Abstract Interpretation

(Cousot, Cousot 1977)

also known as

Data-Flow Analysis

(Kildall 1973)

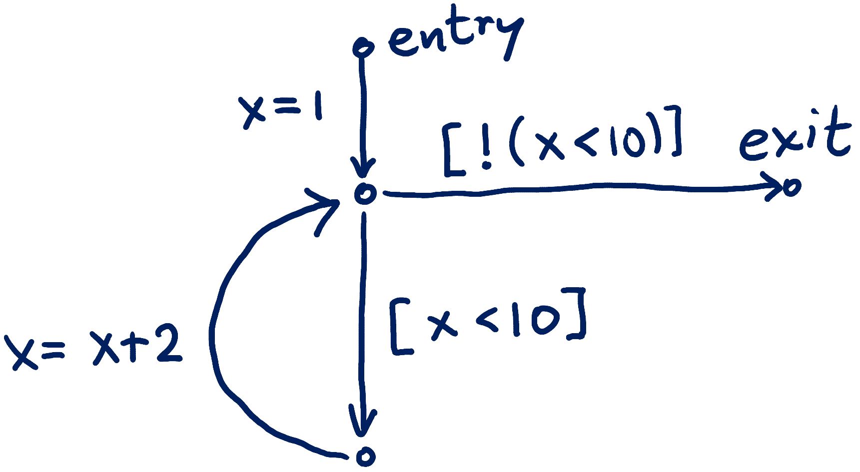
Goal of Data-Flow Analysis

Automatically compute information about the program

- Use it to report errors to user (like type errors)
- Use it to optimize the program

Works on control-flow graphs:
(like flow-charts)

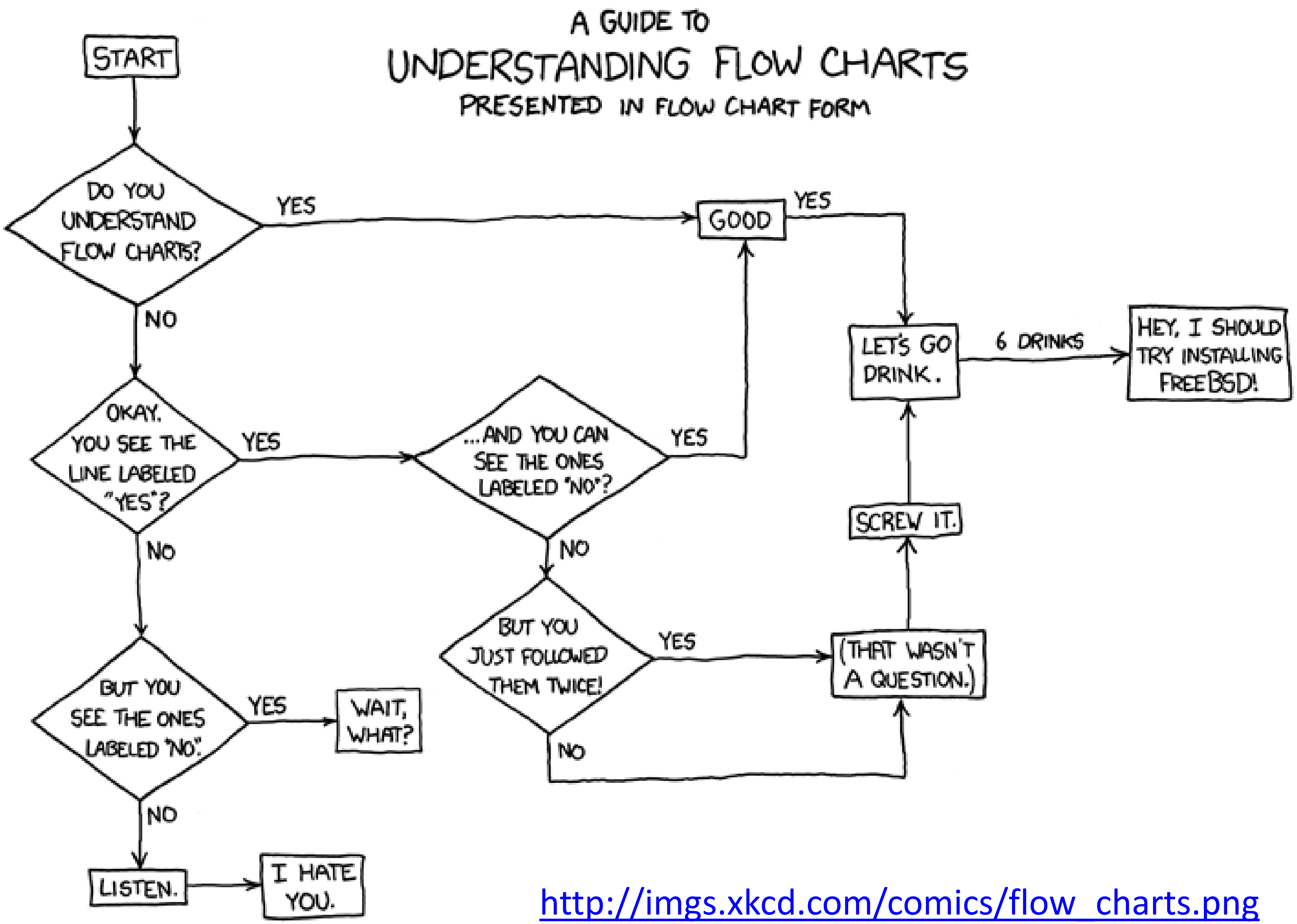
```
x = 1
while (x < 10) {
  x = x + 2
}
```



Why Constant Propagation

```
int a, b, step, i;
 boolean c;
 a = 0;
 b = a + 10;
 step = -1;
 if (step > 0) {
   i = a;
 } else {
   i = b;
 }
 c = true;
 while (c) {
   print(i);
   i = i + step;    // can emit decrement
   if (step > 0) {
     c = (i < b);
   } else {
     c = (i > a); // can emit better instruction here
  } // insert here (a = a + step), redo analysis
}
```

Control-Flow Graphs: Like Flow Charts



http://imgs.xkcd.com/comics/flow_charts.png

Control-Flow **Graph**: (V,E)

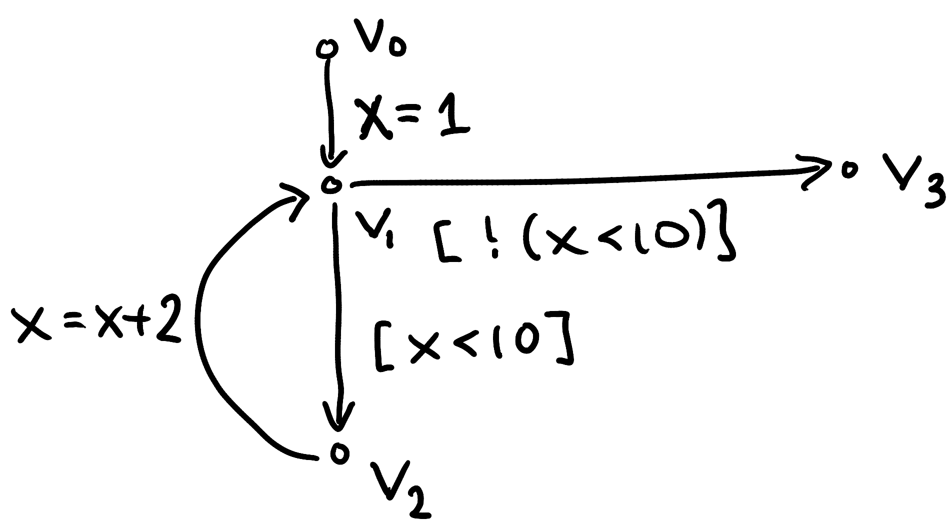
Set of nodes, V

Set of edges, which have statements on them

$(v_1, st, v_2) \in E$

means there is edge from v_1 to v_2 labeled with statement st.

