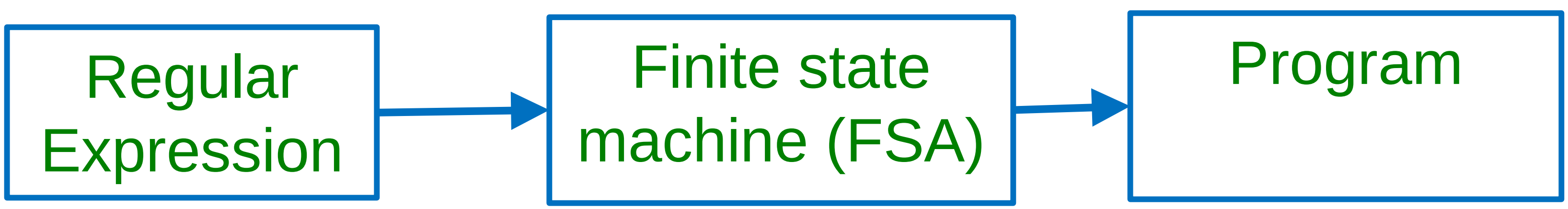
Automating Construction of Lexers

Regular Expression to Programs

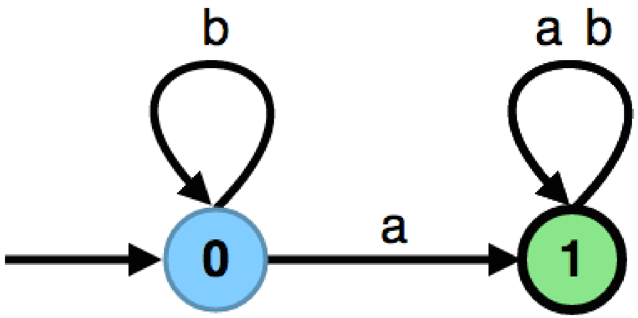
- Not all regular expressions are simple.
- How can we write a lexer for (a*b | a) ?
- Tokenizing aaaab Vs aaaaa



Finite State Automaton (Finite State Machine)

- A = (Σ, Q, q_0, δ, F)

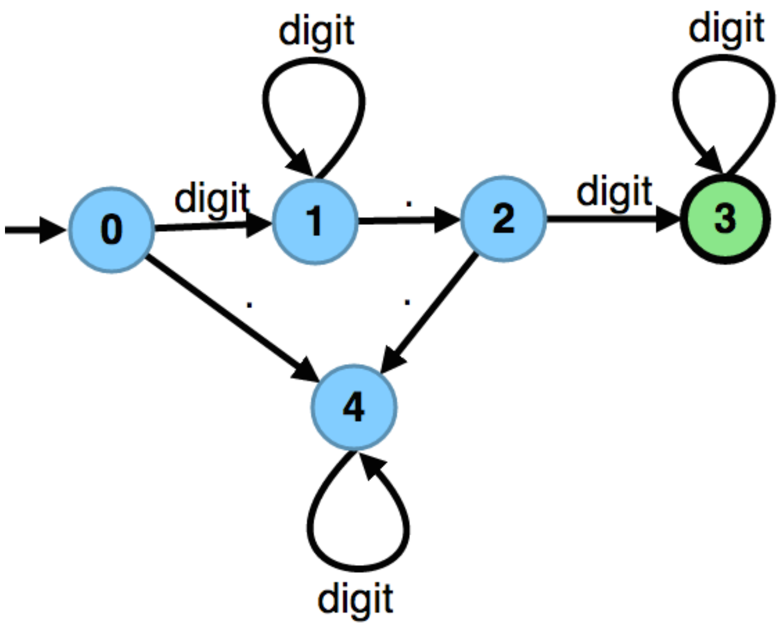
$$\delta \subseteq Q \times \Sigma \times Q,$$
$$q_0 \in Q,$$
$$F \subseteq Q$$



$$q_0 \in Q$$
$$q_1 \subseteq Q$$
$$\delta = \{ (q_0, a, q_1), (q_0, b, q_0), (q_1, a, q_1), (q_1, b, q_1), \}$$

- Σ - alphabet
- Q - states (nodes in the graph)
- q_0 - initial state (with '->' sign in drawing)
- δ - transitions (labeled edges in the graph)
- F - final states (double circles)

Numbers with Decimal Point



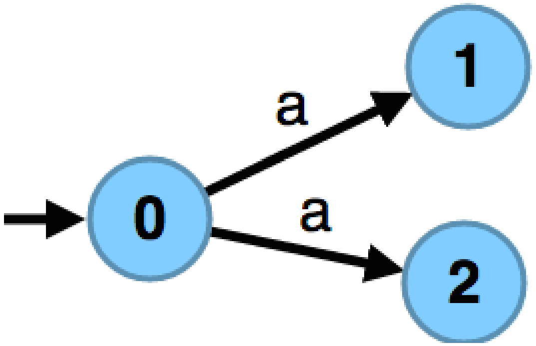
digit digit* . digit digit*

What if the decimal part is optional?

