Type Inference

```
def CONS[T](x:T, lst:List[T]):List[T]={...}
def listInt() : List[Int] = {...}
def listBool() : List[Bool] = {...}

def baz(a, b) = CONS(a(b), b)
def test(f,g) =
      (baz(f,listInt), baz(g,listBool))
```

Solving ‘baz’

```
def CONS[T](x:T, lst:List[T]) : List[T] = {...}
def baz(a:TA, b:TB):TD = CONS(a(b):TC, b:TB):TD
```

TA = (TB => TC)

CONS : T_1 x List[T_1] => List[T_1]

TC = T_1

TB = List[T_1]

TD = List[T_1]

TA = (List[T_1]=>T_1)

Solved form. Generalize over T_1

```
def baz[T1](a: List[T1]=>T1,b:List[T1]):List[T1] =
    CONS[T1](a(b),b)
```

Using generalized ‘baz’

```
def baz[T1](a: List[T1]=>T1,b:List[T1]):List[T1] =
        CONS[T1](a(b),b)
```

```
def test(f,g) = (baz(f,listInt), baz(g,listBool))
```

```
test : (List[Int] => Int) x (List[Bool] => Bool) =>
        List[Int] x List[Bool]
```

Omega (Iteration) Function

Bonus:

```
Type check omega function
def w(f)(x) = f(w(f)(f(x)))

def w(f:TF)(x:TX):TR = f(w(f)(f(x):TA):TB)


TF = TX => TA
TA = TX
omega: TF x TX => TR
TR = TB
TF = TB => TR
Therefore : TF = TX => TX
TX => TX == TB => TB
TB = TX
```

`def w[T](f: T => T, x: T): T`

Find most general types

```
def twice(f:TF,x:TX): TR=f(f(x))
TF = A => B
A  = TX
A  = B
TR = B
```

therefore: def twice[T](f: T => T, x: T): T = f(f(x))

Self-Application Type Check Attempt

def selfApp(f) = f(f)

def selftApp(f:TF):TR = (f:TF)(f:TF)

TF = (TF => TR)

because of the OCCURS CHECK, this constraint has no solutions.

Therefore, we have a type error.

Try to infer types for Y combinator

In λ calculus:

$$\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))$$

In our language:

```
def Y(f) = {
  def repeat(x) = { f(x(x)) }
  repeat(repeat)
}
```

Try to infer types for Y combinator

```
def Y(f:TF) = {
  def repeat(x:TX) = { f(x(x):TA):TB }
  (repeat(repeat:TC)):TD
}
```

So **Y : TF => TD** and **repeat : TX => TB**

x(x):TA gives: **TX = TX => TA** ← **no solution for this constraint, so we should report a type error.**

f(x(x)):TB gives: **TF = TA => TB**

repeat:TC gives: **TC = TX => TB**

repeat(repeat) : **(TX => TB) = (TC => TD)**

Reminder : physical units

A unit expression is defined by following grammar

u, v := b | 1 | u * v | u^{-1}

where u, v are unit expressions themselves and b is a base unit:

b := m | kg | s | A | K | cd | mol

You may use B to denote the set of the unit types

B = { m, kg, s, A, K, cd, mol }

For readability reasons, we use the syntactic sugar

u^n = u * … * u if n > 0

1 if n = 0

u^{-1} * … * u^{-1} if n < 0

and u/v = u * v^{-1}

Reminder: Rules

$$\frac{\Gamma \vdash a : U}{\Gamma \vdash a : \text{simplify}(U)}$$

$$\frac{\Gamma \vdash a : (U*U)^{-1}}{\Gamma \vdash \sqrt{a} : U^{-1}}$$

$$\frac{\Gamma \vdash a : U \qquad \Gamma \vdash b : U}{\Gamma \vdash a + b : U}$$

$$\frac{\Gamma \vdash a : U \qquad \Gamma \vdash b : V}{\Gamma \vdash a * b : U*V}$$

$$\frac{\Gamma \vdash a : U \qquad \Gamma \vdash b : V}{\Gamma \vdash a / b : U/V}$$

