma \vdash t_1 : Boolean \quad \Gamma \vdash t_2 : Boolean}{\Gamma \vdash t_1 == t_2 : Boolean} \qquad \frac{\Gamma \vdash t_1 : Integer \quad \Gamma \vdash t_2 : Integer}{\Gamma \vdash t_1 + t_2 : Integer}$$

$$\frac{\Gamma \vdash t_1 : Boolean \quad \Gamma \vdash t_2 : Boolean}{\Gamma \vdash t_1 \,\&\&\, t_2 : Boolean} \qquad \frac{(x, T) \in \Gamma}{\Gamma \vdash x : T}$$

$$\frac{\Gamma \vdash t_1 : Boolean \quad \Gamma \vdash t_2 : T \quad \Gamma \vdash t_3 : T}{\Gamma \vdash if\ (t_1)\ t_2\ else\ t_3 : T}$$

$$\frac{\Gamma \vdash f : (T_1, \dots, T_n) \Rightarrow T \quad \Gamma \vdash x_1 : T_1 \ \dots \ \Gamma \vdash x_n : T_n}{\Gamma \vdash f(x_1 \dots x_n) : T}$$

Part 2

We denote by

$$e_1 [X := e_2] \mapsto e_3$$

The fact that e_1 with all free occurences of X replaced by e_2 is e_3.

We have the following rules:

$$\frac{}{x\,[x := e] \mapsto e} \qquad \frac{x \neq y}{x\,[y := e] \mapsto x}$$

$$\frac{}{\text{true}\,[x := e] \mapsto \text{true}} \qquad \frac{}{\text{false}\,[x := e] \mapsto \text{false}}$$

$$\frac{}{c_e\,[x := e] \mapsto c_e}$$

$$\frac{e_1 [x := e_3] \mapsto e_1' \qquad e_2 [x := e_3] \mapsto e_2'}{(e_1 == e_2)[x := e_3] \mapsto (e_1' == e_2')}$$

(similarly for $+$, $\&\&$, if-else and function application)

We show that, for any environment Γ, variable x, expressions e_1, e_2, e_3, and types T_1, T_2, if we have that:

1) $\Gamma \vdash e_1 : T_1$
2) $(x, T_2) \in \Gamma$
3) $\Gamma \vdash e_2 : T_2$
4) $e_1 [x := e_2] \mapsto e_3$

Then, we have that $\Gamma \vdash e_3 : T_1$

We prove this by structural induction on e_1.

- Case $e_1 = x$: We must have that $e_3 = e_2$ as well, as the only valid substitution is

$$x [x := e_2] \mapsto e_2$$

Thus, to show $\Gamma \vdash e_3 : T_1$ we show $\Gamma \vdash e_2 : T_1$.

From 1), we have that $\Gamma \vdash x : T_1$, which can only be derived from

$$\frac{(x, T_1) \in \Gamma}{\Gamma \vdash x : T_1}$$

Therefore, we must have that

$$(x, T_1) \in \Gamma$$

We also have that $(x, T_2) \in \Gamma$, from 2). Thus, we must have that $T_1 = T_2$. Since, from 3), $\Gamma \vdash e_2 : T_2$, then also $\Gamma \vdash e_2 : T_1$ #

- Case $e_1 = y \neq x$, or
  $e_1 = true$ or
  $e_1 = false$ or
  $e_1 = c_e$: In those cases, the proof is trivial since e_1 and e_3 must be the same, as the only valid subtitutions are of the form

$$e_1 [x := e_2] \mapsto e_1$$

- Case $e_1 = (e_4 == e_5)$: We have that

  1) $\Gamma \vdash (e_4 == e_5) : T_1$
  2) $(x, T_2) \in \Gamma$
  3) $\Gamma \vdash e_2 : T_2$
  4) $(e_4 == e_5)[x := e_2] \mapsto e_3$