Kinds of Finite State Automata

- DFA: δ is a function : $(Q, \Sigma) \mapsto Q$
- NFA: δ could be a relation

- In NFA there is no unique next state. We have a set of possible next states.



Remark: Relations and Functions

- Relation $r \subseteq B \times C$

  $r = \{ \ldots, (b,c1) , (b,c2) ,\ldots \}$

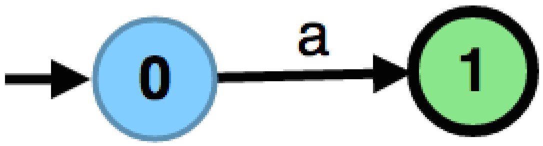
- Corresponding function: $f : B \to 2^C$

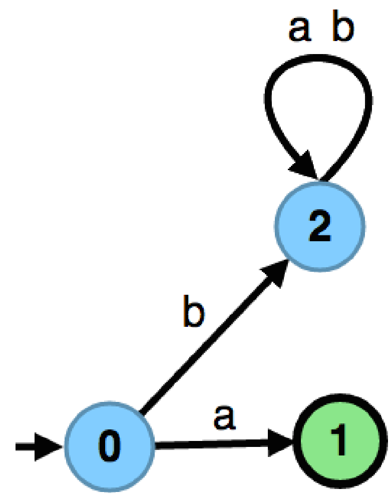
  $f = \{ \ldots (b,\{c1,c2\}) \ldots \}$

  $f(b) = \{ c \mid (b,c) \in r \}$

- Given a state, next-state function returns the set of new states
  - for deterministic automaton, the set has exactly 1 element

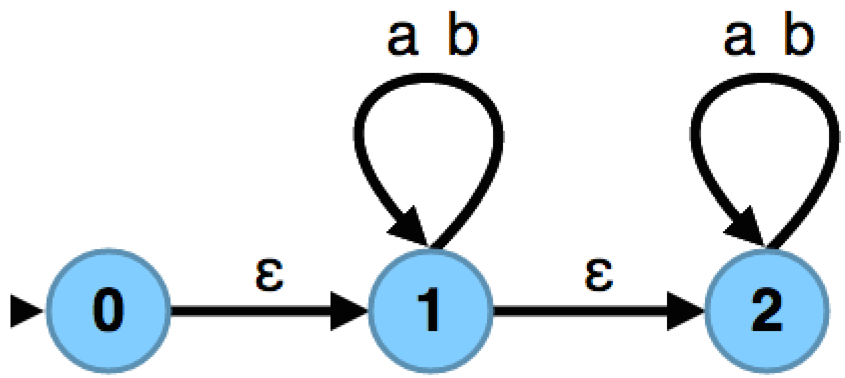
Allowing Undefined Transitions





- Undefined transitions lead to a sink state from where no input can be accepted

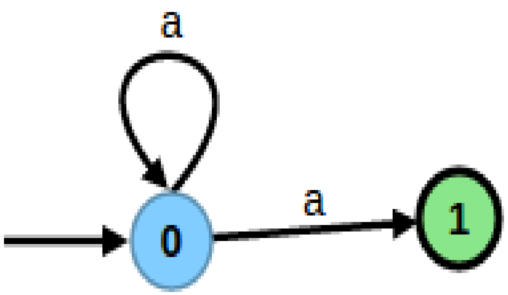
Allowing Epsilon Transitions



- Epsilon transitions:
  - traversing them does not consume anything
- Transitions labeled by a word:
  - traversing them consumes the entire word

Interpretation of Non-Determinism

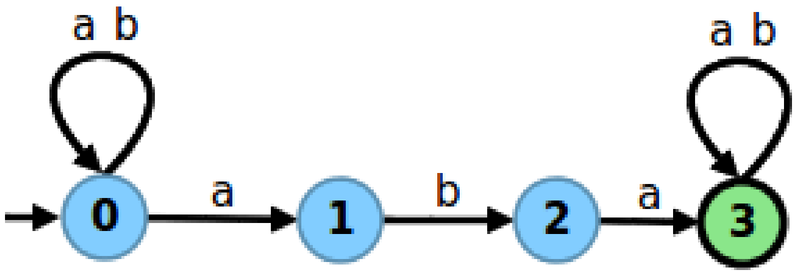
- A word is accepted if there is **a path** in the automaton that leads to an accepting state on reading the word
  Eg.