x = 1
while (x < 10) {
x = x + 2
}



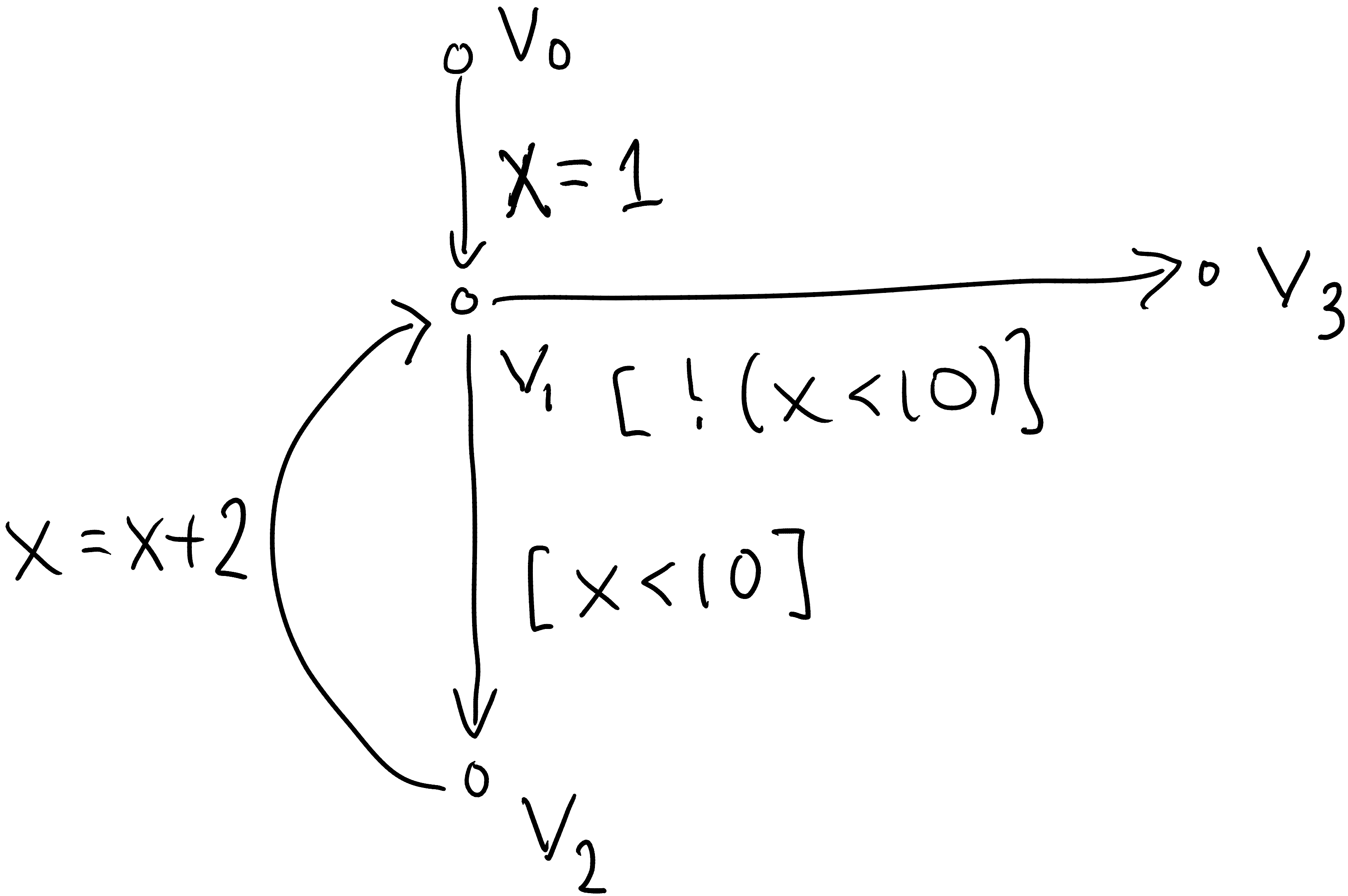
$V = \{v_0, v_1, v_2, v_3\}$

$E = \{(v_0, x=1, v_1), (v_1, [x<10], v_2),$
$(v_2, x=x+2, v_1), (v_1, [!(x<10)], v_3)\}$

Interpretation and Abstract Interpratation

- Control-Flow graph is similar to AST
- We can
  - interpret control flow graph
  - generate machine code from it (e.g. LLVM, gcc)
  - abstractly interpret it: do not push values, but **approximately compute supersets of possible values** (e.g. intervals, types, etc.)

Compute Range of x at Each Point



What we see today

1. How to compile abstract syntax trees into control-flow graphs
2. Lattices, as structures that describe abstractly sets of program states (facts)
3. Transfer functions that describe how to update facts

Generating Control-Flow Graphs

- Start with graph that has one entry and one exit node and label is entire program
- Recursively decompose the program to have more edges with simpler labels
- When labels cannot be decomposed further, we are done

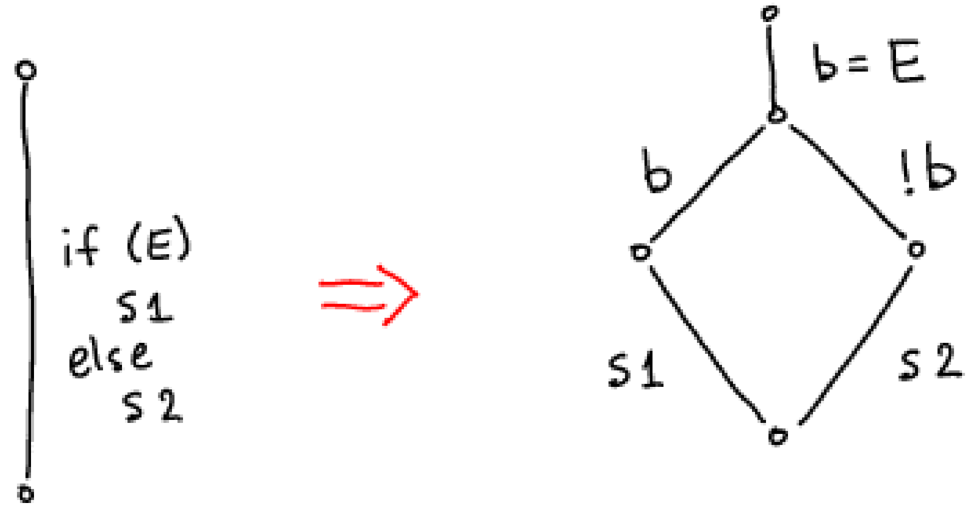
Flattening Expressions for simplicity and ordering of side effects

E_1, E_2 - complex expressions

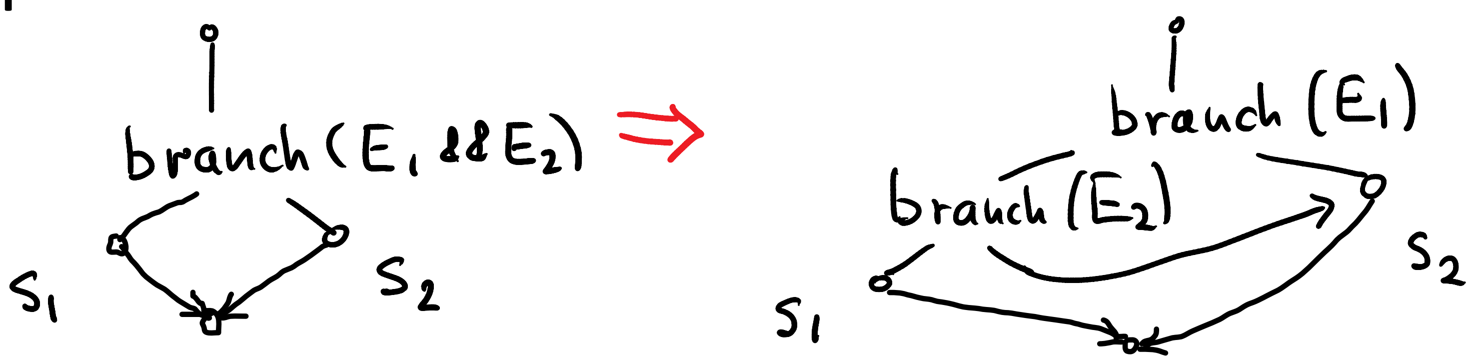
t_1, t_2 - fresh variables

$$x = E_1 * E_2 \quad \Longrightarrow \quad t_1 = E_1;\ \ t_2 = E_2;\ \ x = t_1 * t_2$$