  We must have that $e_3 = (e_4' == e_5')$ for some e_4' and e_5', and that $e_4[x := e_2] \mapsto e_4'$ and $e_5[x := e_2] \mapsto e_5'$ from the only applicable derivation rule:

$$\frac{e_4[x := e_2] \mapsto e_4' \qquad e_5[x := e_2] \mapsto e_5'}{(e_4 == e_5)[x := e_2] \mapsto (e_4' == e_5')}$$

  Moreover, we must have that $\Gamma \vdash (e_4 == e_5) : Boolean$, since the only applicable rules all match T_1 with Boolean. In addition, we have, for some type $T_3 \in \{Integer, Boolean\}$, that $\Gamma \vdash e_4 \vdash T_3$ and $\Gamma \vdash e_5 \vdash T_3$.

From induction hypothesis, we have then that

$$\Gamma \vdash e_4' : T_3 \quad \text{and that} \quad \Gamma \vdash e_5' : T_3$$

(Since conditions 1-4 are satisfied by e_4' and e_5').

Therefore, we can derive that:

$$\frac{\Gamma \vdash e_4' : T_3 \qquad \Gamma \vdash e_5' : T_3}{\Gamma \vdash (e_4' == e_5') : \text{Boolean}}$$

(Since T_3 in {Boolean, Integer})

And thus we have $\Gamma \vdash (e_4' == e_5') : T_1$, which gives us

$$\Gamma \vdash e_3 : T_1 \quad \#$$

- All other cases are similar to the previous one.

#

Part 3

The only rule that differs from lectures is function application:

$$\frac{\begin{array}{c} b \text{ body of } p,\ x_1 .. x_n \text{ args of } p \\ b[x_1 := e_1, \dots, x_n := e_n] \mapsto b' \end{array}}{p(e_1, \dots, e_n) \rightsquigarrow b'}$$

Where $b[x_1 := e_1, \dots, x_n := e_n] \mapsto b'$ denotes that b' is the result of the subtitution of x_1 by e_1, x_2 by e_2 and so on in b.

Note that in previous part we only treated the case when only a single variable is replaced. This is a generalisation for more than one variable.

Part 4

Progress: Proof from lecture barely changes. We just plug-in the new operational semantics rule for function application instead of the old one.

Preservation: We must show that, if $\Gamma \vdash p(e_1, \dots, e_n) : T$ and $p(e_1, \dots, e_n) \rightsquigarrow e$, then $\Gamma \vdash e : T$.

The only applicable typing rule is:

$$\frac{\Gamma \vdash f : (T_1, \dots, T_n) \Rightarrow T \quad \Gamma \vdash e_1 : T_1 \dots \Gamma \vdash e_n : T_n}{\Gamma \vdash f(e_1, \dots, e_n) : T}$$

Therefore $\Gamma \vdash f : (T_1, \dots, T_n) \Rightarrow T \quad \Gamma \vdash e_1 : T_1 \dots \quad \Gamma \vdash e_n : T_n$

Also, the only applicable operational semantics rule is:

$$\frac{b\,[x_1 := e_1, \dots, x_n := e_n] \mapsto b'}{f(e_1, \dots, e_n) \rightsquigarrow b'}$$

where b is the body of f and $x_1 \dots x_n$ its arguments.

We must thus show that $\Gamma \vdash b' : T$

We assume that $\Gamma = \Gamma_0 \oplus \Gamma_1$ for some Γ_1.

(The program environment remains visible in Γ). Let $\Gamma_2 = \{(x_1, T_1), \dots, (x_n, T_n)\}$.

Let's assume $\Gamma_0, \Gamma_1, \Gamma_2$ disjoint.

We have that $\Gamma_0 \oplus \Gamma_2 \vdash b : T$ as the body of f types checks.

We thus have that $\Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \vdash b : T$, as we have just added irrelevant bindings.