- Does the automaton accept ‘a’ ?
  - yes

Exercise

- Construct a NFA that recognizes all strings over {a,b} that contain "aba" as a substring



Running NFA (without epsilons) in Scala

```
def δ(q : State, a : Char) : Set[States] = { ... }
def δ'(S : Set[States], a : Char) : Set[States] = {
  for (q1 <- S, q2 <- δ(q1,a)) yield q2
}

def accepts(input : MyStream[Char]) : Boolean = {
  var S : Set[State] = Set(q0) // current set of states
  while (!input.EOF) {
    val a = input.current
    S = δ'(S,a)        // next set of states
  }
  !(S.intersect(finalStates).isEmpty)
}
```

NFA Vs DFA

- For every NFA there exists an equivalent DFA that accepts the same set of strings
- But, NFAs could be exponentially smaller (succinct)
- There are NFAs such that **every** DFA equivalent to it has exponentially more number of states

Regular Expressions and Automata

Theorem:

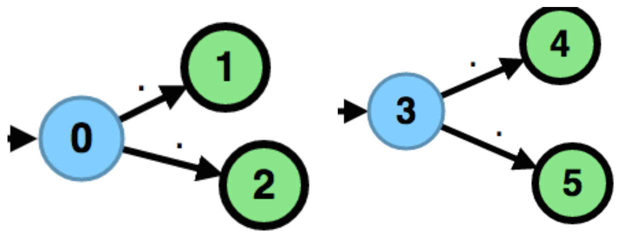
If L is a set of words, it is describable by a regular expression iff (if and only if) it is the set of words accepted by some finite automaton.

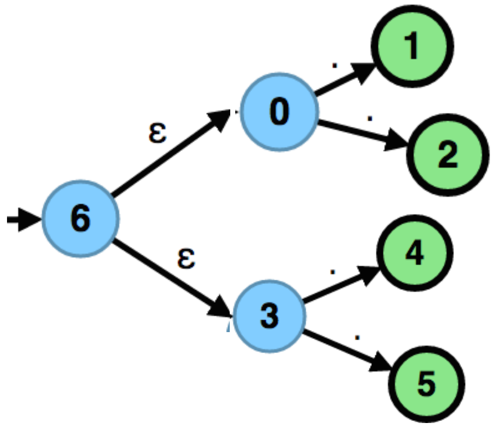
Algorithms:

- regular expression → automaton (important!)
- automaton → regular expression (cool)