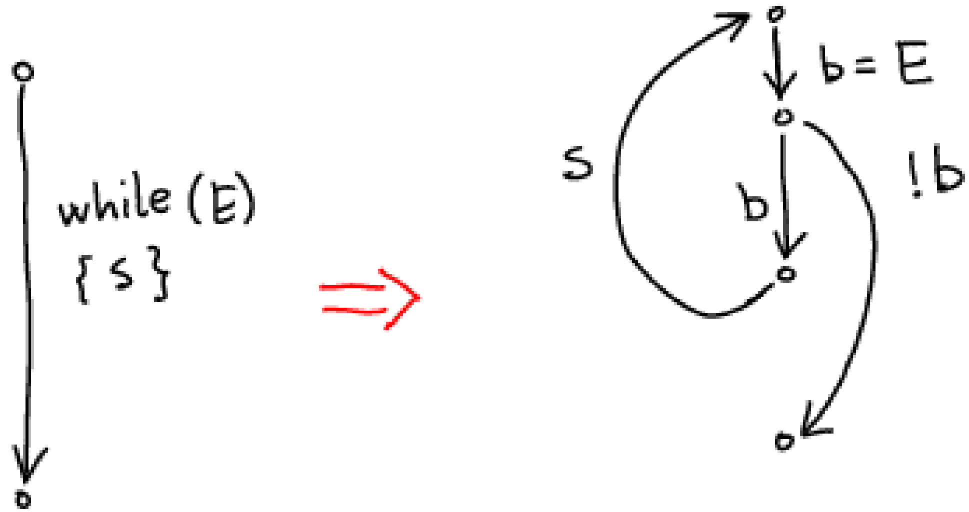
If-Then-Else



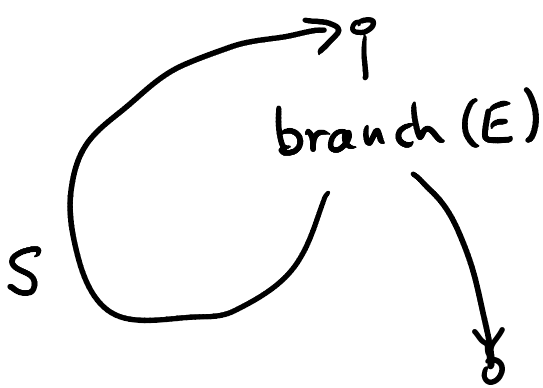
Better translation uses the "branch" instruction approach: have two destinations



While



Better translation uses the "branch" instruction

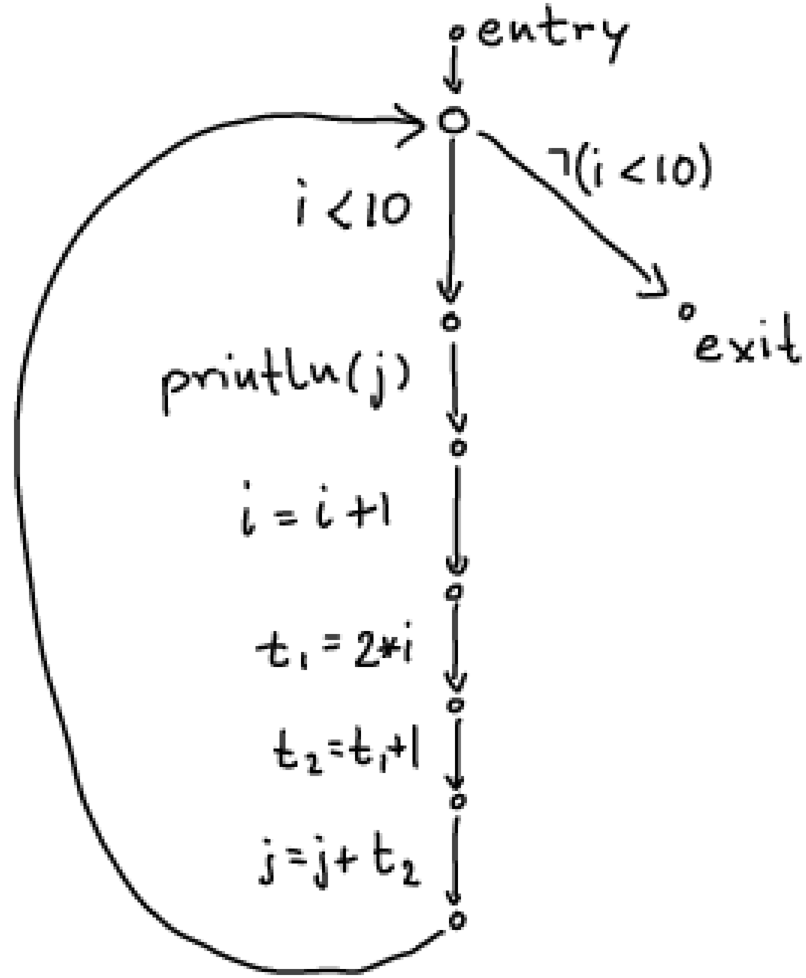


Example 1: Convert to CFG

```
while (i < 10) {
  println(j);
  i = i + 1;
  j = j +2*i + 1;
}
```

Example 1 Result

```
while (i < 10) {
  println(j);
  i = i + 1;
  j = j +2*i + 1;
}
```

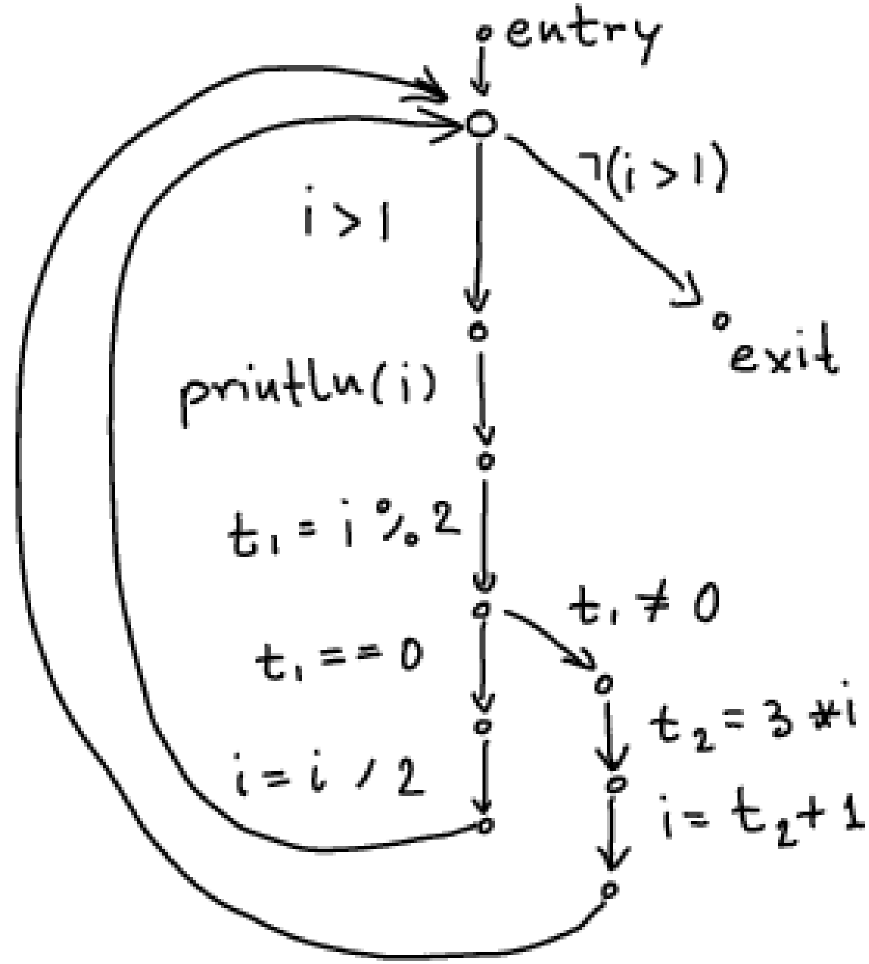


Example 2: Convert to CFG

```
int i = n;
while (i > 1) {
  println(i);
  if (i % 2 == 0) {
    i = i / 2;
  } else {
    i = 3*i + 1;
  }
}
```

Example 2 Result

```
int i = n;
while (i > 1) {
  println(i);
  if (i % 2 == 0) {
    i = i / 2;
  } else {
    i = 3*i + 1;
  }
}
```