We thus have, from (an adapted version of) Part 2's lemma:

$$\Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \vdash b' : T$$

Since after substitution all occurence of $x_1 \dots x_n$ have been replaced in b', we have that

$$\Gamma_0 \oplus \Gamma_1 \vdash b' : T \, ,$$

as all bindings in Γ_2 are unused.

Thus, we have that

$$\Gamma \vdash e : T \qquad \#$$

Ex 2

Part 1

$$\frac{}{\Gamma \vdash c_e : \text{Integer}} \qquad \frac{c_e \text{ strictly pos.}}{\Gamma \vdash c_e : \text{Pos}} \qquad \frac{c_e \text{ strictly neg.}}{\Gamma \vdash c_e : \text{Neg}}$$

$$\frac{\Gamma \vdash t_1 : T \quad \Gamma \vdash t_2 : T}{\Gamma \vdash t_1 + t_2 : T} \qquad \frac{\Gamma \vdash t_1 : \text{Pos} \quad \Gamma \vdash t_2 : \text{Pos}}{\Gamma \vdash t_1 * t_2 : \text{Pos}} \qquad \frac{\Gamma \vdash t_1 : \text{Neg} \quad \Gamma \vdash t_2 : \text{Neg}}{\Gamma \vdash t_1 * t_2 : \text{Pos}}$$

$$\frac{\Gamma \vdash t_1 : \text{Neg} \quad \Gamma \vdash t_2 : \text{Pos}}{\Gamma \vdash t_1 * t_2 : \text{Neg}} \qquad \frac{\Gamma \vdash t_1 : \text{Pos} \quad \Gamma \vdash t_2 : \text{Neg}}{\Gamma \vdash t_1 * t_2 : \text{Neg}}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash t_1 * t_2 : \text{Integer}}$$

$$\frac{\Gamma \vdash t_1 : T \quad \Gamma \vdash t_2 : \text{Pos}}{\Gamma \vdash t_1 / t_2 : \text{Integer}}$$

$$\frac{\Gamma \vdash t_1 : T \quad \Gamma \vdash t_2 : \text{Neg}}{\Gamma \vdash t_1 / t_2 : \text{Integer}}$$

Those are one possible set of rules.
Variations are possible.

Part 2

- $1+1$ can have type Pos or Integer.

$$\frac{\Gamma \vdash 1 : Pos \quad \Gamma \vdash 1 : Pos}{\Gamma \vdash 1+1 : Pos} \quad \text{and} \quad \frac{\Gamma \vdash 1 : Integer \quad \Gamma \vdash 1 : Integer}{\Gamma \vdash 1+1 : Integer}$$

- $-2 * 4$ can have type Neg or Integer.
- $-1 * (2 + -1)$ can only have type Integer.
- $7 / (18 + -1)$ can not be typed.

Part 3

We have :

$$Pos <: Integer$$
$$Neg <: Integer$$

But also, we could have:

$$Pos <: Pos, \quad Neg <: Neg, \quad Integer <: Integer$$

(But we chose not to for simplicity here).

Part 4

$$\frac{\Gamma \vdash e : T_1 \quad T_1 <: T_2}{\Gamma \vdash e : T_2}$$

One could change some rules. For instance, the integer literal rules could be:

$$\frac{}{\Gamma \vdash 0 : Integer} \qquad \frac{c_e > 0}{\Gamma \vdash c_e : Pos} \qquad \frac{c_e < 0}{\Gamma \vdash c_e : Neg}$$

Part 5

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash 7 : Pos} \quad Pos <: Integer}{\Gamma \vdash 7 : Integer} \quad \dfrac{}{\Gamma \vdash 2 : Pos}}{\Gamma \vdash 7/2 : Integer} \quad \dfrac{\dfrac{}{\Gamma \vdash 3 : Pos} \quad Pos <: Integer}{\Gamma \vdash 3 : Integer}}{\Gamma \vdash power(7/2, 3) : Integer}$$

Is one possible type derivation.

If we were to add reflexive subtyping rules, such as $Pos <: Pos$, we could generate more derivations.