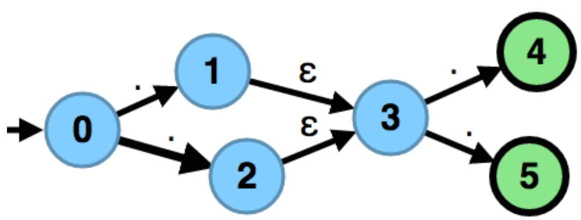
Recursive Constructions

- Union



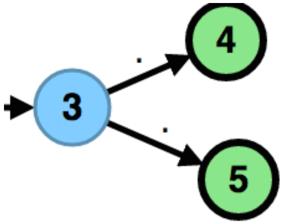


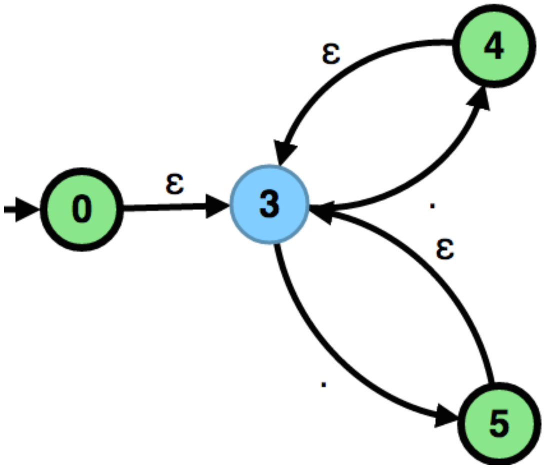
- Concatenation



Recursive Constructions

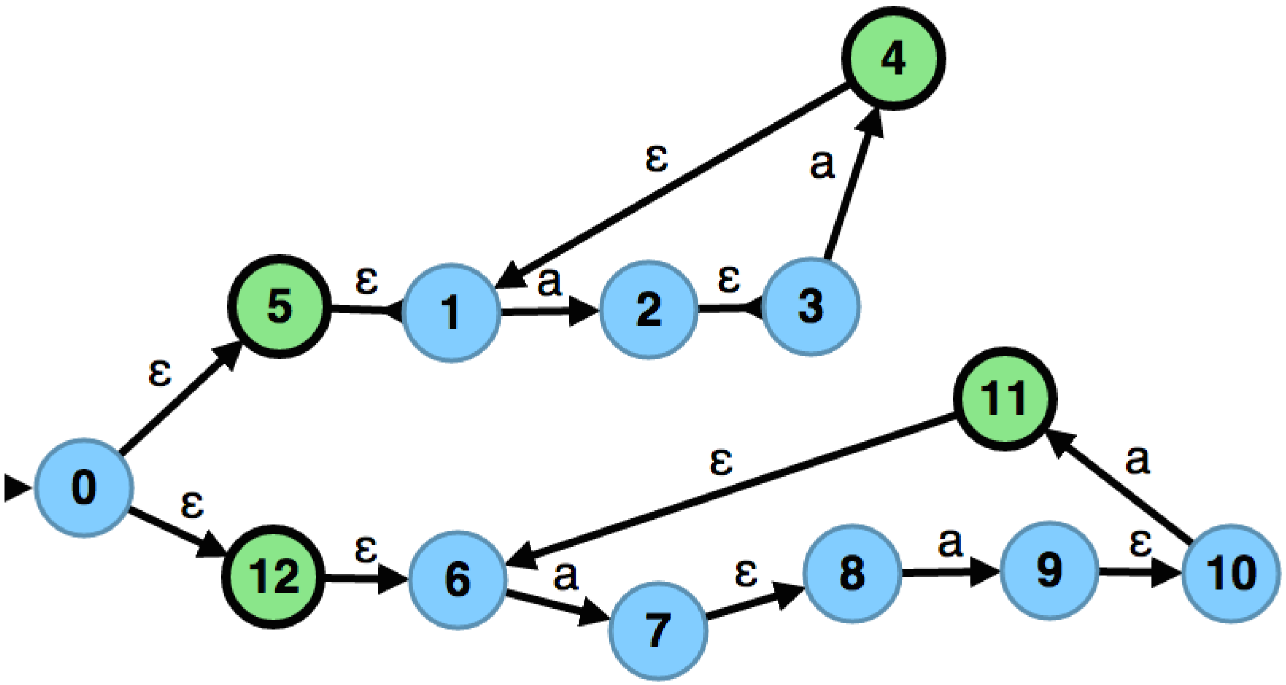
- Star





Exercise: (aa)* | (aaa)*

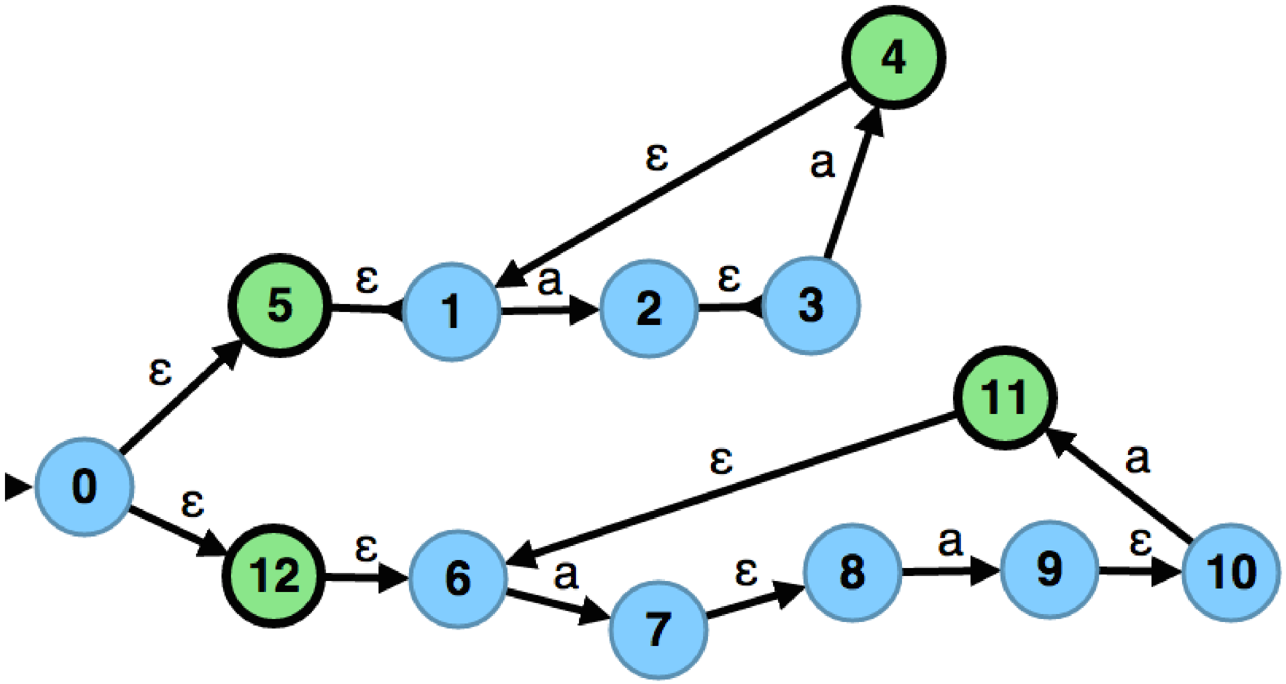
- Construct an NFA for the regular expression