Translation Functions

$[\ s_1\ ;\ s_2\ ]\ v_{source}\ v_{target} =$

$[\ s_1\ ]\ v_{source}\ \ v_{fresh}$

$[\ s_2\ ]\ v_{fresh}\ \ v_{target}$

$[$ **branch**$(x<y)\ ]\ v_{source}\ v_{true}\ v_{false} =$

insert$(v_{source},[x<y],v_{true});$

insert$(v_{source},[!(x<y)],v_{false})$

insert $(v_s,stmt,v_t)=$

$cfg = cfg + (v_s,stmt, v_t)$

$[\ x=y+z\ ]\ v_s\ v_t =$ **insert**$(v_s,x=y+z, v_t)$

when y,z are constants or variables

Analysis Domain (D)
Lattices

Abstract Intepretation Generalizes Type Inference

Type Inference

- computes types
- type rules
  - can be used to compute types of expression from subtypes
- types fixed for a variable

Abstract Interpretation

- computes **facts** from a domain
  - types
  - intervals
  - formulas
  - set of initialized variables
  - set of live variables
- transfer functions
  - compute facts for one program point from facts at previous program points
- facts change as the values of vars change (*flow-sensitivity*)

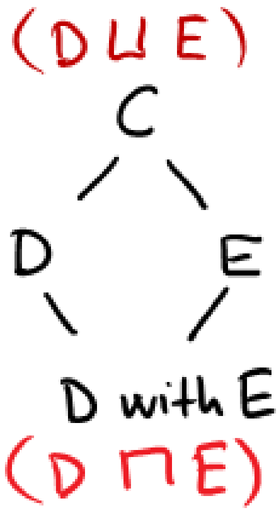
scalac computes types. Try in REPL:

```
class C
class D extends C
class E extends C
val p = false
val d = new D()
val e = new E()
val z = if (p) d else e

val u = if (p) (d,e) else (d,d)
val v = if (p) (d,e) else (e,d)

val f1 = if (p) ((d1:D) => 5) else ((e1:E) => 5)
val f2 = if (p) ((d1:D) => d) else ((e1:E) => e)
```

Finds "Best Type" for Expression

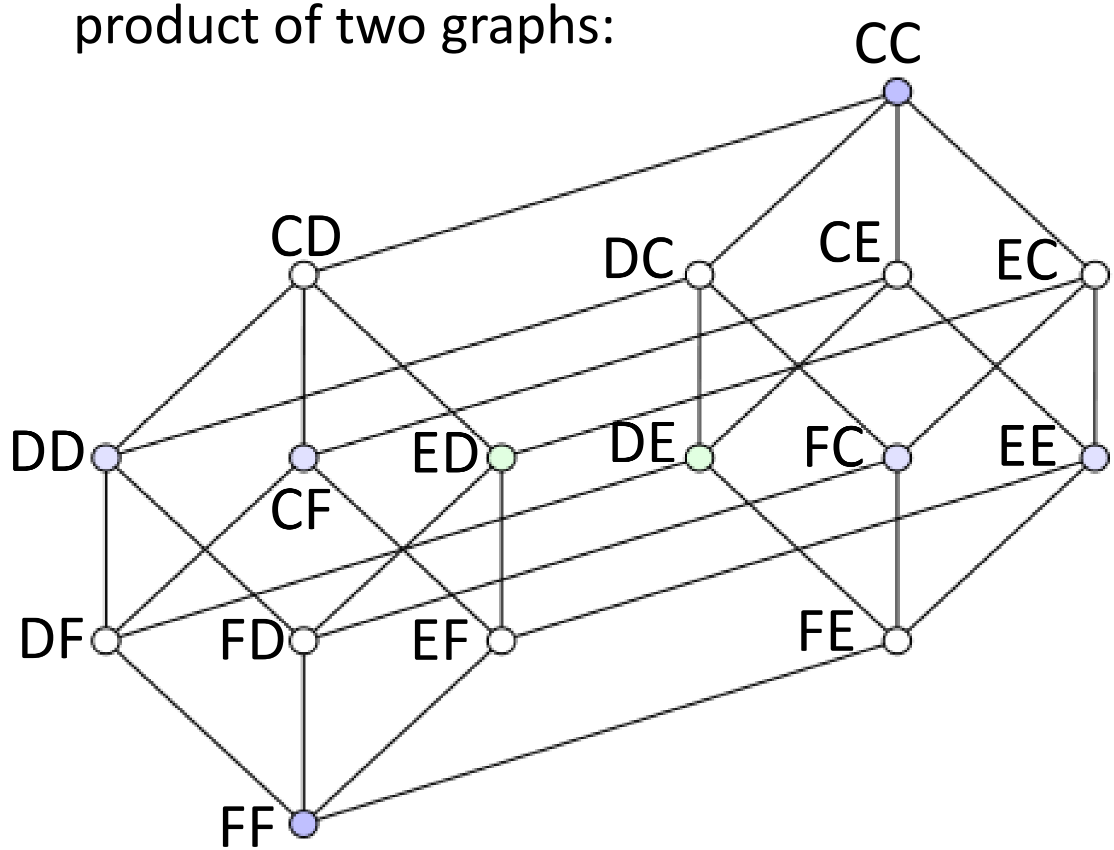
```
class C
class D extends C
class E extends C
val p = false
val d = new D()                                   //  d:D
val e = new E()                                   //  e:E
val z = if (p) d else e                           //  z:C

val u = if (p) (d,e) else (d,d)                   //  u:(D,C)
val v = if (p) (d,e) else (e,d)                   //  v:(C,C)

val f1 = if (p) ((d1:D) => 5) else ((e1:E) => 5)  //  f1: ((D with E) => Int)
val f2 = if (p) ((d1:D) => d) else ((e1:E) => e)  //  f2: ((D with E) => C)
```

Subtyping Relation in this Example



class C
class D **extends** C
class E **extends** C

product of two graphs:

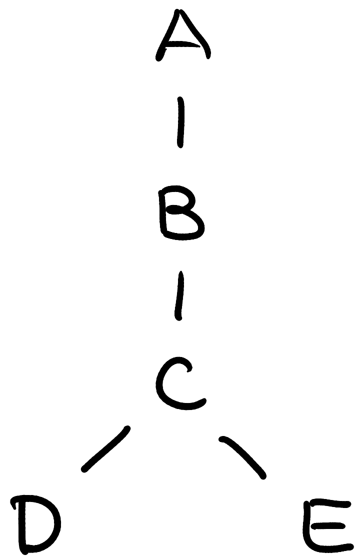


each relation can be visualized in 2D

- two relations: naturally shown in 4D (hypercube)

we usually draw larger elements higher

Least Upper Bound (lub, join)



A,B,C are all upper bounds on both D and E (they are above each of then in the picture, they are supertypes of D and supertypes of E).
Among these upper bounds, C is the least one (the most specific one).
We therefore say C is the **least upper bound**,

$$C = D \sqcup E$$

In any partial order $\leq$, if S is a set of elements (e.g. S={D,E}) then:

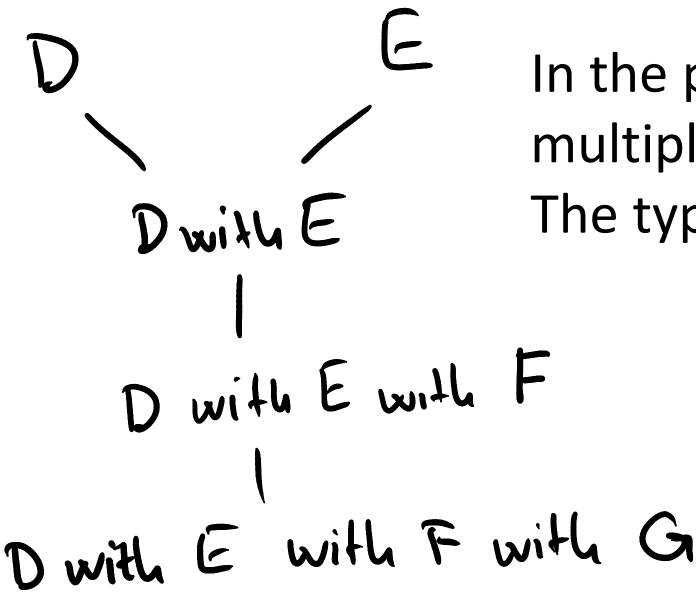
U is **upper bound** on S iff $x \leq U$ for every x in S.