$$\frac{\Gamma \vdash a : U*U}{\Gamma \vdash \sqrt{a} : U}$$

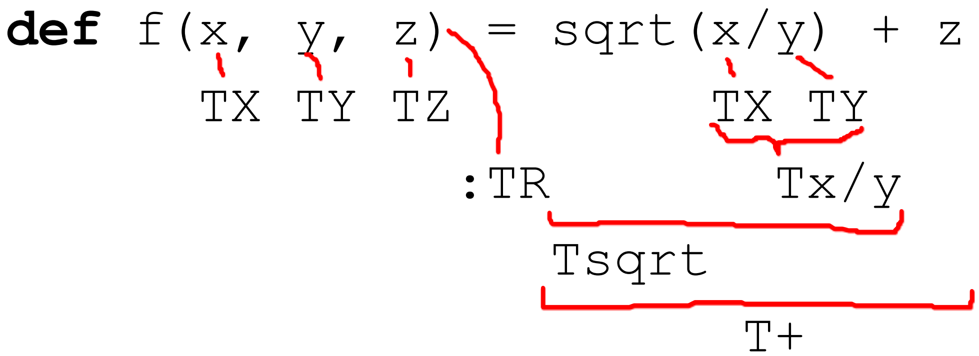
$$\frac{\Gamma \vdash a : 1}{\Gamma \vdash \sin(a) : 1}$$

$$\frac{\Gamma \vdash a : U}{\Gamma \vdash \text{abs}(a) : U}$$

Physical Units Type Inference

```
val g = 9.87.m / (1.s * 1.s)
val href = 1.92.m

def f(x, y, z) = sqrt(x/y) + z
def T(L) = 2*pi*sqrt(L/g) + 0.s
def fall(t, v) = -0.5*g*t*t + t*v + href
def freq(t, w) =
   href * sin(2*pi*t*w) + g*(t/w)
def einstein(E, p, v) = E - p*v*v == 0 &&
   E/1.s - p*g*v == 0 && p > 0.kg
def prof(a, b) = if(b > 1.s)
   sqrt(prof(a+a,b-1.s)/b) else a
```



T+ = TR

Tsqrt = T+

TZ = T+

Tsqrt * Tsqrt = Tx/y

Tx/y = TX / TY

Therefore:

Tsqrt = TR

TZ = TR

TR*TR = TX / TY

so **TX = TR*TR/TY**

def f[T,V](x: T*T/V, y:V, z: T): T

```
def T(L)        = 2pi*sqrt(L/g) + 0.s
      TL  :TR      1         TLG
                           TSQRT
                        TM
                                      TP
```

TR = TP

TP = s

TP = TM

TM = 1 * TSQRT

TSQRT*TSQRT = TLG

TLG = TL / (m/(s*s))

Therefore:

TR = s

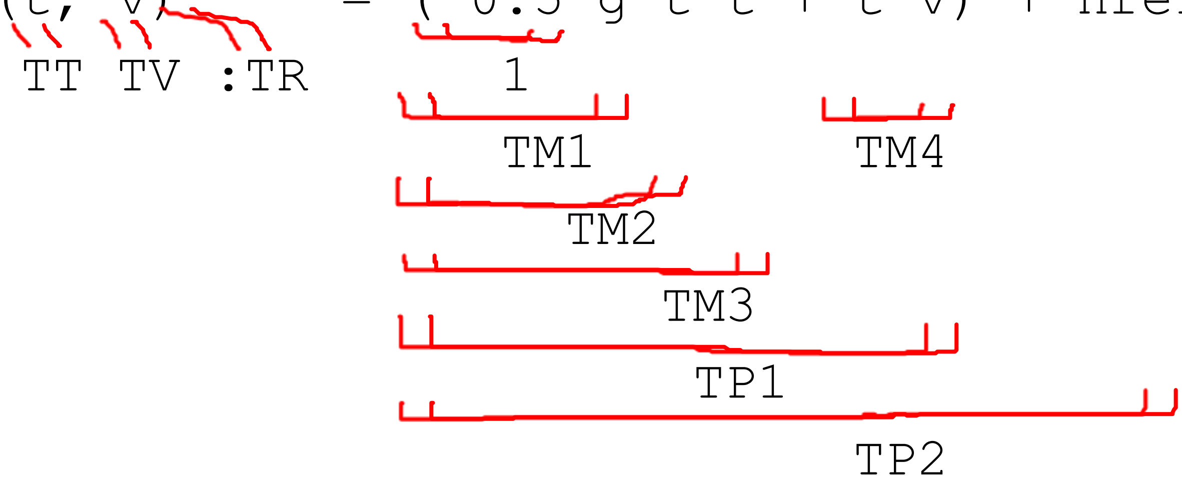
s = TQSRT

s * s = TLG

s*s = TL / (m/(s*s))