NFAs to DFAs (Determinisation)

- keep track of a set of all possible states in which the automaton could be
- view this finite set as one state of new automaton

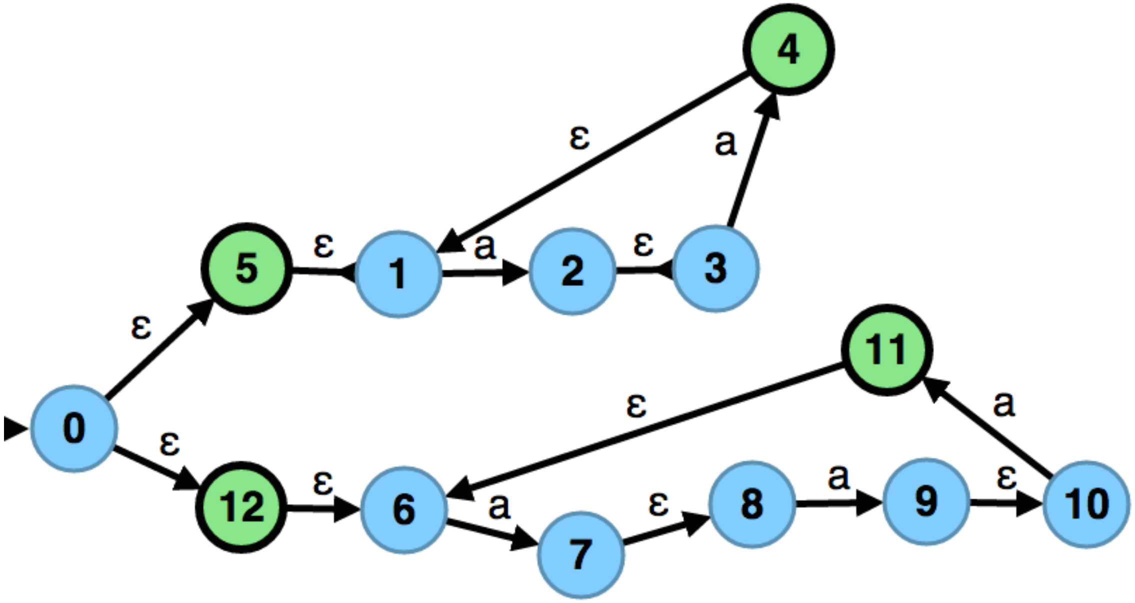
NFA to DFA Conversion



Possible states of the DFA: 2^Q

{ { } , { 0},…{12}, {0,1}, …,{0,12}, …{12, 12}, {0,1,2} …, { 0,1,2…,12 } }

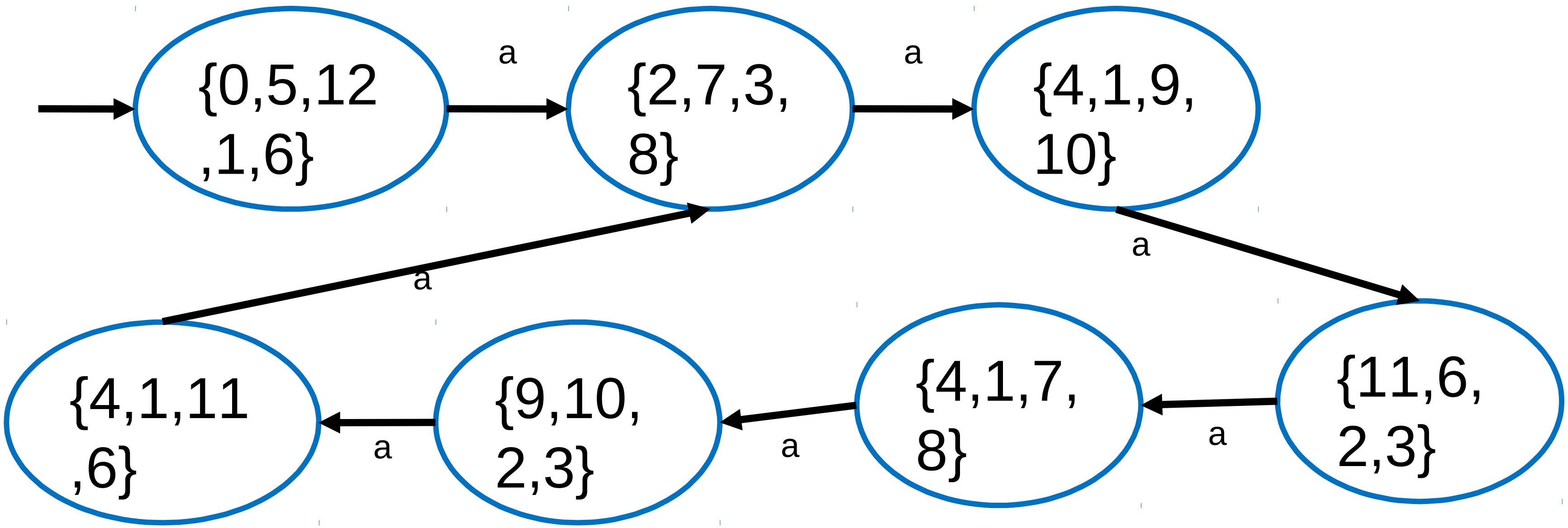
NFA to DFA Conversion



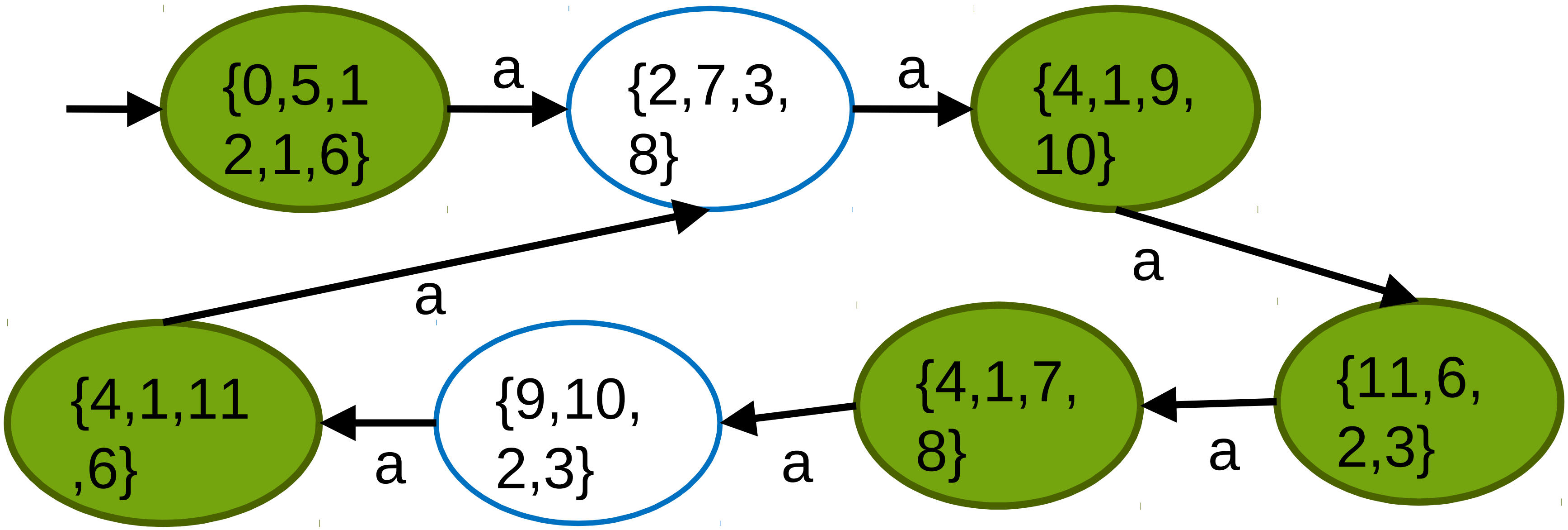
NFA to DFA Conversion

- DFA: $(\Sigma, 2^Q, q_0', \delta', F')$
- $q_0' = E(q_0)$
- $\delta'(q', a) = \bigcup_{\{\exists q_1 \in q', \, \delta(q_1, a, q_2)\}} E(q_2)$
- $F' = \{ \, q' | q' \in 2^Q, q' \cap F \neq \, \emptyset \}$

NFA to DFA Conversion



NFA to DFA Example



Clarifications

- what happens if a transition on an alphabet ‘a’ is not defined for a state ‘q’ ?
- $\delta'(\{q\}, a) = \emptyset$
- $\delta'(\emptyset, a) = \emptyset$

- Empty set represents a state in the NFA
- It is a trap/sink state: a state that has self-loops for all symbols, and is non-accepting.