U_0 is the **least upper bound (lub)** of S, written $U_0 = \bigsqcup S$, or U_0=lub(S) iff:

U_0 is upper bound and

if U is any upper bound on S, then $U_0 \leq U$

Greatest Lower Bound (glb, meet)



In the presence of traits or interfaces, there are multiple types that are subtypes of both D and E. The type (D with E) is the largest of them.

$D \sqcap E$

In any partial order $\leq$, if S is a set of elements (e.g. S={D,E}) then:

L is **lower bound** on S iff $L \leq x$ for every x in S.

L_0 is the **greatest upper bound (glb)** of S, written $L_0 = \bigsqcup S$, or L_0=glb(S), iff:

m_0 is upper bound and

if m is any upper bound on S, then $m_0 \leq m$

Computing lub and glb for tuple and function types

$$(x_1, y_1) \sqcup (x_2, y_2) = (x_1 \sqcup x_2,\ y_1 \sqcup y_2)$$

$$(x_1, y_1) \sqcap (x_2, y_2) = (x_1 \sqcap x_2,\ y_1 \sqcap y_2)$$

$$(x_1 \rightarrow y_1) \sqcup (x_2 \rightarrow y_2) = (x_1 \sqcap y_1) \rightarrow (y_1 \sqcup y_2)$$

$$(x_1 \rightarrow y_1) \sqcap (x_2 \rightarrow y_2) = (x_1 \sqcup y_1) \rightarrow (y_1 \sqcap y_2)$$

Lattice

Partial order: binary relation $\leq$ (subset of some D^2) which is

- reflexive: $x \leq x$
- anti-symmetric: $x \leq y \land y \leq x \rightarrow x=y$
- transitive: $x \leq y \land y \leq z \rightarrow x \leq z$

Lattice is a partial order in which every **two-element** set has **lub** and **glb**

- Lemma: if $(D, \leq)$ is lattice and D is finite, then lub and glb exist for every finite set

Idea of Why Lemma Holds

- $lub(x_1, lub(x_2, \ldots, lub(x_{n-1}, x_n)))$ is $lub(\{x_1, \ldots x_n\})$
- $glb(x_1, glb(x_2, \ldots, glb(x_{n-1}, x_n)))$ is $glb(\{x_1, \ldots x_n\})$
- lub of all elements in D is maximum of D
  - by definition, glb({}) is the maximum of D
- glb of all elements in D is minimum of D
  - by definition, lub({}) is the minimum of D

Graphs and Partial Orders

- If the domain is finite, then partial order can be represented by directed graphs
  - if $x \leq y$ then draw edge from x to y
- For partial order, no need to draw $x \leq z$ if $x \leq y$ and $y \leq z$. So we only draw non-transitive edges
- Also, because always $x \leq x$, we do not draw those self loops
- Note that the resulting graph is acyclic: if we had a cycle, the elements must to be equal

Defining Abstract Interpretation

Abstract Domain D describing which information to compute – this is often a lattice

- inferred types for each variable: x:T1, y:T2
- interval for each variable   x:[a,b], y:[a',b']

Transfer Functions, [[**st**]] for each statement **st**, how this statement affects the facts

- Example:

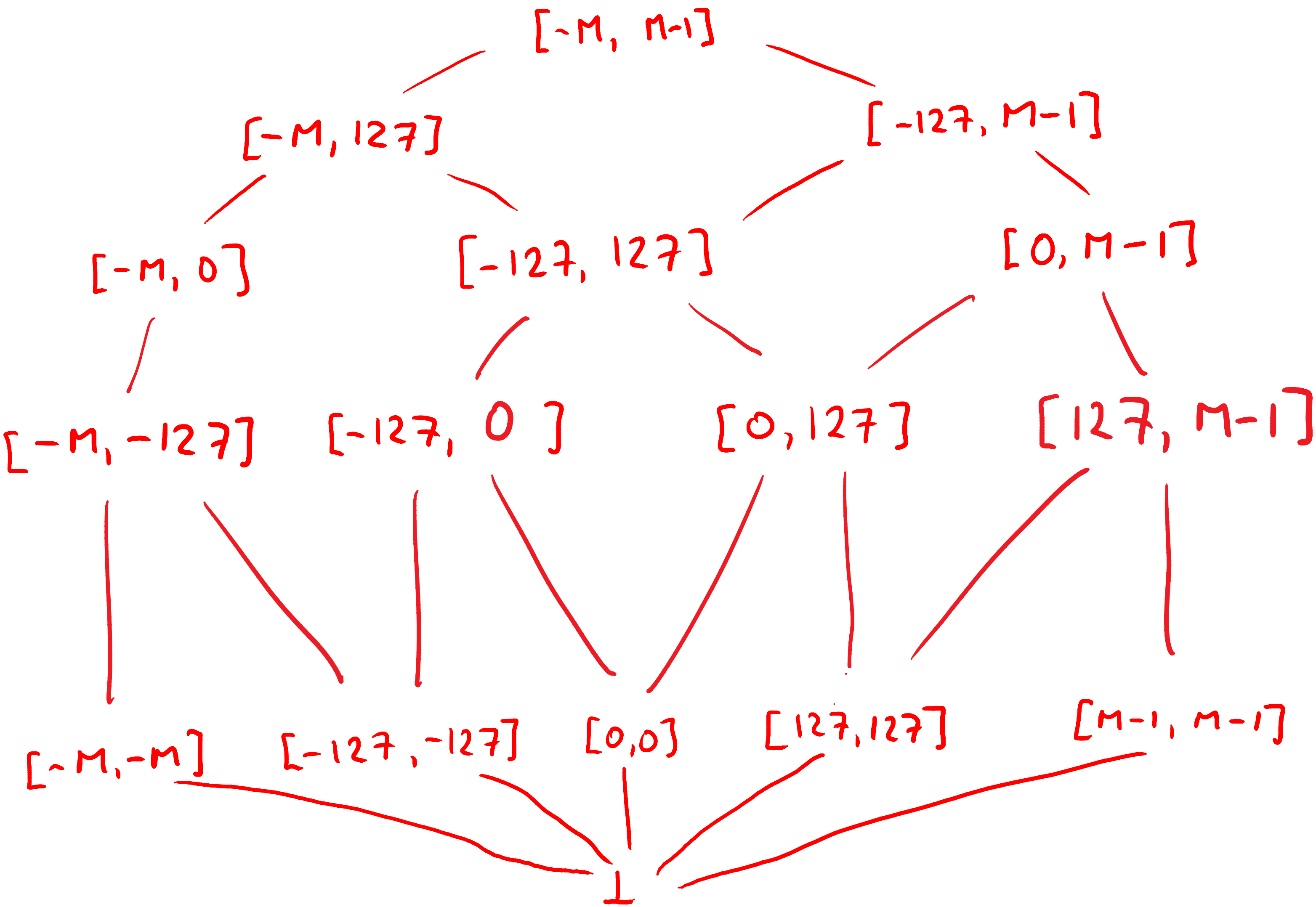
$[\![x = x+2]\!](x:[a,b], \ldots)$
$= (x:[a+2, b+2], \ldots)$

o x:[a,b] y:[c,d]

x = x+2

o x:[a+2, b+2], y:[c,d]

Domain of Intervals [a,b] where a,b$\in${-M,-127,0,127,M-1}



For now, we consider arbitrary integer bounds for intervals

- Really 'Int' should be BigInt, as in Haskell, Go
- Often we must analyze machine integers
  - need to correctly represent (and/or warn about) overflows and underflows
  - fundamentally same approach as for unbounded integers
- For efficiency, many analysis do not consider arbitrary intervals, but only a subset of them
- For now, we consider as the domain
  - empty set (denoted $\bot$, pronounced "bottom")
  - all intervals [a,b] where a,b are integers and $a \leq b$, or where we allow $a= -\infty$ and/or $b = \infty$
  - set of all integers $[-\infty, \infty]$ is denoted T , pronounced "top"