So: **TL = m** and **T(L: <m>): <s>**

```
def fall(t, v)      = (-0.5*g*t*t + t*v) + href
```



TR = TP2, TM1 = 1*m/(s*s), TM2 = TM1*TT, TM3 = TM2*TT,
TM4 = TT*TV, TP1 = TM4, TP1 = TM3, TP2 = TP1, TP2 = m

Therefore:

TM2 = m/(s*s)*TT, TM3 = m/(s*s)*TT*TT, TP1 = TT*TV,
TP1 = m/(s*s)*TT*TT, TT*TV = m/(s*s)*TT*TT

TV = m/(s*s)*TT

TR = m, TP1 = m, so m = m/(s*s)*TT*TT, therefore **TT=s**

TV = m/s

fall(t: <s>): <m>

```
def freq(t, w)      = href * sin(2pi*t*w)    + g*(t/w)
           TT  TW: TR     m       1      1   TT  TW   m/s^2  TTOW
                                               TTW            TM2
                                  TM1
                                                   TP
```

TP = TM1, TM1 = TM2, TM2 = m/s^2*TTOW, TTOW=TT/TW, TTW = TT*TW, TTW = 1, TM1 = m*1, TR = TP

Therefore

TR = m

m = m/s^2 * TTOW, so TT/TW = s^2

1 = TT * TW

so TW = 1/TT and TT^2 = s^2, so **TT = s and TW = 1/s**

def freq(t: <s>, w: <1/s>): <m>

```
def ein(E, p, v): Bool =
        TE TP TV
E - p*v*v == 0 && E/1.s - p*g*v == 0 && p > 0.kg
TE    TM1               TES      TM2

TP = kg, TE = TM1, TM1 = TP * TV^2, TES = TE/s, TES =
TM2, TM2 = TP*m/s^2*TV

TE/s = TM2, TE/s = kg*m/s^2*TV, TE = kg*TV^2

So kg*TV^2/s = kg*m/s^2*TV
therefore: TV = m/s, TE = kg*m^2/s^2

def ein(E: <kg*m^2/s^2>, p: <kg>, v: <m/s>): Bool
```

```
def prof(a, b) =
         TA TB   :TR
if(b > 1.s) sqrt(prof(a+a,b-1.s)/b) else a
    BOOL                TPA TB1
                        TR
                            TRB
                     TSQRT
```

TB = s

TSQRT = TA = TR

TQSRT*TSQRT = TRB

TRB = TR / TB

TA = TA = TA = TA, TB = s = TB

TR * TR = TR / s

so **TR = 1/s, TA = 1/s**

```
def prof(a: <1/s>, b: <s>): <1/s>
```

Theorem

Theorem:

Suppose that the result type T does not contain a base unit b_1 (or, equivalently, this base type occurs only with the overall exponent 0, as b_1^0). If we multiply all variables of type b_1 by a fixed numerical constant K, the final result of the expression does not change.

Question 1: Generalize the theorem.

Question 2: Prove the theorem.

Theorem examples

- Center of mass

$x: <m> = (x1*m1 + x2*m2)/(m1+m2)$

Multiplying all mass variables by 2 does not change the center of mass.

- Gravity estimation

$t_1 = \sqrt{(t_2^2 * h_1/h_2)}$

Changing h_1 and h_2 by any factor will not change the time.

Solution – part I

Lemma:

If we multiply all variables of type B by constant K, and the result has type T where B has exponent N, then the value of the expression is multiplied by K^N

Solution – part II

Suppose that we have proved the lemma for expressions of size < n.

Let give us an expression of size n. If the last applied rule is +, then:

$$\frac{\Gamma \vdash E : U \qquad \Gamma \vdash F : U}{\Gamma \vdash E + F : U}$$

Let us assume that B appears in U with exponent N.

If we multiply all variables of type B in E and F by K, by recurrence E is multiplied by K^n.

Therefore E+F is transformed to (E*K^N + F*K^N) = (E+F)*K^N

Other rules are similar.