Minimizing DFAs to Keep Them Small

- First, throw away all unreachable states: those for which there is no path to them from the initial state

Minimizing DFAs: Procedure

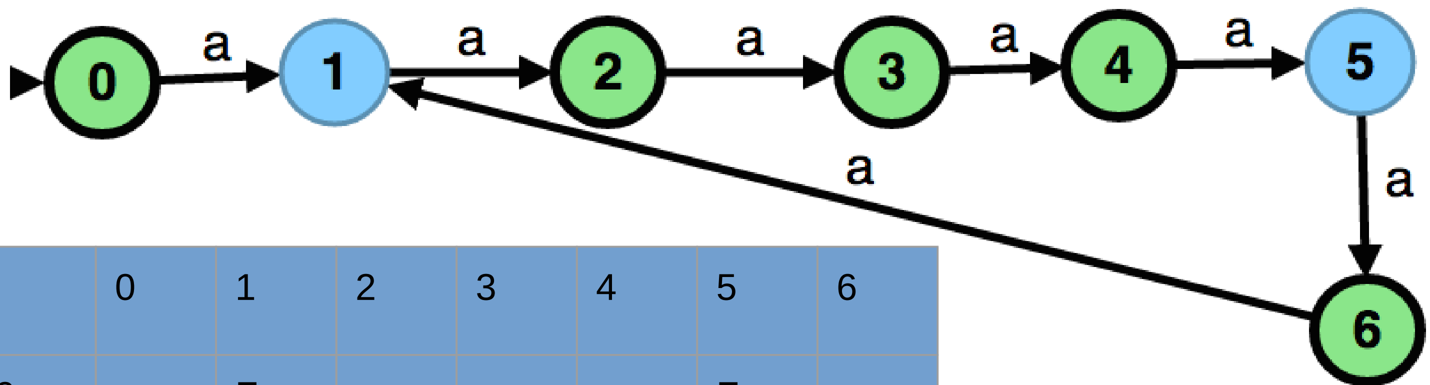
- Write down all pairs of state as a table
- Every cell in the table denotes whether the corresponding states are equivalent

	q1	q2	q3	q4	q5
q1	x	?	?	?	?
q2		x	?	?	?
q3			x	?	?
q4				x	?
q5					x

Minimizing DFAs: Procedure

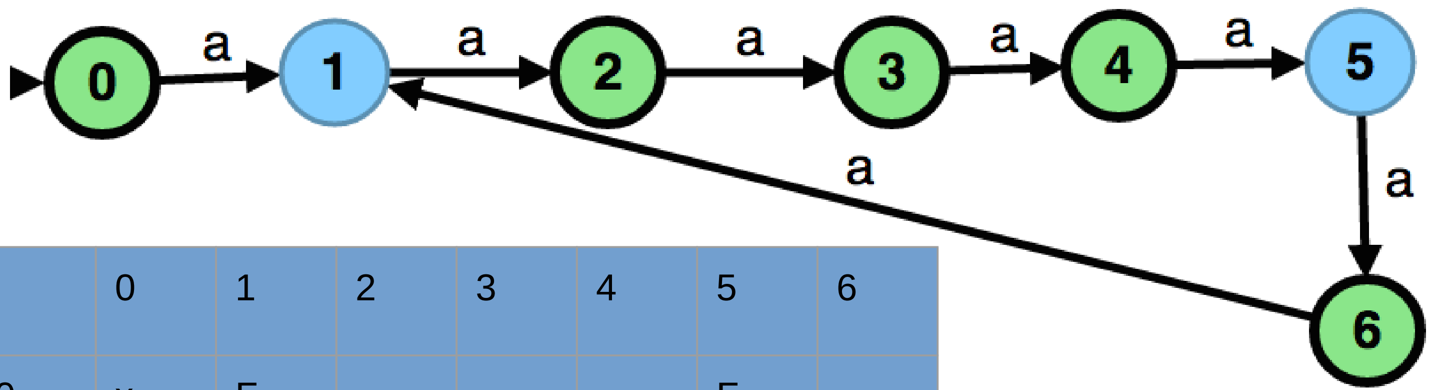
- Inititalize cells (q1, q2) to false if one of them is final and other is non-final
- Make the cell (q1, q2) false, if q1 -> q1' on some alphabet symbol and q2 -> q2' on 'a' and q1' and q2' are not equivalent
- Iterate the above process until all non-equivalent states are found