Find Transfer Function: Plus

Suppose we have only two integer variables: x,y

$x: [a,b] \quad y: [c,d]$

$$\Big\downarrow \; x = x+y$$

$x: [a',b'] \quad y: [c',d']$

If $a \le x \le b \qquad c \le y \le d$

and we execute **x= x+y**

then $x' = x+y$

$y' = y$

so

$\le x' \le$

$\le y' \le$

So we can let

a’= a+c b’ = b+d

c’=c d’ = d

Find Transfer Function: Minus

Suppose we have only two integer variables: x,y

$x : [a, b] \quad y : [c, d]$

$y = x - y$

$x : [a', b'] \quad y : [c', d']$

If

and we execute **y= x-y**

then

So we can let

$a' = a \qquad b' = b$

$c' = a - d \qquad d' = b - c$

Further transfer functions

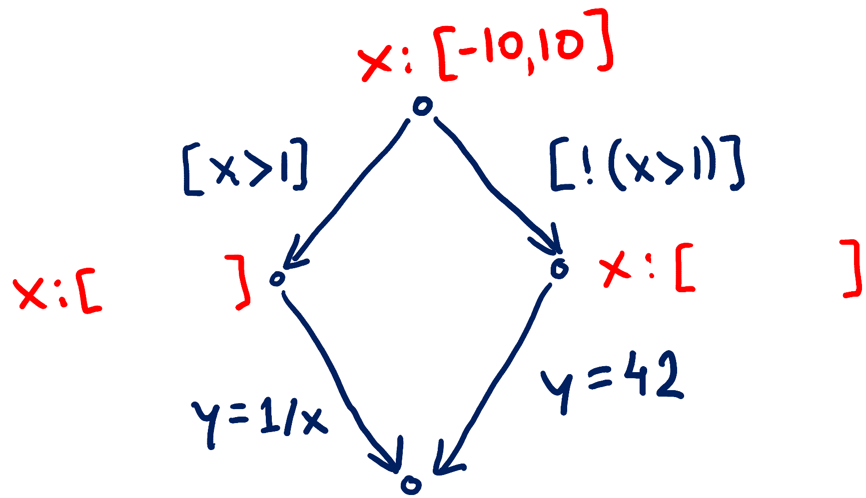
- x=y*z (assigning product)

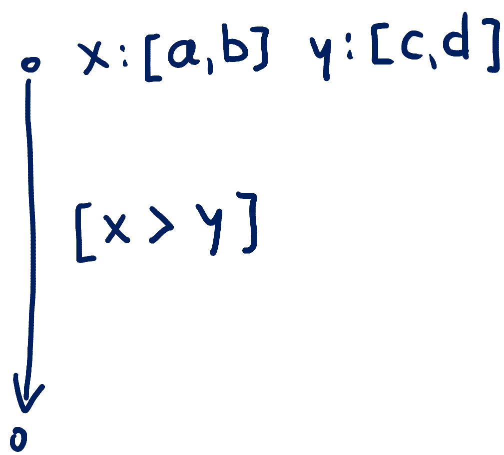
- x=y (copy)

Transfer Functions for **Tests**

Tests e.g. [x>1] come from translating if,while into CFG

```
x : [-10,10]
if (x > 1) {
   x :
   y = 1 / x
} else {
   x :
   y = 42
}
```

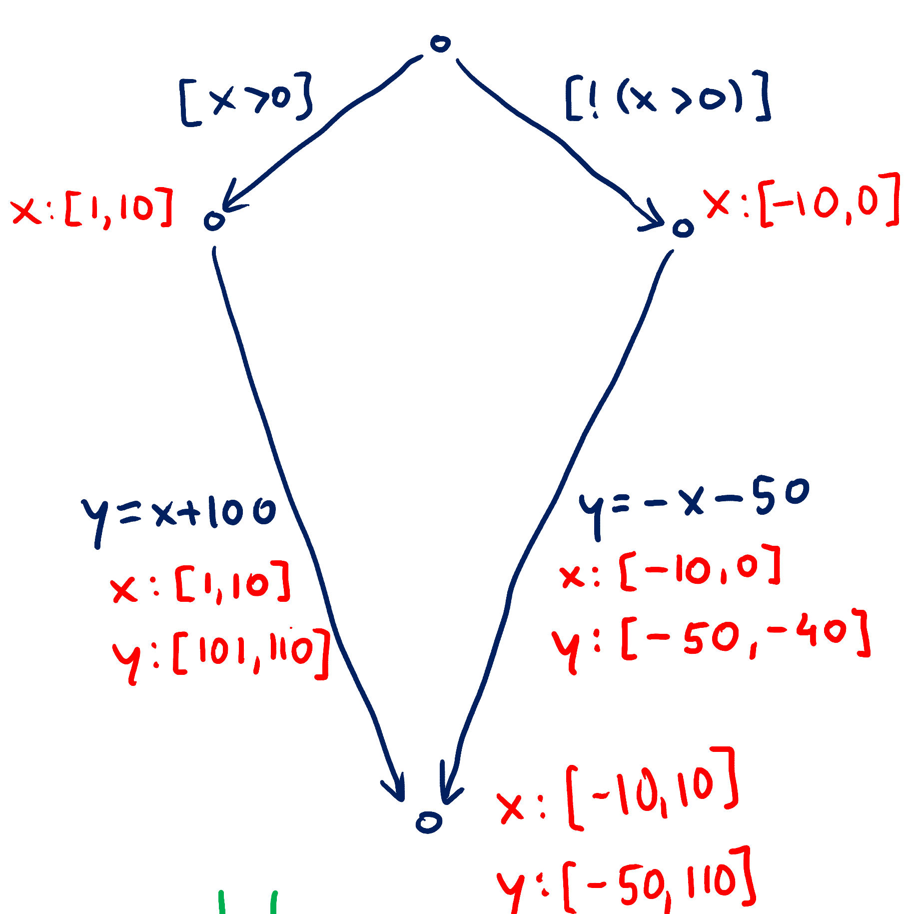




Joining Data-Flow Facts

x: [-10,10]   y: [-1000,1000]

```
if (x > 0) {
```
x: y:
```
  y = x + 100
```
x: y:
```
} else {
```
x: y:
```
  y = -x – 50
```
x: y:
```
}
```
x: y:

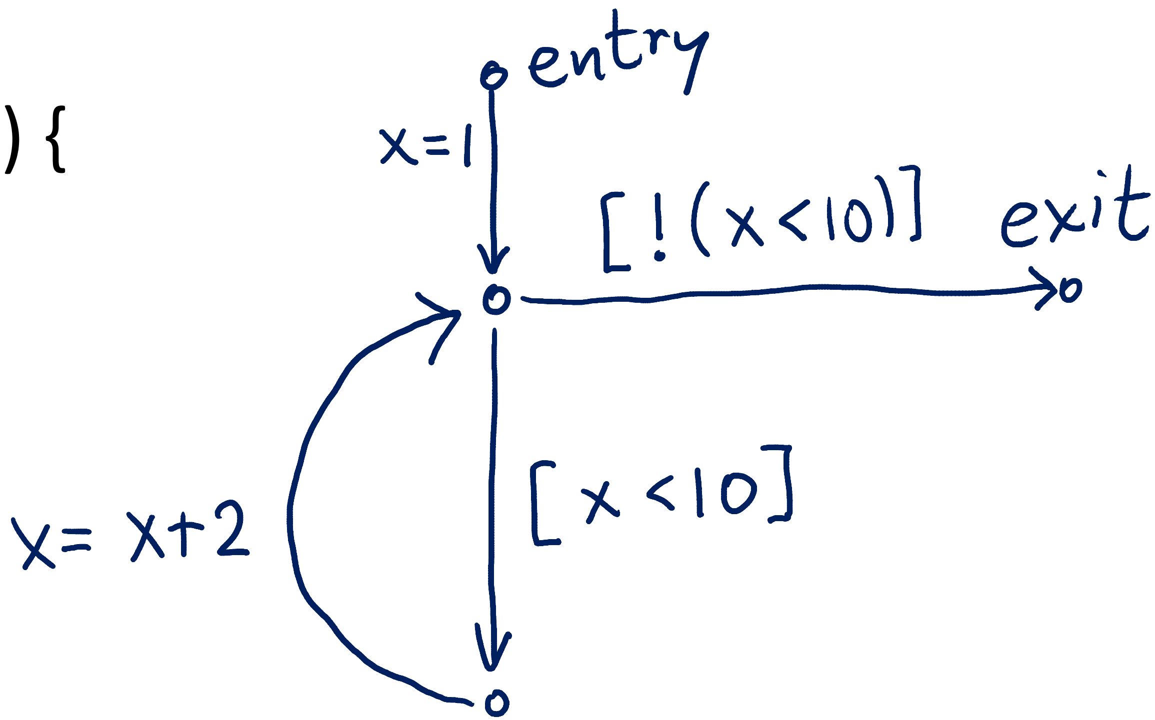


join $\sqcup$

$[a,b] \sqcup [c,d] = [\min(a,c), \max(b,d)]$

Handling Loops: Iterate Until Stabilizes

```
x = 1

while (x < 10) {

  x = x + 2

}
```