Minimizing DFAs: Illustration



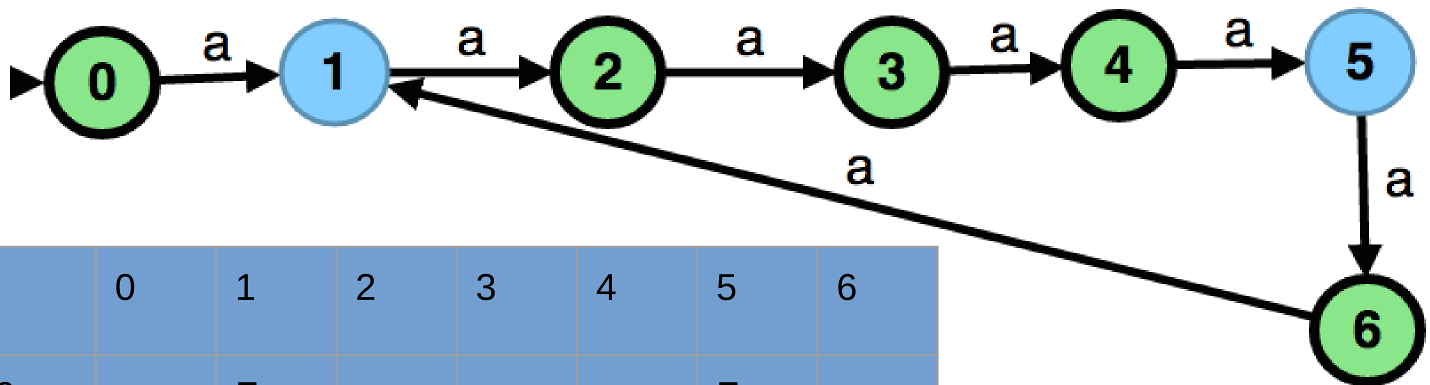
	0	1	2	3	4	5	6
0	x	F				F	
1		x	F	F	F		F
2			x			F	
3				x		F	
4					x	F	
5						x	F
6							x

Minimizing DFAs: Illustration



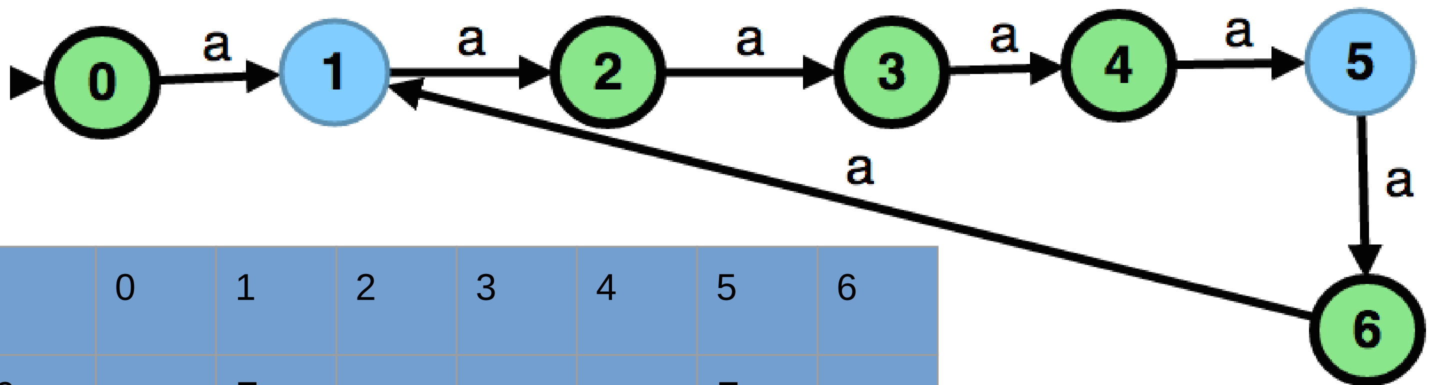
	0	1	2	3	4	5	6
0	x	F				F	
1		x	F	F	F		F
2			x			F	
3				x		F	
4					x	F	
5						x	F
6							x

Minimizing DFAs: Illustration



	0	1	2	3	4	5	6
0	x	F				F	
1		x	F	F	F		F
2			x			F	
3				x		F	
4					x	F	
5						x	F
6							x

Minimizing DFAs: Illustration



	0	1	2	3	4	5	6
0	x	F				F	
1		x	F	F	F		F
2			x			F	
3				x		F	
4					x	F	
5						x	F
6							x

Minimizing DFAs: Procedure

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	q1	q2	q3	q4	q5
q1	x	?	?	?	?
q2		x	?	?	?
q3			x	?	?
q4				x	?
q5					x

Properties of Automata

Complement:

- Given a DFA A, switch accepting and non-accepting states in A gives the complement automaton A^c
- $\mathrm{L}(\mathrm{A}^{\mathrm{c}}) = (\Sigma^* \setminus L(A))$

Note this does not work for NFA

Intersection: $\mathrm{L}(\mathrm{A}') = L(A_1) \cap L(A_2)$

- $A' = (\Sigma, Q_1 \times Q_2, (q_0^1, q_0^2), \delta', F_1 \times F_2)$
- $\delta'\big((q_1, q_2), a\big) = \delta(q_1, a) \times \delta(q_2, a)$

Exercise 0.1: on Equivalence

Prove that (a*b*)* is equivalent to (a|b)*

Sequential Circuits are Automata

$A = (\Sigma, Q, q_0, \delta, F)$

Q – states of flip-flops, registers, etc.

δ – combinational circuit that determines next state