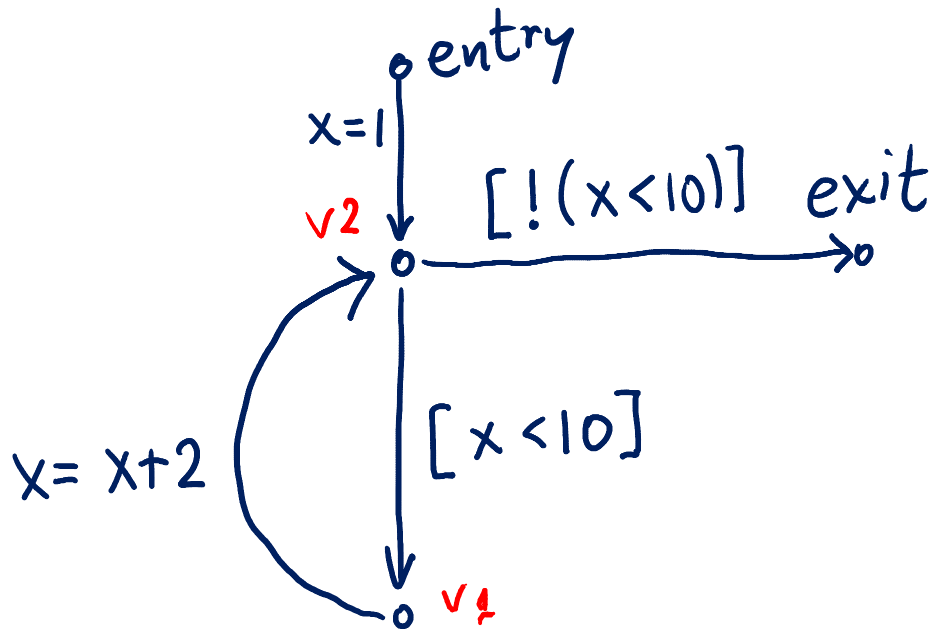
Analysis Algorithm

var facts : Map[Node,Domain] = Map.withDefault(empty)
facts(entry) = initialValues

while (there was change)
  pick edge (v1,statmt,v2) from CFG
      such that facts(v1) has changed
  facts(v2)=facts(v2) **join** transferFun(statmt, facts(v1))
}

$\sqcup$

Order does not matter for the end result, as long as we do not permanently neglect any edge whose source was changed.



```
var facts : Map[Node,Domain] = Map.withDefault(empty)
var worklist : Queue[Node] = empty
  def assign(v1:Node,d:Domain) = if (facts(v1)!=d) {
   facts(v1)=d
   for (stmt,v2) <- outEdges(v1) { worklist.add(v2) }
 }
assign(entry, initialValues)
while (!worklist.isEmpty) {
  var v2 = worklist.getAndRemoveFirst
  update = facts(v2)
  for (v1,stmt) <- inEdges(v2)
    { update = update join transferFun(facts(v1),stmt) }
  assign(v2, update)
}
```

Work List Version

Run range analysis, prove **error** is unreachable

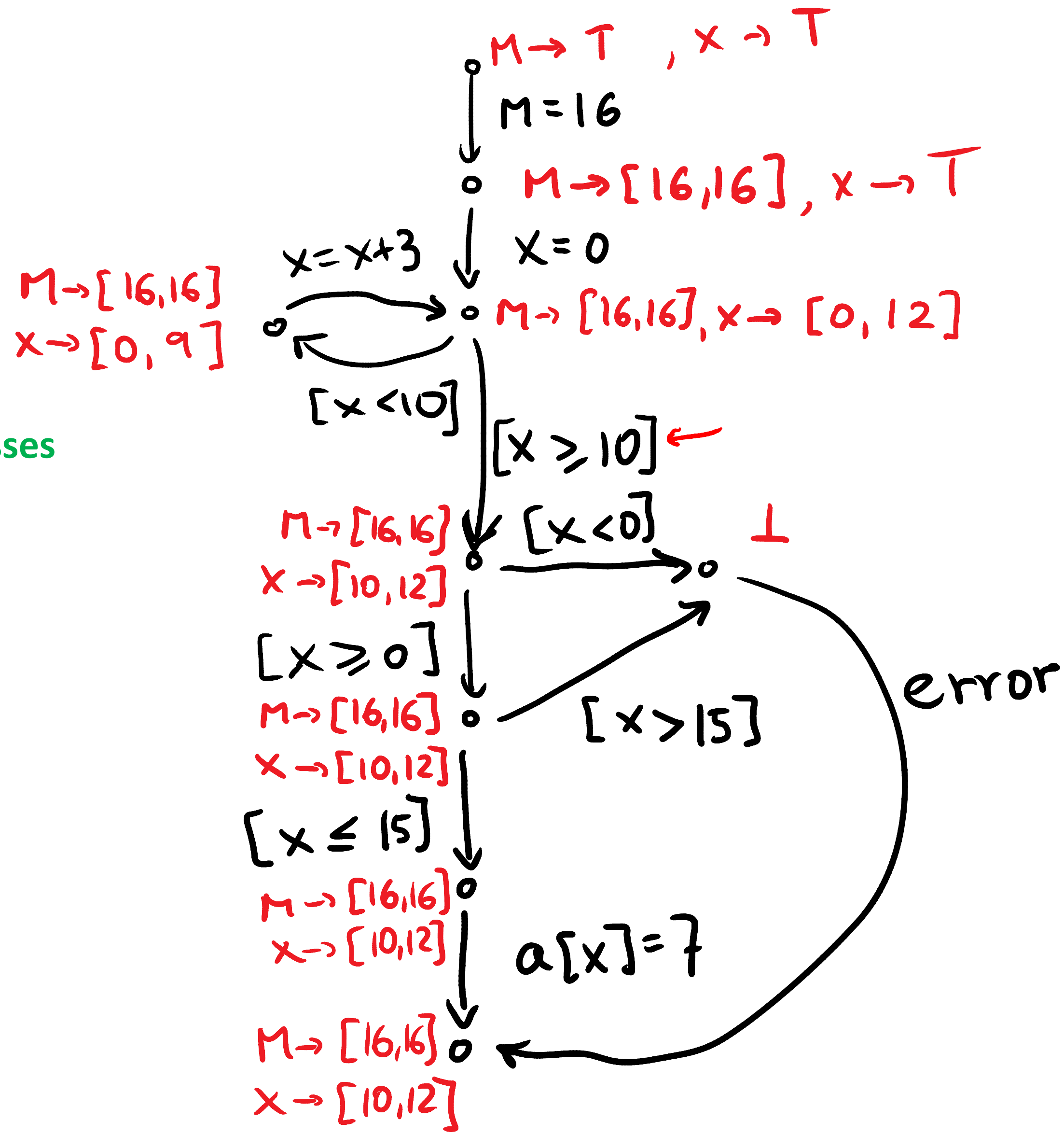
```
int M = 16;
int[M] a;
x := 0;
while (x < 10) {
  x := x + 3;
}
if (x >= 0) {
  if (x <= 15)
    a[x]=7;
  else
    error;
} else {
  error;
}
```

checks array accesses

Range analysis results

```
int M = 16;
int[M] a;
x := 0;
while (x < 10) {
  x := x + 3;
}
if (x >= 0) {
  if (x <= 15)
    a[x]=7;
  else
    error;
} else {
  error;
}
```

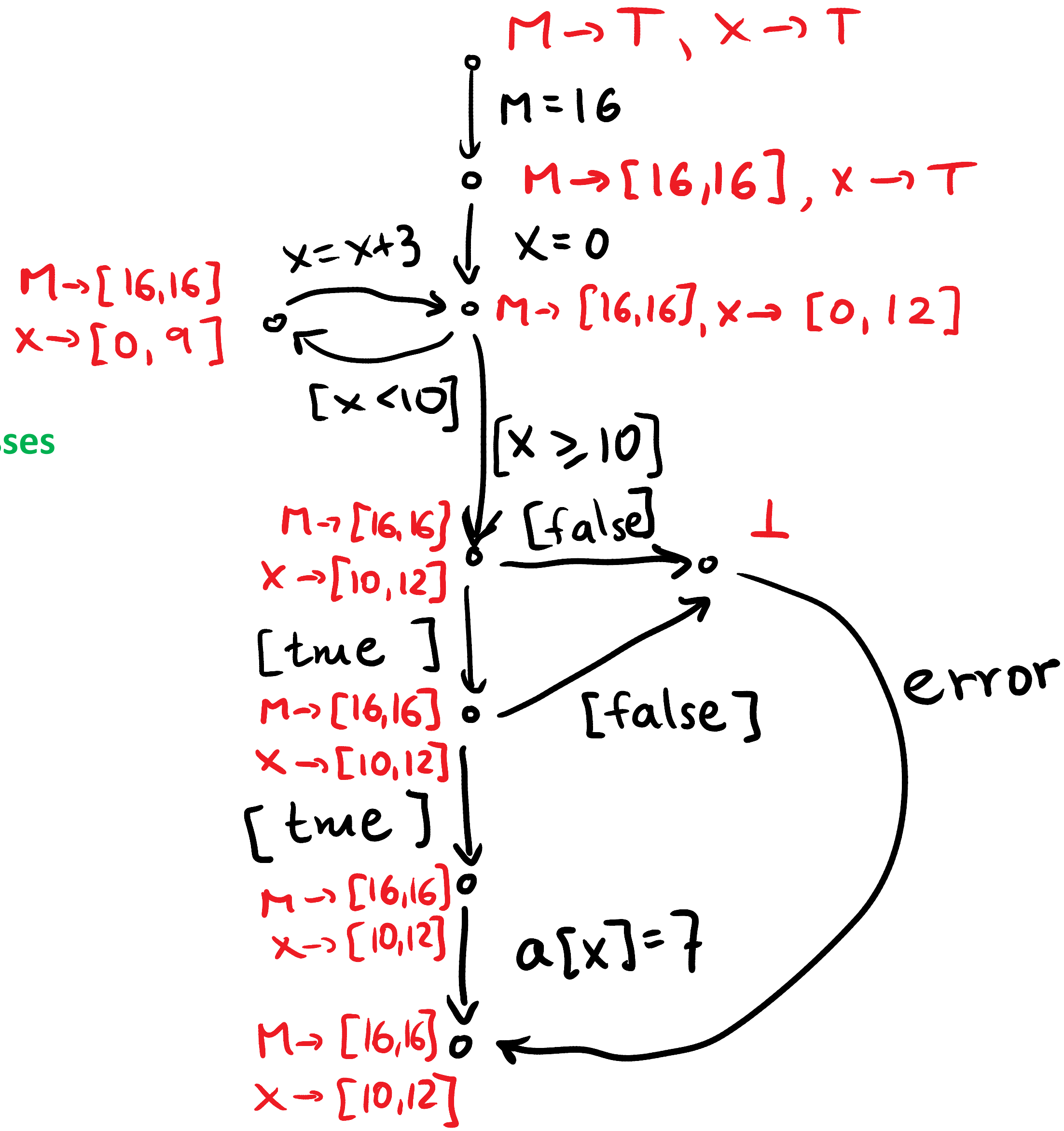
checks array accesses



Simplified Conditions

```
int M = 16;
int[M] a;
x := 0;
while (x < 10) {
  x := x + 3;
}
if (x >= 0) {
  if (x <= 15)
    a[x]=7;
  else
    error;
} else {
  error;
}
```

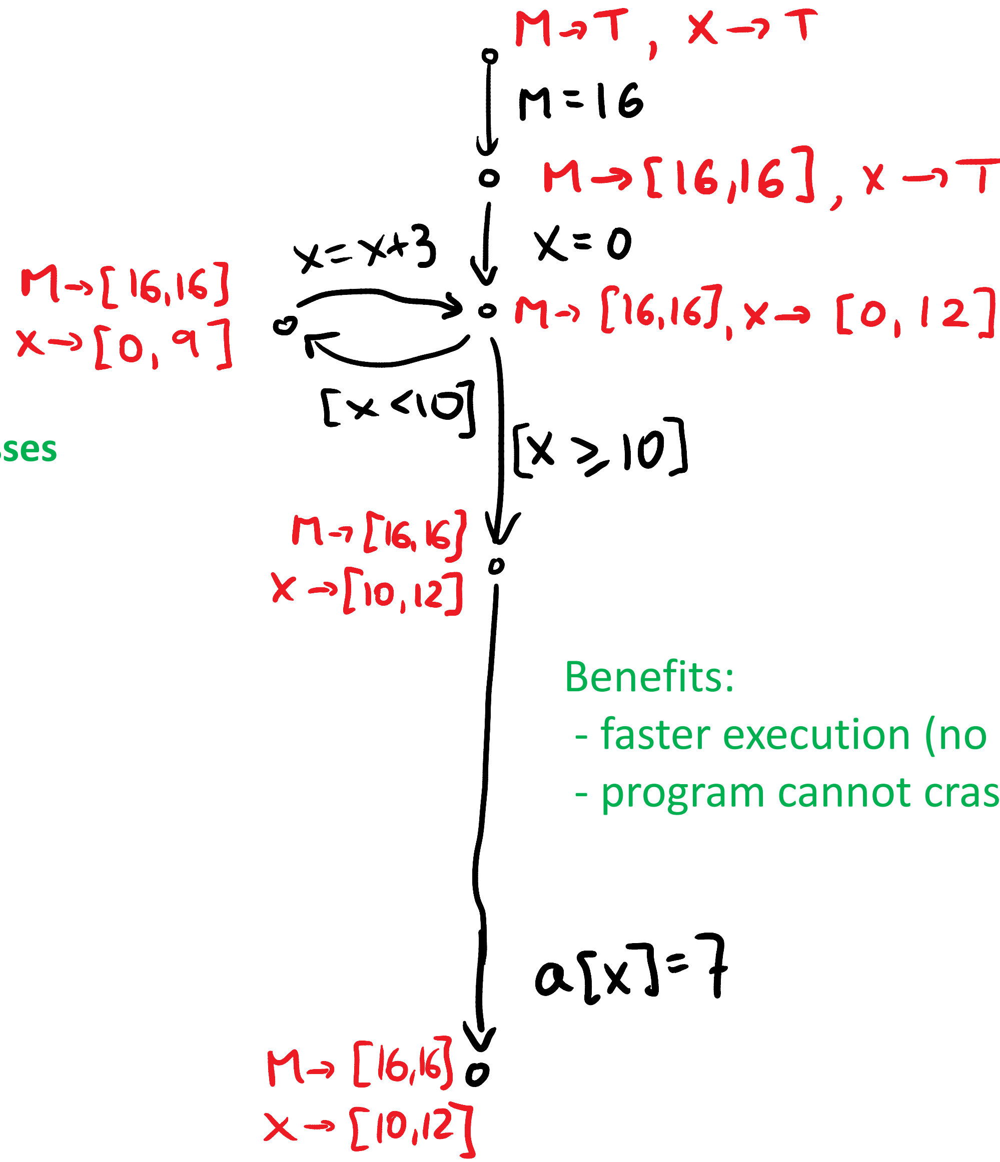
checks array accesses



Remove Trivial Edges, Unreachable Nodes

```
int M = 16;
int[M] a;
x := 0;
while (x < 10) {
  x := x + 3;
}
if (x >= 0) {
  if (x <= 15)
    a[x]=7;
  else
    error;
} else {
  error;
}
```

checks array accesses



Benefits:

- faster execution (no checks)
- program cannot crash with error

Apply Range Analysis and Simplify

For booleans, use this lattice: $D_b = \{ \overset{\perp}{\{\}}, \{false\}, \{true\}, \overset{\top}{\{false,true\}} \}$ with ordering given by set subset relation.

```
int a, b, step, i;
boolean c;
a = 0;
b = a + 10;
step = -1;
if (step > 0) {
  i = a;
} else {
  i = b;
}
c = true;
while (c) {
  process(i);
  i = i + step;
  if (step > 0) {
    c = (i < b);
  } else {
    c = (i > a);
  }
}
```